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REGULAR RETRACTIONS ONTO FINITE DIMENSIONAL CONVEX SETS AND THE AR-PROPERTY FOR ROBERTS SPACES

By

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Abstract. It is proved that if X is an n-dimensional closed convex subset in a linear metric space E, then there is a retraction $r: E \to X$ such that $\|x - r(x)\| \leq 2(n+1)\|x - X\|$ for every $x \in E$. This fact is applied to study the AR-property in linear metric spaces. We identify a class of Roberts spaces with the AR-property. We also give a direct proof that for every $p \in [0,1), L_p$ is a needle point space.

1. Introduction.

Following Roberts [R1] [R2], let us say that a non-zero point a of a linear metric space X is a *needle point* iff for every $\varepsilon > 0$, there exists a finite set $A(a,\varepsilon) = \{a_1, \cdots, a_m\}$, satisfying the following conditions:

(1) $\|a_i\| < \varepsilon$ for every $i = 1, \cdots, m$;

(2) for every $b \in A^+(a,\varepsilon)$, there is an $\alpha \in [0,1]$ such that $\|b - \alpha a\| < \varepsilon$ where $A^+ = \text{conv}(A \cup \{\theta\})$;

(3) $a = \dfrac{1}{m}(a_1 + \cdots + a_m)$.

We say that X is a *needle point space* iff X is complete separable linear metric space in which every non-zero point is a needle point. Roberts [R2] has shown that for every $p \in [0,1)$ the space L_p is a needle point space. We recall that the spaces $L_p, 0 \leq p < 1$, are defined by

$$L_p = \left\{ f : [0,1] \to \boldsymbol{R}; \int_0^1 |f(t)|^p dt < \infty \right\} \text{ for } 0 < p < 1,$$

and

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$$L_0 = \left\{ f : [0,1] \to \mathbf{R}; \int_0^1 \frac{|f(t)|}{1+|f(t)|} dt < \infty \right\}.$$

Other examples of needle point spaces were given in [R1] [KP] [KPR].

Roberts [R2], see also [KPR], showed that if X is a needle point space then for any summable sequence $s = \{s_n\}$ of positive numbers there is a compact convex set $C(s)$ without any extreme points. Therefore, the classical theorem of Krein and Milman [KM] fails to be true for non-locally convex linear metric spaces. We shall describe Roberts' method of constructing $C(s)$.

Let $s = \{s_n\}$ be a summable sequence of positive numbers. Let a_0 be a non-zero point of X. Using the needle point space property of X, we choose by induction, a sequence $\{A_n(s)\}$ of finite subsets of X, where $A_0(s) = \{a_0\}$, with the following properties:

(4) $\|a\| < \varepsilon_n$ for every $a \in A_n(s)$; where

(5) $\varepsilon_n = [m(n-1)]^{-1} s_n$, and $m(n) = \text{card}\, A_n(s)$;

(6) If $A_n(s) = \{a_1^n, \cdots, a_{m(n)}^n\}$ then $A_{n+l}(s)$ is defined by the formula

$$A_{n+l}(s) = \bigcup_{i=1}^{m(n)} A(a_i^n, \varepsilon_{n+1}),$$

where $A(a_i^n, \varepsilon_{n+1}), i = 1, \cdots, m(n)$, are determined by the needle point property of a_i^n, see (1)–(3).

We define

(7) $C(s) = \overline{\bigcup_{n=1}^{\infty} \hat{A}_n(s)} \subset X$; where $\hat{A} = \text{conv}\,(A^+ \cup (-A^+))$, see (2).

Roberts showed in [R2] that $C(s)$ is a compact convex set with no extreme points. We call $C(s)$ a *Roberts space*.

In [NT1], see also [N1], it was shown that every needle point space contains a compact convex AR-set without any extreme points. However, the results of [NT1] and [N1] do not indicate which of Roberts spaces are AR. So far, it is shown that all Roberts spaces have the fixed point property, see [NT2]. Nevertheless, the AR-property for Roberts spaces is still being questioned. Several readers of the papers [NT1] and [N1] have asked the authors to classify Roberts spaces with the AR-property: This would be very important for further study of the AR-property for Roberts spaces. In this note, we identify a class of Roberts spaces with the AR-property. Namely, we shall show that, instead of $\varepsilon_n = (m(n-1))^{-1} s_n$, see (5), we take $\varepsilon_n = (m(n-1))^{-2} s_n$, then the resulting Roberts space $C^*(s)$, defined by (7), is an AR.

NOTATION AND CONVENTIONS. By a linear metric space we mean a topological linear space X which is metrizable. We equip X with an F-norm $\|\cdot\|$ such that, see [Re],

$$\|\lambda x\| \le \|x\| \text{ for every } x \in X \text{ and } \lambda \in \boldsymbol{R} \text{ with } |\lambda| \le 1.$$

Let A be a subset of a linear metric space X. For $x \in X$ we write

$$\|x - A\| = \inf\{\|x - y\| : y \in A\};$$

and for $\mathrm{a}, \mathrm{b} \in X$, we write:

$$[\mathrm{a}, \mathrm{b}] = \{\alpha a + (1-\alpha) b : \alpha \in [0,1]\}.$$

2. **Regular Retractions onto Finite Dimensional Convex Sets**

In this section we establish the following fact which is an extension of [NT2, Lemma 1].

PROPOSITION 1. *Let X be an n-dimensional closed convex set in a linear metric space E. Then there is a continuous retraction $r : E \to X$ such that*

(8) $\|x - r(x)\| \le 2(n+1)\|x - X\|$ *for every* $x \in E$.

PROOF. Let $\{U_s, a_s\}_{s \in S}$ denote a Dugundji system for $E \setminus X$, that is a family $\{U_s, a_s\}_{s \in S}$ with the following properties, see [BP, P. 58],:

(i) $U_s \subset E \setminus X$ and $a_s \in X$ for every $s \in S$;

(ii) $\{U_s\}_{s \in S}$ is a locally finite open cover of $E \setminus X$;

(iii) $\|x - a_s\| \le 2\|x - X\|$ for every $x \in U_s$.

Let $\{\lambda_s\}_{s \in S}$ be a locally finite partition of unity inscribed into $\{U_s\}_{s \in S}$. We define $r : E \to X$ by Dugundji formula:

$$r(x) = \begin{cases} x & \text{if } x \in X; \\ \sum_{s \in S} \lambda_s(x) a_s & \text{if } x \in E \setminus X. \end{cases}$$

Observe that the continuity of r follows from (8). Let us verify (8). Denote $A = \{a_s : s \in S\}$. Then we have $r(x) \in$ conv A for every $x \in E \setminus X$. Let $A(x)$ denote a subset of A of *smallest cardinality* so that $r(x) \in$ conv $A(x)$. It is easy to see that

(9) card $A(x) \le k + 1$, where k = dim span $A(x)$.

In fact assume that card $A(x) \ge k + 2$. Since dim span $A(x) = k$ there exist $a_0, \cdots, a_k \in A(x)$ such that Int $A_k \ne \varnothing$, where A_k = conv $A_k^0, A_k^0 = \{a_0, \cdots, a_k\}$ and Int A denotes the interior of A relative to the space $E(x)$ = span $A(x)$.

Let $B_k^0 = A(x) \setminus A_k^0$ and B_k = conv B_k^0. Since card $A_k^0 = k + 1 <$ card $A(x)$, we have $r(x) \notin A_k$. Therefore there exist $a \in A_k, b \in B_k$ so that $r(x) \in [a,b]$.

Since $\operatorname{Int} A_k \neq \varnothing$ and $r(x) \notin A_k$, there exists a face S of the simplex A_k such that $[a,b] \cap S \neq \varnothing$. Let $S = \operatorname{conv} S^0$,where $S^0 \subset A_k^0$ and card $S^0 < k+1$. Since $r(x) \notin A_k$ we have $r(x) \in \operatorname{conv}(S \cup B_k) = \operatorname{conv}(S^0 \cup B^0{}_k)$. Observe that

$$S^0 \cup B_k^0 \subset A(x) \text{ and card } (S^0 \cup B_k^0) < \operatorname{card}(A_k^0 \cup B_k^0) = \operatorname{card} A(x).$$

This contradiction proves (9).

Let $A(x) = \{a_0, \cdots, a_k\}$. Then we have $r(x) = \sum_{i=0}^k \lambda_i a_i$, where $a_i \in A(x), i = 0, \cdots, 1$ and $\sum_{i=0}^k \lambda_i = 1$.

Then from (iii) we have

$$\begin{aligned} \|r(x) - x\| &\leq \left\| \sum_{i=0}^k \lambda_i a_i - x \right\| \\ &\leq \sum_{i=0}^k \|\lambda_i (a_i - x)\| \\ &\leq \sum_{i=0}^k \|a_i - x\| \\ &\leq 2(k+1)\|x - X\| \leq 2(n+1)\|x - X\|. \end{aligned}$$

The proposition is proved.

Now we are going to apply Proposition 1 to obtain several results on the AR-property in linear metric spaces.

Following [N2], a convex set X has the *locally convex approximation property*, the LCAP, iff there exist a sequence of convex subsets $\{X_n\}$ of X such that each X_n can be affinely embedded into a locally convex space and a sequence of continuous maps $f_n : X \to X_n$ such that for some summable sequence $\{s_n\}$ of positive numbers we have

$$\text{(LC)} \quad \liminf_{n\to\infty} (s_n)^{-1} \|x - f_n(x)\| = 0 \quad \text{for every } x \in X.$$

It was proved in [N2] that any convex set with LCAP is an AR.

In [N1] it was said that a convex set X has the *finite dimensional approximation property*, the FDAP, iff there exists a sequence of continuous maps $\{f_n\}$ from X into finite dimensional subsets X_n of X such that for some summable sequence $\{s_n\}$ of positive numbers we have

$$\text{(FD)} \quad \liminf_{n\to\infty} (s_n)^{-1} (1 + \dim X_n) \|x - f_n(x)\| = 0 \quad \text{for every } x \in X.$$

It was proved in [N1] that if a convex set X has the FDAP then any convex subset of X is an AR.

From Proposition 1, we get the following result:

COROLLARY 1. *Let X be a convex set in a linear metric space. Assume that there exists a sequence of finite dimensional convex subsets $\{X_n\}$ of X such that for some summable sequence $\{s_n\}$ of positive numbers we have*

(10) $\liminf_{n\to\infty} (s_n)^{-1}(1+\dim X_n)^k \|x - X_n\| = 0$ for every $x \in X$.

Then

(i) *if $k = 1$ then X is an AR;*

(ii) *if $k = 2$ then every convex subset of X is an AR.*

PROOF. Since $X_n, n \in N$, are finite dimensional convex sets, we have dim X_n = dim $\overline{X_n}$, where $\overline{Y}$ denotes the closure of Y in X. Therefore we may assume that X_n is closed in X for every $n \in N$. From Proposition 1, it follows that condition (10) for $k = 1$ implies that X has the LCAP and for $k = 2$ implies that X has the FDAP.

QUESTION 1. Does condition (10) for $k = 1$ imply that every convex subset of X is an AR?

3. The AR-Property for Roberts Spaces

Now we define $C^*(s)$ in the same way as $C(s)$, see (7). The only difference is that, instead of $\varepsilon_n = (m(n-1))^{-1} s_n$, see (5), we take $\varepsilon_n = (m(n-1))^{-2} s_n$. We shall prove that the resulting Roberts space $C^*(s)$, defined by (7), is an AR. First we show:

CLAIM 1. $\|x - \hat{A}_n(s)\| < 2\sum_{i=n}^{\infty}(m(i))^{-1} s_{i+1}$ for every $x \in C^*(s)$ and $n \in N$.

PROOF. Let $A_n(s) = \{a_1^n, \cdots, a_{m(n)}^n\}$. Observe that for every $x \in A_{n+1}^+(s)$ there exist $b_i \in A^+(a_i^n, \varepsilon_{n+1})$ and $\lambda_i \ge 0, i = 1, \cdots, m(n)$, with $\sum_{i=1}^{m(n)} \lambda_i \le 1$ such that $x = \sum_{i=1}^{m(n)} \lambda_i b_i$, see (6).

By (2) for every $i = 1, \cdots, m(n)$ there is an $\alpha_i \in [0,1]$ such that

$$\|b_i - \alpha_i a_i^n\| < \varepsilon_{n+1} = (m(n))^{-2} s_{n+1}.$$

Let $y = \sum_{i=1}^{m(n)} \alpha_i \lambda_i a_i^n \in A_n^+(s)$. Then we get

$$\|x - y\| \le \sum_{i=1}^{m(n)} \|b_i - \alpha_i a_i^n\| \le m(n)\varepsilon_{n+1}$$

$$= m(n)(m(n))^{-2} s_{n+1} = (m(n))^{-1} s_{n+1}.$$

Therefore

$$\|x - A_n^+(s)\| \le (m(n))^{-1} s_{n+1} \text{ for every } x \in A_{n+1}^+(s).$$

It follows that

$$\|x \in \hat{A}_n(s)\| \le 2(m(n))^{-1} s_{n+1} \text{ for every } x \in \hat{A}_{n=1}(s).$$

By induction we have

$$\|x - \hat{A}_n(s)\| \le 2 \sum_{i=n}^{n+k} (m(i))^{-1} s_{i+1} \text{ for every } x \in \hat{A}_{n+k}(s) \text{ and } k \in N$$

Consequently, the claim follows from the above inequality.

THEOREM 1. *$C^*(s)$ is an AR.*

PROOF. We aim to verify condition (i) of Corollary 1. Observe that $\{m(n)\}$ is an increasing sequence. Therefore from Claim 1 we get

(11) $\|x - A_n(s)\| < 2(m(n))^{-1} s_{n+1}$ for every $x \in C^*(s)$ and $n \in N$;

where $S_n = \sum_{i=n}^{\infty} s_i$.

Since $\{s_i\}$ is summable, $S_n \to 0$ as $n \to \infty$. It follows that there exists a sequence $\{n_k\} \subset N$ such that $S_{n_k} k^{-1} 2^{-k}$ for every $k \in N$. Therefore from (11) we get, for $x \in C^*(s)$ and $k \in N$

$$\|x - \hat{A}_{n_k}(s)\| < 2(m(n_k))^{-1} S_{n_k} < 2(m(n_k))^{-1} k^{-1} 2^{-k}.$$

It follows that

$$(1 + m(n_k))2^k \|x - \hat{A}_{n_k}(s)\| < 3k^{-1} \text{ for every } k \in N.$$

Since $\dim \hat{A}_{n_k}(s) \le m(n_k)$ and $\{2^{-k}\}$ is a summable sequence we infer that the sequence $\{\hat{A}_{n_k}(s)\}$ satisfies condition (i) of Corollary 1. Consequently, Theorem 1 is proved.

4. The Needle Point Space Property for $L_p, p \in [0,1)$

As we have seen, if we have a needle point space at hands, it is not hard to construct a compact convex set with no extreme points. However, it is quite difficult to give an example of a needle point space. Roberts [R2] showed that for every $p \in [0,1)$, the space L_p is a needle point space. However, the proof of Roberts [R2] as well as other proofs given in [KP] [KPR] (see also [Re]) do not provide a direct proof that the spaces $L_p, 0 \le p < 1$, are needle point spaces. (In [R2] it was proved that $L_p(Q)$, where $Q = [0,1]^\infty$ is the Hilbert cube, is a needle point space and since Q is isomorphic (in measure) to $[0,1]$, it follows that L_p is a needle point space). It would be nicer if we could have a clear picture of a Roberts space directly in L_p. In this section, we give such a direct proof.

First we show:

CLAIM 2. For every $n \in N$, there exists a sequence $\{S_k^n, k=1,2,\cdots\}$ of measurable sets in [0,1] with the following properties:

(12) $\mu(S_k^n)=n^{-1}$ for every $k \in N$;

(13) $\mu(S_k^n \cap S_j^n)=n^{-2}$ for every $k \neq j$.

(μ denotes the Lebesgue measure on [0,1]).

PROOF. For $n \in N$ and $I=[a,b]$, we define

$$S^n(I)=\left[a, a+n^{-1}(b-a)\right).$$

For every $k \in N$, let π_k denote the partition of [0,1] into n^{k-1} equal subintervals of length n^{-k+1}. We define S_k^n by the formula:

$$S_k^n = \bigcup_{I \in \pi_k} S^n(I), \text{ see Figure 1.}$$

It is easy to see that the sequence $\{S_k^n, k=1,2,\cdots\}$ satisfies the required conditions. The claim is proved.

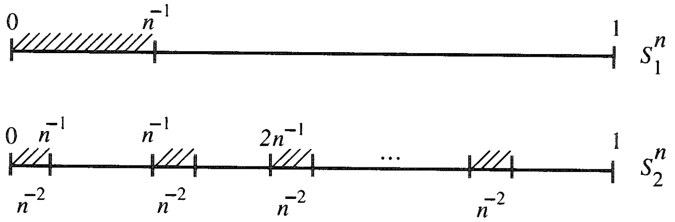

Figure 1

Denote $a_k^n = n\chi_{S_k^n}$ for every $k \in N$, where χ_A is the characteristic function of A.

We have the following simple observation:

CLAIM 3. $\int_0^1 (a_k^n - 1)(a_j^n - 1)\, dt = 0$ for every $k \neq j$.

PROOF. From (12) (13) we get

$$\int_0^1 (a_k^n - 1)(a_j^n - 1)\,dt = \int_0^1 (a_k^n a_j^n - a_k^n - a_j^n + 1)\,dt$$

$$= \int_0^1 (n^2 \chi_{S_k^n \cap S_j^n} - n\chi_{S_k^n} - n\chi_{S_j^n} + 1)\,dt$$

$$= n^2 \mu(S_k^n \cap S_j^n) - n\mu(S_k^n) - n\mu(S_j^n) + 1$$

$$= 1 - 1 - 1 + 1 = 0\,.$$

The claim is proved.

From Claim 3 and from Jensen's inequality we get

$$\left\|\sum_{i=1}^{k} \alpha_i (a_i^n - 1)\right\| \le \left(\int_0^1 \left(\sum_{i=1}^{k} (\alpha_i (a_i^n - 1))\right)^2\right)^{p/2} \tag{14}$$

$$= \left(\int_0^1 \left(\sum_{i=1}^{k} (\alpha_i (a_i^n - 1))\right)^2\right)^{p/2}$$

for any finite sequence $\alpha_i \ge 0, i = 1, \cdots, k$ with $\sum_{i=1}^{k} \alpha_i \le 1$.

From (14) we obtain the following fact, which implies the needle point space property for the spaces L_p, $0 \le p < 1$, see [R2] [KP].

PROPOSITION 2. *For any* $\varepsilon > 0$ *and* $a > 0$ *there exist an* $n \in N$ *and* b, $0 < b < a$, *such that for any finitely non-zero sequence* $\{\alpha_i\}$ *with* $\alpha_i \ge 0$ *and* $\sum \alpha_i \le 1$ *we have* $\|\sum_{\alpha_i \ge a} \alpha_i a_i^n\| < \varepsilon$ *and* $\|\sum_{x_i \le b} \alpha_i (a_i^n - 1)\| < \varepsilon$.

REMARK. Observe that the AR-property for the first-known example of a compact convex set with no extreme points, constructed by Roberts [R1], has been established in [NST]. However, the AR-problem for Roberts spaces has not yet been answered even for the spaces L_p, $0 \le p < 1$.

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