

ON A PROBLEM ABOUT SKOLEM'S PARADOX OF TAKEUTI'S VERSION

By

Masazumi HANAZAWA

§1. Introduction.

In his paper [2], Takeuti gave a theorem as a proof-theoretical interpretation of so-called Skolem's paradox. He showed as a corollary that Gödel-Bernays set theory with the additional axioms

$$\forall X \exists x \in \omega (f(x) = X)$$

is consistent, where f is a newly introduced function symbol.

Concerning it, we are interested here in the question whether we can add consistently the axiom schema

$$\overrightarrow{\forall}(\mathfrak{F}(0) \wedge \forall x(\mathfrak{F}(x) \rightarrow \mathfrak{F}(x+1))) \rightarrow \forall x \in \mathfrak{F}(x))^{1), 2)}$$

bisides with an axiom $\forall x \exists y \in \omega (f(y) = x)$ to any consistent set theory.

We show in this paper that the above question is affirmative for any consistent extension of ZF .³⁾

§2. Results.

2.1. CONVENTION. In the first order predicate calculus with equality, we regard the equality axioms as logical axioms. It means that equality axioms for any language are provided always tacitly.

f, g, h, \dots denote function symbols.

$\mathfrak{s}, \mathfrak{t}, \mathfrak{u}, \dots$ denote terms.

$\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \dots$ denote formulae.

\mathfrak{s} is a unary function symbol and $\mathfrak{s}(\mathfrak{t})$ is abbreviated by \mathfrak{t}' .

seq is a binary function symbol and $\text{seq}(\mathfrak{s}, \mathfrak{t})$ is abbreviated by $\mathfrak{s}\mathfrak{t}$.

P, Q, R, \dots denote predicate symbols.

1) $\overrightarrow{\forall}\mathfrak{A}$ denotes the universal closure of a formula \mathfrak{A} .

2) The formula $\mathfrak{F}(0)$ contains possibly the newly introduced function symbol f .

3) See [1], where the rather folklore result that the question is affirmative for any consistent extension of ZFC is shown.

N is a unary predicate symbol.

Iq is a binary predicate symbol and $Iq(s, t)$ is abbreviated by $s < t$.

2.2. THEOREM 1. Let I' be a consistent theory in which the sentences of the following forms are provable:

$$N(1) \wedge \forall x(N(x) \rightarrow N(x')) \wedge \forall x(x' \neq 1) \wedge \forall x \forall y(x' = y' \rightarrow x = y).$$

$$\vec{\forall}(\mathfrak{A}(1) \wedge \forall x(\mathfrak{A}(x) \rightarrow \mathfrak{A}(x')) \rightarrow \forall x(N(x) \rightarrow \mathfrak{A}(x))) \text{ for any formula } \mathfrak{A}(x),$$

$$\forall x(N(x) \rightarrow \exists x < 1 \wedge 1 < x' \wedge \forall y(N(y) \rightarrow x' < y' \leftrightarrow x < y)),$$

$$\forall x \forall y \forall z(N(z) \rightarrow \exists w(\forall v(N(v) \wedge v < z \rightarrow x'v = w'v) \wedge w'z = y)).$$

Then the theory \mathcal{A} which is obtained from I' by adjoining the following axioms is consistent:

$$\forall x \exists y(N(y) \wedge f(y) = x),$$

$$\vec{\forall}(\mathfrak{F}(1) \wedge \forall x(\mathfrak{F}(x) \rightarrow \mathfrak{F}(x')) \rightarrow \forall x(N(x) \rightarrow \mathfrak{F}(x))),$$

where f is a function symbol not in I' and $\mathfrak{F}(x)$ is any formula containing the function symbol f .

2.3. PROOF of Theorem 1. Let L_I and $L_{\mathcal{A}}$ be the languages of I' and \mathcal{A} respectively. $\vdash_I \mathfrak{A}$ means the formula \mathfrak{A} is provable in I' . $\vdash \mathfrak{A}$ means the formula \mathfrak{A} is provable from only logical axioms. We shall use the following abbreviation:

$\mathbf{Cond}(x)$ is the L_I -formula $N(x'1)$,

$x[y]$ is the L_I -term $(x'2)'y^1$,

$x \subset y$ is the L_I -formula

$$\mathbf{Cond}(x) \wedge \mathbf{Cond}(y) \wedge x'1 \leq y'1 \wedge \forall z(N(z) \wedge z < x'1 \rightarrow x[z] = y[z]).$$

The letters p, q, r are used as variables that range over $\mathbf{Cond}(\ast)$: e.g. $\exists p \mathfrak{A}(p)$ denotes the formula $\exists x(\mathbf{Cond}(x) \wedge \mathfrak{A}(x))$. In the same sense, the letters i, j, k, l, m, n are used as variables ranging over $N(\ast)$.

For each $L_{\mathcal{A}}$ -formula \mathfrak{A} (resp. $L_{\mathcal{A}}$ -term t) we define its rank $\rho(\mathfrak{A})$ (resp. $\rho(t)$) inductively as follows:

- 1) $\rho(x) = 0$;
- 2) $\rho(g(t_1, \dots, t_\mu)) = \rho(t_1) + \dots + \rho(t_\mu) + 1$;
- 3) $\rho(s = t) = \rho(s) + 2 \cdot \rho(t)$;
- 4) For a predicate symbol P different from $=$,
 $\rho(P(t_1, \dots, t_\mu)) = \rho(t_1) + \dots + \rho(t_\mu) + 1$;
- 5) $\rho(\exists x \mathfrak{A}) = \rho(\mathfrak{A}) + 1$;
- 6) $\rho(\mathfrak{A} \wedge \mathfrak{B}) = \rho(\mathfrak{A}) + \rho(\mathfrak{B}) + 1$;
- 7) $\rho(\exists x \mathfrak{A}) = \rho(\mathfrak{A}) + 1$.

For every pair of $L_{\mathcal{A}}$ -formula \mathfrak{A} and an individual variable u , we define an

4) "2" stands for "1".

L_T -formula denoted by $[v \Vdash \mathfrak{A}]$ by induction on the rank $\rho(\mathfrak{A})$ as follows:

- 1) $[v \Vdash x=y]$ is $x=y$;
- 2) $[v \Vdash f(x)=y]$ is $(N(x) \wedge x < v \cdot 1 \wedge v[x]=y) \vee (\neg N(x) \wedge y=1)$;
- 3) For a function symbol g different from f ,
 $[v \Vdash g(x_1, \dots, x_\mu)=y]$ is $g(x_1, \dots, x_\mu)=y$;
- 4) If $\rho(t_1) + \dots + \rho(t_\mu) > 0$, then $[v \Vdash g(t_1, \dots, t_\mu)=x]$ is
 $\exists y_1 \dots \exists y_\mu ([v \Vdash g(y_1, \dots, y_\mu)=x] \wedge [v \Vdash t_1=y_1] \wedge \dots \wedge [v \Vdash t_\mu=y_\mu])$;
- 5) If $\rho(t) > 0$, then
 $[v \Vdash \mathfrak{s}=t]$ is $\exists x ([v \Vdash \mathfrak{s}=x] \wedge [v \Vdash t=x])$;
- 6) For a predicate symbol P different from $=$, $[v \Vdash P(t_1, \dots, t_\mu)]$ is
 $\exists x_1 \dots \exists x_\mu ([v \Vdash t_1=x_1] \wedge \dots \wedge [v \Vdash t_\mu=x_\mu] \wedge P(x_1, \dots, x_\mu))$;
- 7) $[v \Vdash \neg \mathfrak{A}]$ is $\neg (\exists v \supset v) [v \Vdash \mathfrak{A}]$;
- 8) $[v \Vdash \mathfrak{A} \wedge \mathfrak{B}]$ is $[v \Vdash \mathfrak{A}] \wedge [v \Vdash \mathfrak{B}]$;
- 9) $[v \Vdash \exists x \mathfrak{A}(x)]$ is $\exists y [v \Vdash \mathfrak{A}(y)]^5$.

Lemmata 1, 2, 3 and 4 below follow from routine arguments.

LEMMA 1.

- 1) Except for v , all variables occurring free in $[v \Vdash \mathfrak{A}]$ occur free in \mathfrak{A} ;
- 2) If both x and y are different from v , then
 $\text{Sub}_y^x [v \Vdash \mathfrak{A}]$ is $[v \Vdash \text{Sub}_y^x \mathfrak{A}]^6$;
- 3) $\vdash_T x=y \wedge [v \Vdash \mathfrak{A}(x)] \rightarrow [v \Vdash \mathfrak{A}(y)]$;
- 4) $\vdash_T x_1=y_1 \wedge \dots \wedge x_\mu=y_\mu \wedge [v \Vdash \mathfrak{A}(x_1, \dots, x_\mu)] \rightarrow [v \Vdash \mathfrak{A}(y_1, \dots, y_\mu)]$;
- 5) $\vdash_T [v \Vdash t=x] \wedge [v \Vdash t=y] \rightarrow x=y$;
- 6) $\vdash_T [v \Vdash g(t_1, \dots, t_\mu)=\mathfrak{s}] \leftrightarrow$
 $\exists x_1 \dots \exists x_\mu \exists y ([v \Vdash t_1=x_1] \wedge \dots \wedge [v \Vdash t_\mu=x_\mu] \wedge [v \Vdash \mathfrak{s}=y] \wedge [v \Vdash g(x_1, \dots, x_\mu)=y])$;
- 7) $\vdash_T q \supset p \wedge [v \Vdash \mathfrak{A}] \rightarrow [v \Vdash q \supset p]$;
- 8) $\vdash_T \forall p \exists q \supset p \exists w [q \Vdash t=w]$, for each term t ;
- 9) $\vdash_T [v \Vdash t=w] \rightarrow ([v \Vdash \mathfrak{A}(t)] \leftrightarrow [v \Vdash \mathfrak{A}(w)])$;
- 10) If \mathfrak{A} is an L_T -formula, then $\vdash \mathfrak{A} \leftrightarrow [v \Vdash \mathfrak{A}]$.

For an L_A -formula \mathfrak{A} , we shall denote by \mathfrak{A}^* the L_T -formula $\forall p \exists q \supset p [q \Vdash \mathfrak{A}]$.

LEMMA 2. If \mathfrak{A} is an L_T -formula, then $\vdash_T \mathfrak{A} \leftrightarrow \mathfrak{A}^*$.

LEMMA 3. If $\vdash \mathfrak{A}$, then $\vdash_T \mathfrak{A}^*$.

LEMMA 4. If $\vdash_T \mathfrak{A}^*$ and $\vdash_T (\mathfrak{A} \rightarrow \mathfrak{B})^*$, then $\vdash_T \mathfrak{B}^*$.

LEMMA 5. $\vdash_T (\forall x \exists y (f(y)=x))^*$.

5) The variable y is any variable not occurring free in $[v \Vdash \exists x \mathfrak{A}(x)]$,

6) $\text{Sub}_y^x \mathfrak{A}$ means the substitution of y for x (occurring free) in \mathfrak{A} (tacitly assuming that y is free for x in \mathfrak{A}).

PROOF. Since $\vdash_r \forall x \forall y \forall i \exists z (\forall j < i (x'j = z'j) \wedge z'i = y)$, it follows that $\vdash_r \forall x \forall p \exists q \supset p \exists i (q[i] = x)$ which implies $\vdash_r \forall p [p \Vdash \forall x \exists i (f(i) = x)]$.

LEMMA 6. $\vdash_r (\vec{\forall}(\mathfrak{F}(1) \wedge \forall x (\mathfrak{F}(x) \rightarrow \mathfrak{F}(x')) \rightarrow \forall x (N(x) \rightarrow \mathfrak{F}(x)))^*$, for any L_A -formula $\mathfrak{F}(x)$.

PROOF. It suffices to show that for any L_A -formula $\mathfrak{F}(x)$,

$$\vdash_r \vec{\forall}(\exists i \mathfrak{F}(i) \rightarrow \exists i (\mathfrak{F}(i) \wedge \forall j < i \neg \mathfrak{F}(j)))^*.$$

We explain the proof informally to save the space. In general, it holds that $\vdash_r (\forall x \mathfrak{A})^* \leftrightarrow \forall x \mathfrak{A}^*$ and $\vdash_r (\mathfrak{A} \rightarrow \mathfrak{B})^* \leftrightarrow \forall p ([p \Vdash \mathfrak{A}] \rightarrow \exists q \supset p [q \Vdash \mathfrak{B}])$. So it is sufficient to prove $\exists q \supset p [q \Vdash \exists i (\mathfrak{F}(i) \wedge \forall j < i \neg \mathfrak{F}(j))]$ assuming $[p \Vdash \exists i \mathfrak{F}(i)]$. Suppose $[p \Vdash \exists i \mathfrak{F}(i)]$. Note that “ $\exists q \supset p [q \Vdash \mathfrak{F}(i)]$ ” is an L_r -formula. Hence by means of the mathematical induction in I , we obtain the following in I :

$$\exists i (\exists q \supset p [q \Vdash \mathfrak{F}(i)]) \rightarrow \exists i (\exists q \supset p [q \Vdash \mathfrak{F}(i)] \wedge \forall j < i \neg (\exists q \supset p [q \Vdash \mathfrak{F}(j)]).$$

Since $[p \Vdash \exists i \mathfrak{F}(i)]$ implies $\exists i (\exists q \supset p [q \Vdash \mathfrak{F}(i)])$, there exist i and q such that $q \supset p \wedge [q \Vdash \mathfrak{F}(i)] \wedge \forall j < i \forall q \supset p \neg [q \Vdash \mathfrak{F}(j)]$. Then $\forall j < i [q \Vdash \neg \mathfrak{F}(j)]$, which implies $[q \Vdash \forall j < i (\neg \mathfrak{F}(j))]$.

Hence $\exists q \supset p [q \Vdash \exists i (\mathfrak{F}(i) \wedge \forall j < i \neg \mathfrak{F}(j))]$.

LEMMA 7. If an L_A -formula \mathfrak{A} is provable in \mathcal{A} , then $\vdash_r \mathfrak{A}^*$.

PROOF. Besides Lemmata 5 and 6, it holds that $\vdash_r \mathfrak{A}^*$ for every theorem \mathfrak{A} of I because of Lemma 2. Thus, $\vdash_r \mathfrak{A}^*$ for every axiom \mathfrak{A} in \mathcal{A} . So, by Lemmata 3 and 4, $\vdash_r \mathfrak{A}^*$ for every theorem \mathfrak{A} in \mathcal{A} .

CONCLUSION. \mathcal{A} is consistent.

PROOF. If not, I should be inconsistent by Lemmata 2 and 7.

2.4. Remarks.

2.4.1. COROLLARY 1. If I' is any consistent extension of Zermelo's set theory Z but its language is the same as Z , then the theory which is obtained from I' by adjoining the following axiom is consistent:

$$\forall x \exists y \in \omega (f(y) = x)$$

and all the sentences of the form

$$\vec{\forall}(\mathfrak{F}(0) \wedge \forall x (\mathfrak{F}(x) \rightarrow \mathfrak{F}(x+1)) \rightarrow \forall x \in \omega \mathfrak{F}(x)),$$

where f is a newly introduced function symbol and $\mathfrak{F}(x)$ is any formula of the language $L_r \cup \{f\}$.

2.4.2. Following the proof of Theorem 1, we obtain the following model theoretical theorem which seems to have an independent interest:

THEOREM 2. Let $\mathfrak{M} = \langle \mathcal{M}, \mathcal{N}, 0, *, ** \rangle$ be a countable model satisfying

- 1) $\mathfrak{M} \models (\text{Peano arithmetic})^N$,
- 2) $\mathfrak{M} \models \vec{\forall}(\mathfrak{A}(0) \wedge \forall x(\mathfrak{A}(x) \rightarrow \mathfrak{A}(x')) \rightarrow \forall x(\mathcal{N}(x) \rightarrow \mathfrak{A}(x)))$ for all formulae \mathfrak{A} of the language for \mathfrak{M} ,
- 3) $\mathfrak{M} \models \forall a \forall x \forall i \exists y (\forall j < i (x^j = y^j) \wedge y^i = a)$.

Then there exists a function $\nearrow: \mathcal{M} \rightarrow \mathcal{M}$ such that

- 1) $\nearrow \upharpoonright \mathcal{N}: \mathcal{N} \rightarrow \mathcal{M}$ is surjective.
- 2) $(\mathfrak{M}, \nearrow) \models \vec{\forall}(\mathfrak{F}(0) \wedge \forall x(\mathfrak{F}(x) \rightarrow \mathfrak{F}(x')) \rightarrow \forall x(\mathcal{N}(x) \rightarrow \mathfrak{F}(x)))$ for all formulae $\mathfrak{F}(x)$ of the language for (\mathfrak{M}, \nearrow) .

PROOF. Call an ordered pair (i, p) a forcing condition if $i \in \mathcal{N}$ and $p \in \mathcal{M}$. Define the relation $(i, p) \rightarrow (j, q)$ by " $\mathfrak{M} \models i < j$ and $\forall k \in \mathcal{N} (\mathfrak{M} \models (k \leq i \rightarrow p^k = q^k))$ ". Define the forcing condition $(i, p) \Vdash \mathfrak{A}$ between a forcing condition (i, p) and a sentence \mathfrak{A} of the language for $(\mathfrak{M}, \nearrow, a)_{a \in \mathfrak{M}}$ as follows:

- 1) $(i, p) \Vdash \underline{v} = \underline{w} \iff v = w$;
- 2) $(i, p) \Vdash \underline{f}(t) = \underline{u} \iff (\exists v, w \in \mathcal{M}) ((i, p) \Vdash (t = \underline{v} \wedge u = \underline{w}))$ and $((v \in \mathcal{N}$ and $\mathfrak{M} \models (v < i \wedge p^v = \underline{w}))$ or $(v \notin \mathcal{N}$ and $w = 0))$;
- 3) If g is a function symbol different from f , then $(i, p) \Vdash g(t_1, \dots, t_n) = \underline{u} \iff (\exists v_1, \dots, v_n, w \in \mathcal{M}) (\mathfrak{M} \models g(v_1, \dots, v_n) = \underline{w}$ and $(i, p) \Vdash (t_1 = \underline{v}_1 \wedge \dots \wedge t_n = \underline{v}_n \wedge u = \underline{w}))$;
- 4) $(i, p) \Vdash P(t_1, \dots, t_n) \iff (\exists v_1, \dots, v_n \in \mathcal{M}) (\mathfrak{M} \models P(v_1, \dots, v_n) \wedge (i, p) \Vdash (t_1 = \underline{v}_1 \wedge \dots \wedge t_n = \underline{v}_n))$;
- 5) $(i, p) \Vdash \mathfrak{A} \wedge \mathfrak{B} \iff (i, p) \Vdash \mathfrak{A}$ and $(i, p) \Vdash \mathfrak{B}$;
- 6) $(i, p) \Vdash \neg \mathfrak{A} \iff \forall j \in \mathcal{N} \forall q \in \mathcal{M} ((i, p) \rightarrow (j, q) \Rightarrow \neg (j, q) \Vdash \mathfrak{A})$;
- 7) $(i, p) \Vdash \exists x \mathfrak{A}(x) \iff \exists v \in \mathcal{M} ((i, p) \Vdash \mathfrak{A}(v))$.

Now, take a complete sequence $(i_0, p_0) \rightarrow (i_1, p_1) \rightarrow \dots \rightarrow (i_n, p_n) \rightarrow \dots$ and define $\nearrow: \mathfrak{M} \rightarrow \mathfrak{M}$ by " $\nearrow(x) = y \iff \exists v ((i_n, p_n) \Vdash \underline{f}(x) = \underline{y})$ ". Then the function \nearrow has the desired property as easily seen from the proof of Theorem.

2.4.3. As a by-product of the result, we obtain the result below, which concerns the problem of elimination of ε -symbol in the first order predicate calculus.

In the following, the ε -predicate calculus means the system obtained from the first order predicate calculus by adjoining the ε -symbol.

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- 7) $(\dots)^N$ means the relativization by N .

THEOREM 3. Let Γ be any theory satisfying the condition mentioned in Theorem 1. Let Γ_ε be the theory in ε -predicate calculus which is obtained from Γ by adjoining an axiom schema of mathematical induction of the strengthened form

$$\vec{\forall}(\mathfrak{G}(0) \wedge \forall x(\mathfrak{G}(x) \rightarrow \mathfrak{G}(x')) \rightarrow \forall x(\mathcal{N}(x) \rightarrow \mathfrak{G}(x)))$$

where $\mathfrak{G}(x)$ is any formula of ε -predicate calculus.

Then, if a sentence \mathfrak{S} is provable in Γ_ε and is ε -symbol free, \mathfrak{S} is provable in Γ .

PROOF. It suffices to prove Γ_ε is consistent. By Theorem 1, the theory $\mathcal{A} = \Gamma \cup \{\forall x \exists y(\mathcal{N}(y) \wedge f(y) = x)\} \cup \{\vec{\forall}(\mathfrak{F}(0) \wedge \forall x(\mathfrak{F}(x) \rightarrow \mathfrak{F}(x')) \rightarrow \forall x(\mathcal{N}(x) \rightarrow \mathfrak{F}(x))) \mid \mathfrak{F}: \text{a formula of } L_\Gamma \cup \{f\}\}$ is consistent. Besides $\varepsilon x \mathfrak{A}(x)$ can be interpreted in \mathcal{A} as follows:

$$\varepsilon x \mathfrak{A}(x) = {}^{\mathcal{A}}f \iota x ((\exists z \mathfrak{A}(z) \rightarrow \mathfrak{A}(x)) \wedge \forall y(\mathcal{N}(y) \wedge (\exists z \mathfrak{A}(z) \rightarrow \mathfrak{A}(f(y))) \rightarrow \exists w(\mathcal{N}(w) \wedge w \leq y \wedge f(w) = x))).$$

The following corollary is put to assess the meaning of this assertion:

COROLLARY 2. $ZF + \neg AC + \{\vec{\forall}(\varphi(0) \wedge \forall x(\varphi(x) \rightarrow \varphi(x+1)) \rightarrow \forall x \in \omega \varphi(x)) \mid \varphi: \text{a formula possibly containing } \varepsilon\text{-symbols}\}$ is consistent in ε -predicate calculus.⁸⁾

This is an extension of the well-known fact that $ZF + \neg AC$ is consistent even in ε -predicate calculus.⁹⁾

References

- [1] Hanazawa, M., An Interpretation of Skolem's Paradox in the Predicate Calculus with ε -symbol, Science Reports of the Saitama University, Series A Vol. 9, No. 1 (1979) 11-19.
- [2] Takeuti, G., On Skolem's theorem, J. Math. Soc. Japan, 9 (1957) 71-76.

Department of Mathematics
Faculty of Science
Saitama University
Urawa, Japan

8) We assume here the consistency of ZF .

9) Of course, the axiom schema of replacement must be understood to be a set of axioms, i.e. that schema is not applied to formulae containing ε -symbols.