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# ON A SPECTRAL PROPERTY OF ANALYTIC OPERATORS

By

In Ho JEON*, Jin Chun KIM** and Woo Young LEE*

**Abstract.** If $T \in \mathscr{L}(X)$ then $T$ is analytic if and only if $(\lambda - T)^{-1}$ has a pole for at least a $\lambda \in \sigma(T)$. Furthermore, every analytic operator has a non-trivial invariant subspace.

## 1. Introduction

An operator $T$ is called *algebraic* if there exists a non-zero polynomial $p$ such that $p(T) = 0$(cf.[1],[2]). As a natural extension of algebraicity, Halmos ([5] Problem 97) introduced the concept of analyticity (only for a quasinilpotent operator). In this paper we formulate the definition of analyticity of bounded linear operators and then give a spectral property of analytic operators.

Throughout this paper suppose $X$ is a Banach space and write $\mathscr{L}(X)$ for the set of all bounded linear operators on $X$. If $T \in \mathscr{L}(X)$, write $\rho(T)$ and $\sigma(T)$ for the *resolvent set* and the *spectrum* of $T$, respectively. If $K$ is a subset of $\boldsymbol{C}$, write $\overline{K}$ ,$\partial K$, $accK$ and for the closure, the topological boundary, the accumulation points and the isolated points of $K$, respectively. If there exists an integer $k$ such that $(T^k)^{-1}(0) = (T^{k+1})^{-1}(0)$, we say that $T$ has *finite ascent*. In that case the smallest such integer $k$ is denoted by $a(T)$. If there exists an integer $k$ such that $T^k(X) = T^{k+1}(X)$, we say that $T$ has *finite descent*. In that case the smallest such integer $k$ is denoted by $d(T)$. It is known ([1], [4]) that for every compact $K \subset \boldsymbol{C}$ and open $\Omega \supset K$ there exists an open set $\Delta$ such that

(i) $K \subset \Delta \subset \overline{\Delta} \subset \Omega$;

(ii) $\Delta$ has at most a finite number of components $\{\Phi_i\}_{i=1}^n$;

(iii) every component $\Phi_i$ has a boundary formed by a finite number of simple rectifiable Jordan curves $\Gamma_{ij}$;

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(iv) $K \cap \Gamma_{ij} = \emptyset$ for all $i, j$.

Then

$$\Gamma = \bigcup_{i,j} \Gamma_{ij}$$

is called a *Cauchy* (or an *admissible*) *contour* contained in $\Omega \setminus K$ and surrounding $K$. We recall that if $T \in \mathcal{L}(X)$ and if $f$ is analytic on an open neighborhood $\Omega$ of $\sigma(T)$ then we define

$$f(T) = \frac{1}{2\pi i} \int_\Gamma f(\lambda)(\lambda - T)^{-1} d\lambda,$$

where $\Gamma$ is a Cauchy contour contained in $\Omega \setminus \sigma(T)$ and surrounding $\sigma(T)$.

## 2. Analytic operators

We begin with:

DEFINITION 1. An operator $T \in \mathcal{L}(X)$ will be called *analytic* if there exists a non-zero function $f$ analytic on an open neighborhood $\Omega$ of $\sigma(T)$ such that $f(T) = 0$.

Evidently, we have

(2.1) $$T \text{ is algebraic } \Rightarrow T \text{ is analytic.}$$

However the converse of (2.1) is not true in general: for example, consider a Riesz operator whose spectrum is infinite (see below Corollary 4).

Analyticity gaurantees the existence of an isolated point of the spectrum.

LEMMA 2. *If $T \in \mathcal{L}(X)$ is analytic then $\sigma(T)$ has an isolated point.*

PROOF. Suppose $T$ is analytic. Thus there exists a non-zero function $f$ analytic on an open neighborhood of $\sigma(T)$ such that $f(T) = 0$. Then the spectral mapping theorem implies that all spectral values of $T$ are zeros of $f$. Thus, if all spectral values of $T$ are accumulation points of $\sigma(T)$ then it follows from the Identity Theorem in the elementary complex analysis that $f \equiv 0$ on $\sigma(T)$, which leads a contradiction.

The converse of Lemma 2 is not true in general. We however have:

THEOREM 3. *If $T \in \mathcal{L}(X)$, then $T$ is analytic if and only if $(\lambda - T)^{-1}$ has a* *pole for at least a* $\lambda \in \sigma(T)$.

PROOF. ($\Leftarrow$): Without loss of generality, suppose that $\lambda = 0$ is a pole of $(\lambda - T)^{-1}$ of order $n \neq 0$. Thus $a(T) = d(T) = n \neq 0$ and hence $X$ can be written by ([3], [4])

$$X = T^n(X) \oplus (T^n)^{-1}(0).$$

In this case we can find a Riesz projection $P_0$ corresponding to 0: namely,

$$P_0 = \frac{1}{2\pi i}\int_{\partial B_0} (\lambda - T)^{-1} d\lambda,$$

where $B_0$ is an open disk of center 0 which contains no other points of $\sigma(T)$. We also have

$$TP_0 = P_0 T, \ P_0^{-1}(0) = T^n(X), \text{ and } P_0(X) = (T^n)^{-1}(0).$$

Thus we see that $T^n P_0 = 0$. In particular, the Riesz projection $P_0$ is equal to $f(T)$, where $f$ is a function which takes the value 1 on $B_0$ and the value 0 on an open neighborhood $\Omega \setminus B_0$ of the complement $\sigma(T) \setminus \{0\}$ such that $\overline{B_0} \cap \overline{\Omega \setminus B_0} = \emptyset$. If we define $\tilde{f} : \Omega \to C$ by setting

$$\tilde{f}(\lambda) = \lambda^n f(\lambda)$$

then $\tilde{f}$ is analytic on $\Omega$ and does not vanish on $B_0$, and $\tilde{f}(T) = 0$. This says that $T$ is analytic.,

($\Rightarrow$): Suppose $T$ is analytic. Thus there exists a non-zero function $g$ analytic on an open neighborhood $\Omega$ of $\sigma(T)$ such that $g(T) = 0$. In view of Lemma 2, we may assume without loss of generality that $0 \in \operatorname{iso}\sigma(T)$. Then there is an open disk $B_0$ of center 0 which contains no other points of $\sigma(T)$ and $g$ does not vanish on $B_0$. Also, we may assume that $\overline{B_0} \cap \overline{\Omega \setminus B_0} = \emptyset$. If $P_0$ is the corresponding Riesz projection as above, then the spectral mapping theorem implies that $TP_0$ is quasinilpotent. Since $TP_0 = P_0 T$, it follows that $T$ is reduced by the decomposition $P_0(X) \oplus P_0^{-1}(0)$. Thus $P_0(X)$ is invariant under $(\lambda - T)^{-1}$ for $\lambda \in \rho(T)$ and hence under $g(T)$. Therefore, by the functional calculus,

$$0 = g(T)\big|_{P_0(X)} = g(T\big|_{P_0(X)}) = g(TP_0).$$

This says that $TP_0$ is analytic because $\sigma(TP_0) = \{0\}$ and $g$ is non-zero on $B_0$. But, since the only analytic quasinilpotent operator is nilpotent (cf. [5] Problem 97), it follows that $T^n P_0 = 0$ for some $n \in N$. If we define $h : \Omega \to C$ by setting

$$h(\lambda) = \begin{cases} 1 \setminus \lambda & \text{if} \quad \lambda \in \Omega \setminus B_0 \\ 0 & \text{if} \quad \lambda \in B_0 \end{cases},$$

then $h$ is analytic on $\Omega$ and the functional calculus gives

$$1 - P_0 = Th(T) = h(T)T.$$

We thus have

$$T^n = T^n(1 - P_0) = T^{n+1}h(T) = h(T)T^{n+1},$$

which implies

$$T^n(X) = T^{n+1}h(T)(X) \subseteq T^{n+1}(X) \subseteq T^n(X)$$

and

$$(T^n)^{-1}(0) = (h(T)T^{n+1})^{-1}(0) \supseteq (T^{n+1})^{-1}(0) \supseteq (T^n)^{-1}(0),$$

which says that $a(T) = d(T) = n \neq 0$. Thus $\lambda = 0$ is a pole of $(\lambda - T)^{-1}$ of order $n \neq 0$.

COROLLARY 4. *Every Riesz operator having non-zero spectral values is analytic.*

PROOF. If $T \in \mathcal{L}(X)$ is a Riesz operator then $T - \lambda$ has finite ascent and finite descent for every non-zero $\lambda$ (cf. [3] (3.1)). Thus the result follows from Theorem 3.

COROLLARY 5. *If $T \in \mathcal{L}(X)$ is analytic then $aT + b$ is analytic for any $a(\neq 0), b \in \boldsymbol{C}$.*

PROOF. This follows from the fact that if $(\lambda - T)^{-1}$ has a pole then so does $\{(b + a\lambda) - (aT + b)\}^{-1}$.

COROLLARY 6. *Every analytic operator has a non-trivial invariant subspace.*

PROOF. Suppose $T$ is analytic. If $\sigma(T) = \{\lambda\}$, then it follows from Theorem 3 that $T - \lambda$ is nilpotent, so that $T$ has a non-trivial invariant subspace. If $\sigma(T)$ is not a singleton set the range of the Riesz projection for an isolated point of $\sigma(T)$ is a non-trivial invariant subspace for $T$.

THEOREM 7. *If $T \in \mathcal{L}(X)$ is analytic and $N \in \mathcal{L}(X)$ is nilpotent commuting with T, then T+N is also analytic.*

PROOF. Without loss of generality suppose that $\lambda = 0$ is a pole of $(\lambda - T)^{-1}$ of order $n \neq 0$. Thus we can write $T$ as a $2\times 2$ operator matrix:

$$T = \begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix} : T^n(X) \oplus T^{-n}(0) \longrightarrow T^n(X) \oplus T^{-n}(0),$$

where $T_1$ is invertible and $T_2$ is nilpotent. Since $NT = TN$, $N$ can be also written as the following operator matrix:

$$N = \begin{bmatrix} N_2 & 0 \\ 0 & N_2 \end{bmatrix} : T^n(X) \oplus T^{-n}(0) \longrightarrow T^n(X) \oplus T^{-n}(0).$$

We note that $N_1$ and $N_2$ are both nilpotent, $T_1 N_1 = N_1 T_1$ and $T_2 N_2 = N_2 T_2$. It thus follows $T_1 + N_1$ is invertible and $T_2 + N_2$ is nilpotent. Therefore we can conclude that $T+N$ has finite ascent and finite descent, and hence by Theorem 3, $T+N$ is analytic.

It is well known that similarity preserves algebraicity. We can prove more:

THEOREM 8. *Similarity preserves analyticity.*

PROOF. Let $S, T \in \mathcal{L}(X)$ be similar; thus there is an invertible operator $U \in \mathcal{L}(X)$ such that $S = U^{-1}TU$. Suppose $T$ is analytic, say, $f(T) = 0$ for a non-zero function $f$ analytic on an open neighborhood $\Omega$ of $\sigma(T)$. If $\Gamma$ is a Cauchy contour contained in $\Omega \setminus \sigma(T)$ and surrounding $\sigma(T)$ then it follows from the functional calculus and the fact that $\sigma(S) = \sigma(T)$ that

$$f(S) = f(U^{-1}TU) = U^{-1} f(T) U = 0,$$

which says that $S$ is analytic.

## 3. Concluding remarks

(a) Let $T_i \in \mathcal{L}(X_i), i = 1,2$. Even if $T_1$ and $T_2$ are both analytic, $T_1 \oplus T_2$ may not be analytic. For example, if $N$ is nilpotent on $\ell_2$ and $U$ is the unilateral shift on $\ell_2$, then $T_1 := N \oplus (2+U)$ and $T_2 := (2+N) \oplus U$ are both analytic. But $\sigma(T_1 \oplus T_2)$ has no isolated points and therefore $T_1 \oplus T_2$ is not analytic. Of course, if $\sigma(T_1) \cap \sigma(T_2) = \emptyset$ then $T_1 \oplus T_2$ is analytic whenever the one of them is analytic.

(b) It is known ([4] Theorem II.4.1) that if $T \in \mathcal{L}(X)$ and if $\Omega$ is a neigborhood of $\sigma(T)$ then there exists $\varepsilon > 0$ such that $\sigma(S) \subset \Omega$ for any operator $S$ in $\mathcal{L}(X)$ with $\|T - S\| < \varepsilon$ (This property is called the "upper semicontinuity of spectra"). Thus we might conjecture that the set of all analytic operators on $X$ is an open subset of $\mathcal{L}(X)$. But this is not true in general. For example, let

$W : \ell_2 \to \ell_2$ be defined by setting

$$W(\xi_1, \xi_2, \xi_3, \cdots) = (0, \xi_1, \xi_2/2, \xi_3/3, \cdots, \xi_n/n, \cdots). \tag{3.1}$$

Then $W$ is quasinilpotent but not nilpotent. Now consider the operators

$$T_n = W^n : \ell_2 \to \ell_2 \quad \text{for each } n \in N.$$

Then each $T_n$ is not analytic and $T_n \to 0$, while 0 is analytic.

(c) The topological boundary of the set of algebraic operators may not be analytic operators. For example, consider the operators $S_n : \ell_2 \to \ell_2$ defined by setting

$$S_n(\xi_1, \xi_2, \xi_3, \cdots) = (0, \xi_1, \xi_2/2, \cdots, \xi_n/n, 0, 0, \cdots) \qquad \text{for each } n \in N.$$

Then each $S_n$ is nilpotent and hence algebraic. However observe that $S_n \to W$, where $W$ is defined as in (3.1).

(d) From the punctured neighborhood theorem ([4], [8]), we can see that if $T \in \mathcal{L}(X)$ then

$$\partial\sigma(T) \setminus \sigma_e(T) \neq \emptyset \Rightarrow T \text{ is analytic}, \tag{3.2}$$

where $\sigma_e(T)$ denotes the essential spectrum of $T$. We tried to extend (3.2) to the absence of index:

$$\text{iso}\, \sigma(T) \cap \Omega(T) \neq \emptyset \Rightarrow T \text{ is analytic}, \tag{3.3}$$

where $\Omega(T)$ denotes the set of all $\lambda \in C$ such that $T - \lambda$ is 'decomposably regular', in the sense ([6], [7]) that there is $T'_\lambda \in \mathcal{L}(X)$ for which $T - \lambda = (T - \lambda)T'_\lambda(T - \lambda)$ and $T'_\lambda$ is invertible. However, unfortunately, (3.3) fails. For example, consider the operator

$$T = \begin{bmatrix} W & 0 \\ I & 0 \end{bmatrix} : \ell_2 \oplus \ell_2 \to \ell_2 \oplus \ell_2,$$

where $W$ is defined as in (3.1). Then $T$ is decomposably regular with the invertible operator

$$T'_0 = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix},$$

and 0 is the isolated point of $\sigma(T)$. However $T$ is quasinilpotent, and hence it is not analytic.

(e) The obvious extension of polynomials in an operator seems to be "infinite polynomials", more precisely, power seies; that is, if $f$ is a non-zero analytic function on a simply connected domain (or an open disk) containing $\sigma(T)$, then we would like to call $T$ an analytic operator when $f(T)=0$. However, this definition does not make sense (If we concern with only quasinilpotent then the argument of Halmos ([5] Problem 97) make sense.): in that case, in fact, analyticity is equivalent to algebraicity. To see this, appeal to Theorem 3. If $T$ is analytic in the above sense then by Theorem 3, $T$ has a finite spectrum, for whose elements, $(\lambda - T)^{-1}$ has poles. Thus, via an argument of Riesz projection, $T$ may be expressed as

$$T = T_1 \oplus \cdots \oplus T_n,$$

where if $\sigma(T_i)=\{\lambda_i\}$, then $T_i - \lambda_i$ is nilpotent for each $i=1,\cdots,n$. Then $T_i$ is algebraic and hence $T$ is algebraic. (Perhaps Aupetit ([1] P.67) would assert this fact in the above viewpoint.)

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*Department of Mathematics
Sung Kyun Kwan University
Suwon 440-746, Korea

**Department of Mathematics
Semyung University
Jechon 21-1, Korea