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A NOTE ON Ω-STABILITY

Dedicated to Professor Yukihiro Kodama on his 60th birthday

By

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By Mañé [2] and Palis [3] it was proved that Ω-stability of a C^1-diffeomorphism of a compact boundaryless manifold implies Axiom A and no cycle conditions. The converse was already proved by Smale [6]. Our aim is to give a simple proof for the following Smale's theorem.

THEOREM. *If a C^1-diffeomorphism of a compact boundaryless manifold satisfies Axiom A and no cycle conditions, then it is Ω-stable.*

Axiom A is defined by the following (1) the non-wandering set $\Omega(f)$ is hyperbolic (i.e. there exist a Riemannian metric $\langle\,,\rangle$ and a continuous splitting $T_{\Omega(f)}M=E^s\oplus E^u$ such that $Df(E^\sigma)=E^\sigma$, $\sigma=s$, u and $\|Df|_{E^s}\|\leqq\lambda$ and $\|Df^{-1}|_{E^u}\|\leqq\lambda$ for some $0<\lambda<1$) and (2) the set of all periodic points, $\mathrm{Per}(f)$, is dense in $\Omega(f)$. By Axiom A condition $\Omega(f)$ splits into the finite disjoint union $\Omega(f)=\Omega_1\cup\cdots\cup\Omega_k$ of closed subsets Ω_i, so-called basic sets, such that for $1\leqq i\leqq k$ $f(\Omega_i)=\Omega_i$, $\mathrm{Per}(f|_{\Omega_i})$ is dense in Ω_i and $f|_{\Omega_i}$ is topologically transitive. The cycle condition is defined as follows: there exist $n\geqq 1$ and $\{\Omega_{i_j}\}_{j=0}^n\subset\{\Omega_i\}$ such that $\Omega_{i_j}\neq\Omega_{i_l}$ $(0\leqq j\neq l\leqq n)$, $\Omega_{i_{n+1}}=\Omega_{i_0}$ and $W^s(\Omega_{i_j})\cap W^u(\Omega_{i_{j+1}})\neq\phi$ $(0\leqq j\leqq n)$. Here $W^s(\Omega_i)$ is denoted by $W^s(\Omega_i)=\{y\in M: d(f^n(y), \Omega_i)\to 0$ as $n\to\infty)$ and also $W^u(\Omega_i)=\{y\in M: d(f^{-n}(y), \Omega_i)\to 0$ as $n\to\infty\}$ where d is the metric on M induced by the Riemannian metric. For the space $\mathrm{Diff}^1(M)$ of C^1-diffeomorphisms with the uniform C^1-topology, $f\in\mathrm{Diff}^1(M)$ is called Ω-stable if there is a neighborhood $\mathcal{U}(f)$ in $\mathrm{Diff}^1(M)$ such that every $g\in\mathcal{U}(f)$ is Ω-conjugate to f.

If compact f-invariant set Λ is hyperbolic, then there exists $\varepsilon_0>0$ such that $W_\varepsilon^\sigma(x)$ is a C^1-submanifold, $T_xW_\varepsilon^\sigma(x)=E^\sigma(x)$ and $W_\varepsilon^\sigma(x)$ varies continuously with x $(x\in\Lambda$, $0<\varepsilon\leqq\varepsilon_0$ and $\sigma=s$, $u)$, and such that there exists $0<\lambda_0<1$ such that $y\in W_{\varepsilon_0}^s(x)$ $(x\in\Lambda)$ implies $d(f(x), f(y))\leqq\lambda_0 d(x, y)$ and $y\in W_{\varepsilon_0}^u(x)$ $(x\in\Lambda)$ implies $d(f^{-1}(x), f^{-1}(y))\leqq\lambda_0 d(x, y)$. Here $W_\varepsilon^s(x)$ is denoted by $W_\varepsilon^s(x)=\{y\in M: d(f^j(x), f^j(y))\leqq\varepsilon, j\geqq 0\}$ and also $W_\varepsilon^u(x)=\{x\in M: d(f^{-j}(x), f^{-j}(y))\leqq\varepsilon, j\geqq 0\}$.

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Using Axiom A condition, we can check that there exist neighborhoods V of $\Omega(f)$, $\mathcal{V}(f)$ of f and $c>0$ such that for $g\in\mathcal{V}(f)$ and $x, y\in\bigcap_{i\in Z} g^i(V)$ if $d(g^i(x), g^i(y))\leqq c$ for $i\in Z$ then $x=y$ (this remark will be discussed in the last part). For $\varepsilon>0$ small enough there exists $0<\gamma<\varepsilon$ such that $d(x, y)<\gamma(x, y\in\Omega(f))$ implies that $W_\varepsilon^s(x)\cap W_\varepsilon^u(y)$ is one point set and its intersection is transversal. By λ-lemma we have $W_\varepsilon^s(x)\cap W_\varepsilon^u(y)\in\Omega(f)$ which ensures that there exist a neighborhood $U\subset V$ and $\delta>0$ satisfying the following: for every δ-pseudo orbit $\{x_i\}\subset U$ (i.e. $d(f(x_i), x_{i+1})<\delta$ for all $i\in Z$) there is a unique point $x\in M$ such that $d(f^i(x), x_i)<\varepsilon$ $(i\in Z)$. In fact notice that $x\in\Omega(f)$. Therefore $\bigcap_{i\in Z} f^i(U)=\Omega(f)$ holds and f has a filtration for $\Omega(f)$ (i.e. a sequence $\phi=M_0\subset M_1\subset\cdots\subset M_k=M$ of smooth, compact codimension 0 submanifolds with boundary of M such that $f(M_i)\subset\operatorname{int} M_i$ and $\bigcap_{n\in Z} f^n(M_i-M_{i-1})=\Omega_i$) (see [6]). Thus $\Omega(f)$ coincides with the chain recurrent set $R(f)$ (which means the set of all points x such that for $\alpha>0$ there is a finite α-pseudo orbit $\{x_0, x_1, \cdots, x_n\}$ with $x_0=x_n=x$).

Choose a neighborhood $\mathcal{U}(f)\subset\mathcal{V}(f)$ of f in $\mathrm{Diff}^1(M)$ such that $\max_{x\in M} d(f(x), g(x))<\delta$ and $R(g)\subset U$ for every $g\in\mathcal{U}(f)$. Then we have $R(g)\subset\bigcap_{i\in Z} g^i(U)$ because $R(g)$ is g-invariant set. Let $x\in\bigcap_{i\in Z} g^i(U)$, then $\{g^i(x)\}$ is a δ-pseudo orbit of f. Thus there is a unique point $h(x)\in R(f)=\Omega(f)$ such that $d(f^i(h(x)), g^i(x))<\varepsilon$ for $i\in \boldsymbol{Z}$, and then $h:\bigcap_{i\in Z} g^i(U)\to R(f)$ is injective and $h\circ g=f\circ h$ holds. It is not difficult to see that h is continuous. By Hartman's theorem we have $\#\{x\in M: f^m(x)=x\}=\#\{x\in M: g^m(x)=x\}<\infty$ for $m>0$ and $g\in\mathcal{U}(f)$. Then $R(f)=h(\bigcap_{i\in Z} g^i(U))$. Since h is a homeomorphism and $\mathrm{Per}(f)$ is dense in $\Omega(f)$, we have $\Omega(g)=R(g)=\bigcap_{i\in Z} g^i(U)$.

Finally we explain the above remark. Since f satisfies Axiom A, there is $m>0$ such that for $x\in\Omega(f)$ and $v\in T_xM$, $\|Df^n(v)\|\geqq 6\|v\|$ for $n=m$ or $n=-m$. And there exists an extended continuous splitting $T_VM=\tilde{E}^s\oplus\tilde{E}^u$. Then we can find a neighborhood $\mathcal{V}(f)$ such that for $g\in\mathcal{V}(f)$ and $x\in\bigcap_{m}^{m} g^i(V)$

$$D_xg^m=\begin{pmatrix} A^{ss} & A^{us} \\ A^{su} & A^{uu}\end{pmatrix}: \tilde{E}^s(x)\oplus\tilde{E}^u(x)\longrightarrow\tilde{E}^s(g^m(x))\oplus\tilde{E}^u(g^m(x))$$

$$D_xg^{-m}=\begin{pmatrix} B^{ss} & B^{us} \\ B^{su} & B^{uu}\end{pmatrix}: \tilde{E}^s(x)\oplus\tilde{E}^u(x)\longrightarrow\tilde{E}^s(g^{-m}(x))\oplus\tilde{E}^u(g^{-m}(x))$$

$$\|A^{ss}\|\leqq 1/5,\qquad |A^{uu}|\geqq 5,\qquad \|B^{uu}\|\leqq 1/5,\quad |B^{ss}|\geqq 5$$

$$\max\}\|A^{us}\|, \|A^{su}\|, \|B^{us}\|, \|B^{su}\|\}\leqq 1/5$$

where $|E|=\|E^{-1}\|^{-1}$. Since $g^i(x)\in V$ for $|i|\leqq m$, when $\|v^u\|\geqq\|v^s\|$ for $v=v^s+v^u \in\tilde{E}^s(x)\oplus\tilde{E}^u(x)$ we have $\|v\|\leqq 2\|v^u\|$ and then

$$\begin{aligned}\|Dg^m(v)\| &= \|A^{ss}(v^s)+A^{su}(v^s)+A^{us}(v^u)+A^{uu}(v^u)\| \\ &\geqq 5\|v^u\|-(2\|v^s\|/5+\|v^u\|/5) \\ &\geqq 4\|v^u\|\geqq 2\|v\| .\end{aligned}$$

For the case when $\|v^s\|\geqq\|v^u\|$ we have also $\|Dg^{-m}(v)\|\geqq 2\|v\|$.

Take and fix $\varepsilon>0$ such that $\varepsilon(1+K+\cdots+K^{m-1})<1/2$ where $K=\sup\{\|D_x g\| : x\in M,\ g\in\mathcal{V}(f)\}$. Then there is $c>0$ such that for $g\in\mathcal{V}(f)$

$$\|\exp^{-1}_{g^\sigma(x)}\circ g^\sigma\circ\exp_x(v)-D_x g^\sigma(v)\|\leqq\varepsilon\|v\| \qquad (x\in M)$$

whenever $\|v\|\leqq c$ ($\sigma=1, -1$). Let $g\in\mathcal{V}(f)$ and $x, y\in\bigcap_{i\in\boldsymbol{Z}} g^i(V)$. Then we assume $d(g^i(x), g^i(y))\leqq c$ ($i\in\boldsymbol{Z}$) does not ensure $x=y$. Take δ with $0<\delta\leqq c_1/4$ where $c_1=\sup\{d(g^i(x), g^i(y)) : i\in\boldsymbol{Z}\}$. Obviously $c_1-\delta<d(g^k(x), g^k(y))\leqq c_1$ for some $k\in\boldsymbol{Z}$. Put $u=g^k(x)$, $v=g^k(y)$ and $w=\exp_u^{-1}(v)$. Then $c_1-\delta<\|w\|=d(u,v)$ and $\|Dg^n(w)\|\geqq 2\|w\|$ for some $|n|=m$. We deal with only the case $\|Dg^m(w)\|\geqq 2\|w\|$ (because the case $\|Dg^{-m}(w)\|\geqq 2\|w\|$ follows from a similar way). Since $\|w\|=d(u, v)\leqq c$ we have $\|\exp^{-1}_{g(u)}\circ g\circ\exp_u(w)-D_u g(w)\|\leqq\varepsilon\|w\|$, from which $\|D_u g(w)\|\leqq c_1(1+\varepsilon)$, and $\|\exp^{-1}_{g^2(u)}\circ g^2\circ\exp_u(w)-D_u g^2(w)\|\leqq c_1\varepsilon+Kc_1\varepsilon=c_1\varepsilon(1+K)$. Since $\|\exp^{-1}_{g^2(u)}\circ g^2\circ\exp_u(w)\| = d(g^2(u), g^2(v))\leqq c_1$, we have $\|D_u g^2(w)\|\leqq c_1(1+\varepsilon(1+K))$ and by induction we have $2\|w\|\leqq D_u g^m(w)\|\leqq c_1(1+\varepsilon(1+K+\cdots K^{m-1}))$. Consequently $c_1-\delta<\|w\|\leqq 3c_1/4$, i.e. $c_1/4<\delta$ which is a contradiction.

REMARK. By the same method of the above proof it is easily proved that Anosov diffeomorphisms are structurally stable. In fact, an Anosov diffeomorphism f has the pseudo orbit tracing property and every diffeomorphism in some neighborhood of f has the same expansive constant.

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