

**ON THE LONG-RANGE SCATTERING FOR ONE- AND
TWO-PARTICLE SCHRÖDINGER OPERATORS
WITH CONSTANT MAGNETIC FIELDS***

By

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1. Introduction

In this paper we prove the asymptotic completeness for the following 1-particle Schrödinger operator with a constant magnetic field $B \in \mathbf{R}^3$, $B \neq 0$:

$$H = H_0 + V = \frac{1}{2} \left(p - \frac{1}{2} B \times r \right)^2 + V(r),$$

where $r \in \mathbf{R}^3$ and $p = -i\nabla_r$. The real-valued smooth function $V(r)$ is a long-range potential, that is, we impose the following decay condition on $V(r)$:

(V) As $|r| \rightarrow \infty$,

$$(1.1) \quad |V(r)| + |r_{\parallel} \partial_{\parallel} V(r)| = o(1).$$

Moreover, for some $\delta_0 > 0$,

$$(1.2) \quad |\partial_{\perp} V(r)| \leq C \langle r_{\parallel} \rangle^{-1-\delta_0},$$

$$(1.3) \quad |\partial_{\parallel}^l V(r)| \leq C_l \langle r_{\parallel} \rangle^{-l-\delta_0} \quad \text{for any integer } l \geq 0.$$

Here $r_{\parallel} = r \cdot B / |B|$, and r_{\perp} denotes the component of r perpendicular to B ; ∂_{\perp} and ∂_{\parallel} denote the partial differentials with respect to the variables r_{\perp} and r_{\parallel} , respectively. We use $\langle r \rangle$ for $(1+|r|^2)^{1/2}$ throughout this paper. As will be easily seen from below, $V(r)$ is allowed to include short-range parts with some local singularities.

In the absence of magnetic fields, scattering theory is quite well understood for 2-body Schrödinger operators with a large class of long-range potentials and, recently, the long-range scattering for constant electric fields have been intensively investigated by several authors (cf. [5, 8, 9, 16]). On the other hand, the asymptotic completeness for long-range Schrödinger operators with constant magnetic fields was first proved by Avron-Herbst-Simon in [1]; they

* This work is partially supported by Grant-in-Aid for Encouragement of Young Scientist A-0570090 from Ministry of Education, Culture and Science.

Received October 14, 1993. Revised March 7, 1994.

treated only azimuthally symmetric potentials, though their results contains the case for potentials unbounded along the directions prependicular to the magnetic fields. They employed a general argument of Kuroda [11] (see also [4]) to reduce the completeness to the existence of modified wave operators. After we had completed this work, we learned the work of Łaba [13]. Her restriction on the long-range potential V_L : $|\partial^\alpha V_L(r)| \leq C_\alpha \langle r \rangle^{-1/\alpha + \nu_0}$, $\nu_0 > 0$, is stronger than ours and the approach is based on the estimates of the growth of the angular momentum $B \times r \cdot p$, that is, on the fact that scattering states do not essentially propagate in the space-time region $|r_\perp| \geq |t|^\nu$ with some $\nu < \nu_0$.

We shall now state our main result in this paper. For simplicity we write $r = r_\parallel$ and $p_z = -i\partial_\parallel$. Let

$$(1.4) \quad W(z) = V(r)|_{r_\perp=0},$$

and let $\chi(s) \in C^\infty(\mathbf{R}^1)$ such that $0 \leq \chi \leq 1$, $\chi(s) = 1$ if $|s| \geq 2$, and $=0$ if $|s| \leq 1$. Define

$$\chi(t, z) = \chi\left(\frac{\log \langle t \rangle}{\langle t \rangle} z\right), \quad W(t, z) = \chi(t, z)W(z).$$

Then the smooth function $W(t, z)$ satisfies

$$|\partial_t^l \partial_z^k W(t, z)| \leq C_{lk} \langle (t, z) \rangle^{-l-k-\delta_1},$$

for some positive $\delta_1 < \delta_0$. Let $S(t, \xi)$ be a solution to the Hamilton-Jacobi equation:

$$\frac{\partial S}{\partial t}(t, \xi) = \frac{1}{2} \xi^2 + W\left(t, \frac{\partial S}{\partial \xi}(t, \xi)\right), \quad \xi \in \mathbf{R}^1$$

(see [10, 12]). Let $H_{0\perp} = H_0 - (1/2)p_z^2$.

THEOREM. *Assume that the condition (V) is obeyed. Then, the modified wave operators Ω_\pm defined by*

$$\Omega_\pm = s - \lim_{t \rightarrow \pm\infty} e^{itH} e^{-iS(t, p_z) - itH_{0\perp}}$$

exist and are complete.

The present work is new in the sense that all the results for the 1-particle operator obtained in this paper remain true for a 2-particle system $H(k)$ with the center-of-mass removed (see [2]):

$$\begin{aligned} H(k) &= H_0(k) + V(r - \beta) \\ &= \frac{1}{2\mu} \left[p - \frac{m_2 - m_1}{2(m_1 + m_2)} B \times r \right]^2 + \frac{1}{2M} |B \times r|^2 + \frac{1}{2M} k_\parallel^2 + V(r - \beta), \end{aligned}$$

where m_1 and m_2 stand for the masses of the first and the second particle, respectively, μ the reduced mass, and M the total mass; we have denoted by r the relative coordinate of the two particles and by p the conjugate momentum. The parameter $k \in \mathbf{R}^3$ is the total pseudomomentum and $\beta = k \times B / |B|^2$. The Hamiltonian $H(k)$ is derived under the condition of the total charge zero (see also [3, 7]).

Note also that this paper covers such a potential that is not azimuthally symmetric and slowly decays as $|r| \rightarrow \infty$ along the direction perpendicular to B . For example, if $B = (0, 0, B)$ and $r = (x, y, z)$, then a function

$$V(r) = [\log \{(1+x^2+z^2)(1+y^2+z^2)\}]^{-1} \langle z \rangle^{-\delta_0}, \quad \delta_0 > 0$$

satisfies the condition (V) and the property mentioned above.

Our strategy is similar to that of [13] and is to see the asymptotic completeness from the point of view of propagation and non-propagation estimates. We follow the idea of Sigal [14] for propagation estimates. We do not use, anywhere, the exponential decay property of the eigenfunctions for $H_{0\perp}$; both in the proof of the existence of wave operators and in that of inverse wave operators, we use the non-propagation estimate (see Theorem 2.4 and the remark following it) that is our essential result. Our non-propagation estimate is much simpler than that of [13] and, in the case of a 1-particle problem, is based on the mechanics of classical particles and the commutativity of two operators, the pseudomomentum and the free Hamiltonian:

$$\left[p + \frac{1}{2} B \times r, \left(p - \frac{1}{2} B \times r \right)^2 \right] = 0.$$

In the case of a two-particle problem, the estimate is a direct consequence of the property of the domain $D(H(k))$.

2. Nonpropagation estimates

For simplicity of the arguments below, we may restrict ourselves to the magnetic field $B = (0, 0, B)$, $B > 0$. Then we write $r = (r_\perp, z) = (x, y, z)$, $p = (p_\perp, p_z) = (p_x, p_y, p_z)$, and

$$H_0 = H_{0\perp} \otimes Id + Id \otimes H_{0\parallel} \quad \text{on } L^2(\mathbf{R}_{x,y}^2) \otimes L^2(\mathbf{R}_z^1),$$

$$H_{0\parallel} = \frac{1}{2} p_z^2,$$

$$H_{0\perp} = \frac{1}{2} \left(p_\perp - \frac{1}{2} B \times r \right)^2$$

$$= -\frac{1}{2} \mathcal{A}_{x,y} + \frac{B^2}{8} (x^2 + y^2) - \frac{B}{2} (-y p_x + x p_y).$$

Let \mathcal{T}_0 be the set of pure point spectrum of $H_{0\perp}$ given by

$$\mathcal{T}_0 = \left\{ \frac{B}{2} (2n+1) \mid n=0, 1, 2, \dots \right\}.$$

Define $A = (1/2)(z \cdot p_z + p_z \cdot z)$ and

$$d(\lambda) = \begin{cases} \text{dist}(\lambda, (-\infty, \lambda] \cap \mathcal{T}_0) & \text{if } \lambda \geq \frac{B}{2}, \\ 0 & \text{otherwise.} \end{cases}$$

Then, under the conditions (1.1) and (1.3) we have (cf. [7])

LEMMA 2.1. (1) *For every $\varepsilon > 0$ and $\lambda \in \mathbb{R}^1$, there exist an open interval I , containing λ , and a compact operator K such that the Mourre estimate holds:*

$$(2.1) \quad E_I(H)i[H, A]E_I(H) \geq 2(d(\lambda) - \varepsilon)E_I(H) + K,$$

where $E_I(H)$ is the spectral projection for H associated with interval I .

- (2) *The set $\sigma_p(H)$ of point spectrum of H is discrete in $\mathbb{R}^1 \setminus \mathcal{T}_0$.*
- (3) *Define $\mathcal{T} = \sigma_p(H) \cup \mathcal{T}_0$. Let I be a compact interval in $\mathbb{R}^1 \setminus \mathcal{T}$ and for any integer $n > 0$, let $s > n - 1/2$. Then there exists a constant $C > 0$ such that*

$$\sup_{0 < \kappa \leq 1, \lambda \in I} \| \langle z \rangle^{-s} \{(H - (\lambda \pm i\kappa))^{-1}\}^n \langle z \rangle^{-s} \| \leq C.$$

A direct consequence of Lemma 2.1 is

LEMMA 2.2. *Let $f(\lambda) \in C_0^\infty(\mathbb{R}^1 \setminus \mathcal{T})$. Then, for any s, s' with $0 < s' < s$, there exists $C > 0$ such that*

$$(2.2) \quad \| \langle z \rangle^{-s} e^{-itH} f(H) \langle z \rangle^{-s'} \| \leq C \langle t \rangle^{-s'}.$$

With the Mourre estimate (2.1) in mind we can apply the argument of Skibsted [15, Exs. 1, 2] to obtain the following minimal velocity estimates.

LEMMA 2.3. *Let $\lambda > B/2$ and $\lambda \notin \mathcal{T}$. Let I be a compact interval containing λ such that $I \cap \mathcal{T} = \text{empty}$. Then there exists a constant $m > 0$ such that for any $f \in C^\infty(\mathbb{R}^1)$ supported in I and for any $s > s' > 0$,*

$$(2.3) \quad F\left(\frac{z^2}{t^2} \leq m\right) e^{-itH} f(H) \langle z \rangle^{-s} = O(t^{-s'}) \quad \text{as } t \rightarrow \infty,$$

where $F(S)$ denotes the projection onto the set S .

THEOREM 2.4. *Suppose that the condition (V) is obeyed. For any compact*

interval $I \subset \mathbf{R}^1 \setminus \mathcal{T}$, let $\phi = E_1(H)\phi \in L^2(\mathbf{R}^3)$ and $\phi \in D(|p|) \cap D(|r|^{1+\varepsilon})$, $\varepsilon > 0$. Then there exists a positive constant C_ϕ , independent of t such that for any $t \in \mathbf{R}^1$,

$$(2.4) \quad \|r_\perp e^{-itH}\phi\| \leq C_\phi.$$

PROOF. We first note that

$$(2.5) \quad \left[H_0, p_x - \frac{B}{2}y \right] = \left[H_0, p_y + \frac{B}{2}x \right] = 0.$$

We will only show that $\|ye^{-itH}\phi\|$ is bounded in $t \geq 0$. The case $t < 0$ and the boundedness of $\|xe^{-itH}\phi\|$ can be verified in a similar manner. We set $\phi_t = e^{-itH}\phi$ and $I(t) = \|(p_x - (B/2)y)\phi_t\|^2$. Since $\|H_0\phi_t\|$ is bounded in t , we see that $\|(p_x + (B/2)y)\phi_t\|$ is also bounded. So, to prove the boundedness of $\|y\phi_t\|$, it suffices to show that $I(t)$ is bounded in t .

We use (2.5) to compute:

$$\begin{aligned} \frac{d}{dt} I(t) &= 2 \operatorname{Re} \left(i \left[H, p_x - \frac{B}{2}y \right] \phi_t, \left(p_x - \frac{B}{2}y \right) \phi_t \right) \\ &= 2 \operatorname{Re} \left(i [V, p_x] \phi_t, \left(p_x - \frac{B}{2}y \right) \phi_t \right) \\ &= 2 \operatorname{Re} \left(-V_x \phi_t, \left(p_x - \frac{B}{2}y \right) \phi_t \right), \end{aligned}$$

where we have put $V_x = \partial_x V$. The Schwartz inequality implies that

$$\frac{d}{dt} I(t) \leq \|V_x \phi_t\| + \|V_x \phi_t\| I(t).$$

We can apply Gronwall's inequality to get

$$I(t) \leq \exp \left(\int_0^t \|V_x \phi_s\| ds \right) \left(I(0) + \int_0^t \|V_x \phi_s\| ds \right).$$

The propagation estimate (2.2) is combined with the condition (1.2) and the assumption $\phi \in D(|r|^{1+\varepsilon})$ to imply that $\|V_x \phi_t\| = O(t^{-1-\delta})$ as $t \rightarrow \infty$ for some $\delta > 0$. Hence we obtain with some positive constants C_ϕ and C'_ϕ ,

$$I(t) \leq C_\phi \left(\left\| \left(p_x - \frac{B}{2}y \right) \phi \right\|^2 + C'_\phi \right),$$

which completes the proof.

REMARK. (1) In the case of two-particle problems, the estimate (2.4) is a direct consequence of the fact that $|B \times r|^2 (H(k) + i)^{-1}$ is bounded.

(2) We consider a classical free one-particle Hamiltonian: $h_0(r, \xi) = (1/2)(\xi - (1/2)B \times r)^2$. Then the Hamilton flow $(r(t), \xi(t))$ for h_0 has the property:

$\xi(t)_\perp + (1/2)B \times r(t) = \text{constant}$. This is the motivation for introducing $I(t)$ in the proof of the theorem.

3. Asymptotic completeness

To make our contribution clear we split the proof of Theorem 1.1 into two steps by introducing the following intermediate Hamiltonian :

$$H_{0W} = H_0 + W,$$

where the function $W = W(z)$ is given by (1.4). We note that the assertions of Lemma 2.3 and Theorem 2.4 also hold for H_{0W} if \mathcal{T} is replaced by $\mathcal{T}_W := \sigma_p(H_{0W}) \cup \mathcal{T}_0$.

As the first step, we obtain

THEOREM 3.1. *The following inverse wave operators exist :*

$$(3.2_\pm) \quad s - \lim_{t \rightarrow \pm\infty} e^{itH_{0W}} e^{-itH} (I - E_p(H)),$$

where $E_p(H)$ denotes the projection onto the point spectral subspace of H .

PROOF. We shall prove the assertion only for ‘+’ case. Let I be any compact interval in $\mathbb{R}^1 \setminus \mathcal{T}$. If we show that the limit (3.2₊) exists for any $\phi \in L^2(\mathbb{R}^3)$ such that $E_I(H)\phi = \phi$ and $\phi \in D(|p|) \cap D(|r|^{1+\varepsilon})$, $\varepsilon > 0$, then we can obtain the assertion by the density argument. With $V(t, r) = \chi(t, z)V(r)$, we compute the derivative :

$$\begin{aligned} (3.3) \quad \frac{d}{dt} e^{itH_{0W}} e^{-itH} \phi &= ie^{itH_{0W}} (H_{0W} - H) e^{-itH} \phi \\ &= ie^{itH_{0W}} (W(z) - W(t, z)) e^{itH} \phi + ie^{itH_{0W}} (W(t, z) - V(t, r)) e^{-itH} \phi \\ &\quad + ie^{itH_{0W}} (V(t, r) - V(r)) e^{-itH} \phi \\ &= T_1(t) + T_2(t) + T_3(t). \end{aligned}$$

Since $(\log \langle t \rangle)^{-1} = o(1)$ as $t \rightarrow \infty$, we can use the minimal velocity estimate (2.3) to obtain

$$(W(z) - W(t, z)) e^{-itH} \langle z \rangle^{-1-\varepsilon} = O(t^{-1-\delta}),$$

$$(V(t, r) - V(r)) e^{-itH} \langle z \rangle^{-1-\varepsilon} = O(t^{-1-\delta})$$

for some $\delta > 0$, and hence

$$(3.4) \quad \|T_1(t)\| = O(t^{-1-\delta}) \quad \text{and} \quad \|T_3(t)\| = O(t^{-1-\delta}).$$

We examine $T_2(t)$. Note that

$$\begin{aligned} W(t, z) - V(t, r) &= V(t, 0, z) - V(t, r_\perp, z) \\ &= \int_0^1 \nabla_\perp V(t, \theta r_\perp, z) d\theta \cdot r_\perp, \end{aligned}$$

and that $\nabla_\perp V(t, \theta r_\perp, z) = O(t^{-1-\delta_1})$ by condition (1.2). Then we can apply Theorem 2.4 to get

$$\|W(t, z) - V(t, r)\| e^{-itH} \phi \leq C \langle t \rangle^{-1-\delta_1} \|r_\perp e^{-itH} \phi\| \leq C' \langle t \rangle^{-1-\delta_1},$$

and conclude that $\|T_z(t)\| = O(t^{-1-\delta_1})$. Combining this with (3.4) implies that (3.3) is integrable over $[0, \infty)$, and therefore the limit (3.2_+) exists. Q.E.D.

The same argument as in the proof of Theorem 3.1 yields

THEOREM 3.2. *The wave operators*

$$s - \lim_{t \rightarrow \pm\infty} e^{itH} e^{-itH_0W} (I - E_p(H_0W))$$

exist.

Thus it remains to prove

THEOREM 3.3. *The modified wave operators*

$$(3.5) \quad s - \lim_{t \rightarrow \pm\infty} e^{itH_0W} e^{-iS(t, p_z) - itH_{0\perp}}$$

and the inverse operators

$$(3.6) \quad s - \lim_{t \rightarrow \pm\infty} e^{iS(t, p_z) + itH_{0\perp}} e^{-itH_0W} (I - E_p(H_0W)),$$

exist.

PROOF. Since H_{0W} has a form of direct sum

$$H_{0W} = H_{0\parallel} \otimes Id + Id \otimes (H_{0\perp} + W) \quad \text{on } L^2(\mathbf{R}_{x,y}^2) \otimes L^2(\mathbf{R}_z^1),$$

the existence of limit (3.5) can be reduced to the existence of modified wave operators for 1-dimensional Schrödinger operator $H_{0\parallel} + W$ and it is a direct consequence of Hörmander [6]. The existence of limit (3.6) can be verified similarly as in Sigal [14], by using sharp propagation estimates for e^{-itH_0W} and we omit it. Q.E.D.

Now the assertion of Theorem 1.1 can be verified by combining Theorems 3.1-3.3.

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