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THE TYPE NUMBER OF REAL HYPERSURFACES IN $P_n(C)$

Dedicated to Professor Tsunero Takahashi on his sixtieth birthday

By

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Introduction

We denote by $P_n(C)$ an n-dimensional complex projective space with the Fubini-Study metric of constant holomorphic sectional curvature $4c$ and M a real hypersurface in $P_n(C)$ with the induced metric.

The problem with respect to the type number t, that is, the rank of the second fundamental form of real hypersurfaces in $P_n(C)$ has been studied by many geometers ([1], [2], [3] and [4] etc.). The second named author [4] proved that there is a point p on M such that $t(p)\geq 2$ and M. Kimura and S. Maeda [2] gave an example of real hypersurface in $P_n(C)$ satisfying $t=2$, which is non-complete. Recently, Y. J. Suh [3] showed that there is a point p on a complete real hypersurface M in $P_n(C)$ $(n\geq 3)$ such that $t(p)\geq 3$.

In this paper we shall prove the following

MAIN THEOREM. *Let M be a complete real hypersurface in $P_n(C)$. Then there exists a point p on M such that $t(p)\geq n$.*

1. Preliminaries.

Hereafter let $M_n(c)$ $(n\geq 2)$ be a complex space form with the metric of constant holomorphic sectional curvature $4c$ and M be a real hypersurface in $M_n(c)$. Choose a local field of orthonormal frames $\{e_1,\cdots,e_{2n}\}$ in $M_n(c)$ such that $e_1,\cdots,e_{2n-1}$ are tangent to M. We use the following convention on the range of indices unless otherwise stated: $A,B,\cdots=1,\cdots,2n$ and $i,j,\cdots=1,\cdots,2n-1$. We denote by θ_A and θ_{AB} the canonical 1-forms and the connection forms respectively. Then they satisfy

$$d\theta_A+\sum\theta_{AB}\wedge\theta_B=0,\quad \theta_{AB}+\theta_{BA}=0. \tag{1.1}$$

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We restrict the forms under consideration to M. Then we have $\theta_{2n}=0$ and by Cartan's lemma we may write as

$$\phi_i \equiv \theta_{2n,i} = \sum h_{ij}\theta_j,\ h_{ij}=h_{ji}. \tag{1.2}$$

The quadratic form $\sum h_{ij}\theta_i \cdot \theta_j$ is called the second fundamental form of M for e_{2n}. Moreover, the curvature forms Θ_{ij} of M are defined by

$$\Theta_{ij} = d\theta_{ij} + \sum \theta_{ik}\wedge\theta_{kj}. \tag{1.3}$$

We denote by $\tilde{J}$ the complex structure of $M_n(c)$. Let (J_{ij}, f_k) be the almost contact metric structure of M, i.e., $\tilde{J}(e_i)=\sum J_{ji}e_j + f_i e_{2n}$. Then (J_{ij}, f_k) satisfies

$$\begin{aligned} &\sum J_{ik}J_{kj} = f_i f_j - \delta_{ij},\ \sum f_j J_{ji}=0,\\ &\sum f_i^2 = 1,\quad J_{ij}+J_{ji}=0. \end{aligned} \tag{1.4}$$

The parallelism of $\tilde{J}$ implies

$$\begin{aligned} dJ_{ij} &= \sum(J_{ik}\theta_{kj} - J_{jk}\theta_{ki}) - f_i\phi_j + f_j\phi_i,\\ df_i &= \sum(f_j\theta_{ji} - J_{ji}\phi_j). \end{aligned} \tag{1.5}$$

The equations of Gauss and Codazzi are given by

$$\Theta_{ij} = \phi_i\wedge\phi_j + c\theta_i\wedge\theta_j + c\sum(J_{ik}J_{jl}+J_{ij}J_{kl})\theta_k\wedge\theta_l, \tag{1.6}$$

$$d\phi_i = -\sum\phi_j\wedge\phi_{ji} + c\sum(f_iJ_{jk}+f_jJ_{ik})\theta_j\wedge\theta_k, \tag{1.7}$$

respectively.

2. Formulas.

Let M be a real hypersurface in $M_n(c)$, $c\neq 0$. In this section, we assume that the rank of the second fundamental form is not larger than m on an open set U. In the sequel, we use the following convention on the range of indices: $a,b,\cdots=1,\cdots,m$ and $r,s,\cdots=m+1,\cdots,2n-1$. Then for an arbitrary point p in U we can take a local field of orthonormal frames $\{e_1,\cdots,e_{2n-1}\}$ on a neiborhood of p such that the 1-forms ϕ_i can be written as

$$\begin{aligned} &\phi_a = \sum h_{ab}\theta_b,\ h_{ab}=h_{ba},\\ &\phi_r = 0. \end{aligned} \tag{2.1}$$

Here, we put

$$\theta_{ar} = \sum A_{arb}\theta_b + \sum B_{ars}\theta_s. \tag{2.2}$$

Taking the exterior derivative of $\phi_r = 0$ and using (1.7) and (2.1), we have

$$\sum h_{ab}\theta_b \wedge \theta_{ar} - c\sum(f_r J_{ij} + f_i J_{rj})\theta_i \wedge \theta_j = 0,$$

which, together with (2.2), implies

$$\sum(h_{ac}A_{crb} - h_{bc}A_{cra}) - cf_a J_{rb} + cf_b J_{ra} - 2cf_r J_{ab} = 0, \tag{2.3}$$

$$\sum h_{ab}B_{brs} - cf_a J_{rs} + cf_s J_{ra} - 2cf_r J_{as} = 0, \tag{2.4}$$

$$f_s J_{rt} - f_t J_{rs} + 2f_r J_{st} = 0. \tag{2.5}$$

The above equation (2.5) is equivalent to

$$f_r J_{st} = 0. \tag{2.6}$$

Similarly, taking the exterior derivative of $\phi_a = \sum h_{ab}\theta_b$ and making use of (1.1), (1.7), (2.1), (2.2) and (2.4), we get

$$\begin{aligned}&\sum\{dh_{ab} - \sum(h_{ac}\theta_{cb} + h_{bc}\theta_{ca} - \sum h_{ac}A_{crb}\theta_r - cf_b J_{ac}\theta_c + cf_c J_{ab}\theta_c\\ &-2cf_a J_{bc}\theta_c) + c\sum(f_b J_{ar}\theta_r - f_r J_{ab}\theta_r + 2f_a J_{br}\theta_r)\} \wedge \theta_b = 0,\end{aligned}$$

which yields

$$\begin{aligned}&dh_{ab} - \sum(h_{ac}\theta_{cb} + h_{bc}\theta_{ca} - \sum h_{ac}A_{crb}\theta_r)\\ &\qquad + c\sum(f_b J_{ar}\theta_r - f_r J_{ab}\theta_r + 2f_a J_{br}\theta_r) \equiv 0 \ (mod\ \theta_a)\end{aligned} \tag{2.7}$$

Now, we quote two Lemmas.

LEMMA 2.1 ([3]). *Assume that $J_{rs}(p) = 0$ at a point p on M. Then $t(p) \geq n-1$. Furthermore, the equality holds if and only if $f_a = 0$ and $J_{ab} = 0$ at p.*

Here, we denote by T the maximal value of the type number t.

LEMMA 2.2 ([3]). *If $J_{rs} = 0$ on U, then $T \geq n$ on U.*

PROOF. If $T < n$, then owing to Lemma 2.1, we see that $T = n-1$, $f_a = 0$ and $J_{ab} = 0$ on U. For a suitable choice of a field $\{e_r\}$ of orthonormal frames, we can set $f_{2n-1} = 1$ and $f_r = 0$ for $r = n, \cdots, 2n-2$. Then, by means of (1.5), we get

$$0 = df_a = \theta_{2n-1,a},$$

where we have used (2.1). Thus, taking account of (2.2), we find $B_{a,2n-1,s} = 0$. On the other hand, if we put $r = 2n-1$ and $s \neq 2n-1$ in (2.4), then we have $J_{as} = 0$ for $s \neq 2n-1$, which contradicts the fact that rank $J = 2n-2$. □

REMARK. Lemma 2.2 was proved in [3] but the proof is incomplete.

In the remainder of this section, we shall obtain further formulas. First of all, we define the open set V_T by

$$V_T = \{p \in M \mid t(p) = T\}.$$

Next, in order to prove our theorem we shall lead a contradiction by assuming the following:

$$\forall p \in V_T,\ \forall U(p),\ \exists q \in U(p)\ \textit{such that}\ J_{rs}(q) \neq 0, \tag{2.8}$$

where $U(p)$ denotes a neighborhood of a point p.

Moreover, we consider the open set V_T' defined by

$$V_T' = \{p \in V_T \mid J_{rs}(p) \neq 0\}.$$

Since V_T' is dense subset of V_T by the assumption (2.8), any equality obtained on V_T' holds also on V_T. Hence, we may assume $V_T' = V_T$ whenever we treat equalities. Therefore, from (2.6) it follows that $f_r = 0$ on V_T. Consequently, we may set $f_1 = 1$ and $f_a = 0$ for $a = 2, \cdots, T$. This and (1.4) show

$$J_{1a} = 0,\ J_{1r} = 0. \tag{2.9}$$

Furthermore, the fact that $df_a = 0$ and $df_r = 0$ tells us

$$\theta_{1a} = -\sum J_{ab}\phi_b, \tag{2.10}$$

$$A_{1ra} = \sum h_{ab} J_{br}, \tag{2.11}$$

$$B_{1rs} = 0, \tag{2.12}$$

where we have used (1.5), (2.1) and (2.2).

From (2.4), we have

$$\sum h_{ab} B_{brs} = c f_a J_{rs}. \tag{2.13}$$

On the other hand, if we take the exterior derivative of (2.10) and make use of (1.3)~(1.7), (2.1), (2.2), (2.7) and (2.9)~(2.13), then we find

$$c\theta_1 \wedge \theta_a = \sum J_{ar} h_{be} A_{brd} \theta_d \wedge \theta_e + 2c \sum J_{ab} J_{bd} \theta_d \wedge \theta_1.$$

Pick out the coefficients of $\theta_c \wedge \theta_1$ in the above equation. Then from (1.4) and (2.3) we can get

$$\sum J_{ab} J_{bc} = 0$$

and so

$$J_{ab} = 0. \tag{2.14}$$

This and (2.10) give

$$\theta_{1a} = 0. \tag{2.15}$$

Moreover, from (2.12) and (2.13) it follows that (cf.[3])

$$\det(h_{ab}) = 0 \quad (a, b = 2, \cdots, T). \tag{2.16}$$

Thus, for a suitable choice of a field $\{e_a\}$ of orthonormal frames, we may set

$$h_{ab} = \lambda_a \delta_{ab} \quad (a, b = 2, \cdots, T). \tag{2.17}$$

Combining (2.17) with (2.16), we can set $\lambda_2 = 0$. Since $\det(h_{ab}) = -h_{12}^2 \lambda_3 \cdots \lambda_T$, it follows that

$$h_{12} \neq 0 \ \ and \ \ h_{aa} = \lambda_a \neq 0 \quad (a = 3, \cdots, T) \tag{2.18}$$

because $\det(h_{ab})$ does not vanish on V_T.

On the other hand, the equation (2.11), together with (2.9) and (2.17), yields

$$A_{1r2} = 0. \tag{2.19}$$

Now, put a = 2 and $b \geq 3$ in (2.3). Then, using (2.11), (2.17) and (2.18), we find

$$A_{br2} = h_{12} J_{br} \quad (b \geq 3). \tag{2.20}$$

Similarly, put a = 1 and b = 2 in (2.3) and use (2.8). Then we obtain

$$\Sigma(h_{1a} A_{ar2} - h_{2a} A_{ar1}) + cJ_{2r} = 0.$$

It follows from (2.11), (2.17), (2.19) and (2.20) that the above equation can be reformed as

$$h_{12} A_{2r2} = h_{12} \Sigma h_{1a} J_{ar} - h_{12} \sum_{a \geq 3} h_{1a} J_{ar} - cJ_{2r}. \tag{2.21}$$

We put a = 2 and $b \geq 3$ in (2.7) and take account of (2.14), (2.15) and (2.17). Then we have

$$h_{bb} \theta_{b2} - h_{12} \Sigma A_{1rb} \theta_r \equiv 0 \ (mod\, \theta_a),$$

which, together with (2.9), (2.11) and (2.18), leads to

$$\theta_{b2} \equiv h_{12} \Sigma J_{br} \theta_r \ \ (mod\, \theta_a) \ \ for \ \ b \geq 3. \tag{2.22}$$

Last, put a = 1 and b = 2 in (2.7). Then from (2.14) and (2.15) it follows that

$$dh_{12} - \Sigma(h_{1b}\theta_{b2} - \Sigma h_{1b} A_{br2} \theta_r) + 2c \Sigma J_{2r} \theta_r \equiv 0 \ (mod\ \theta_a).$$

Combining this equation with (2.9), (2.15) and (2.19)~(2.22), we get a key equation

(2.23) $$dh_{12}+(h_{12}{}^2+c)\Sigma J_{2r}\theta_r \equiv 0 \ (mod\ \theta_a).$$

3. Lemmas.

In this section, we use the same notion as one in section 2 unless otherwise stated. From now on, we suppose that M is complete. For simplicity, we put $F=h_{12}$. Then the equation (2.23) is equivalent to

(3.1) $$dF+(F^2+c)\Sigma J_{2r}\theta_r \equiv 0 \ (mod\ \theta_a).$$

Here, we note that $J_{2r}\neq 0$ everywhere on V_T because of (2.9), (2.14) and the fact that rank $J = 2n-2$.

Let p be any point of V_T and let $\alpha: I \to V_T$ be a maximal integral curve of the unit vector field $\Sigma J_{2r}e_r$ on V_T through p. Assume that I has an infimum or a superemum, say t_0. Then we have

LEMMA 3.1.
$$\lim_{t\to t_0} h_{aa}(\alpha(t))\neq 0 \quad (a=3,\cdots,T)$$

PROOF. Put $a=b(\geq 3)$ in (2.7). Then from (2.14), we get
$$dh_{aa}-2\Sigma h_{ac}\theta_{ca}+\Sigma h_{ac}A_{cra}\theta_r \equiv 0 \ (mod\ \theta_a).$$

From (2.9), (2.11), (2.15) and (2.17), it follows that

(3.2) $$dh_{aa}+h_{aa}\,\Sigma(h_{a1}J_{ar}+A_{ara})\theta_r \equiv 0 \ (mod\ \theta_a).$$

We restrict the forms under consideration to α. Then (3.2) becomes
$$\frac{dh_{aa}}{dt}+h_{aa}\,\Sigma(h_{a1}J_{ar}+A_{ara})J_{2r}=0,\ t\in I.$$

On the other hand, since M is complete, there exists a limit point $\lim_{t\to t_0}\alpha(t)$ on M. Suppose that $\lim_{t\to t_0} h_{aa}(\alpha(t))=0$. Then from the above differential equation, we have $h_{aa}=0$ on V_T. This contradicts the fact (2.18). □

LEMMA 3.2.
$$\lim_{t\to t_0} F(\alpha(t))=0.$$

PROOF. Assume that $\lim_{t\to t_0} F(\alpha(t))\neq 0$. Owing to Lemma 3.1 and the definition of the open set V_T, we see that $\alpha(t_0)\in V_T$, which contradicts the maximality of the integral curve α. □

4. The proof of Main Theorem.

In this section, we keep the notion in sections 2 and 3. Put $t_1 = \inf I (\geq -\infty)$ and $t_0 = \sup I (\leq \infty)$. Then there are four possibilities of an open interval (t_1, t_0). Namely, the interval I is one of the following:

(1) $-\infty < t_1, t_0 < \infty$,
(2) $-\infty = t_1, t_0 < \infty$,
(3) $-\infty < t_1, t_0 = \infty$,
(4) $-\infty = t_1, t_0 = \infty$.

On the other hand, by virtue of (3.1) the function F defined on an open interval (t_1, t_0) satisfies

$$\frac{dF}{F^2 + c} + dt = 0. \tag{3.3}$$

Here, we consider the case where $c > 0$. Then solving this differential equation (3.3), we have

$$F(\alpha(t)) = -\sqrt{c} \tan \sqrt{c}(t - t_2), \tag{3.4}$$

where $t_2 = t_1$ or t_0 in the cases (1)~(3) and t_2 is some constant in the case (4).

In order to prove our theorem, it suffices to show that we lead a contradiction at any case because of Lemma 2.2 and the assumption (2.8).

Combining Lemma 3.2 with the fact that $J_{2r} \neq 0$ everywhere on V_T, we see that the case (1) can not occur. In fact, owing to Lemma 3.2 it is seen that there exists a real number t' such that $t_1 < t' < t_0$, $dF = 0$ at $\alpha(t')$ Then the differential equation (3.3) gives $J_{2r} = 0$. This contradicts.

Moreover, in the cases (2)~(4) we note that the function *tan* of the solution (3.4) can not be defined for all $t \in \boldsymbol{R}$ but $F(\alpha(t))$ is defined on (t_1, t_0), where t_1 or t_0 is ∞. Thus, from Lemma 3.2 it follows that the cases (2)~(4) can not occur too.

It completes the proof of Main Theorem.

REMARK. In the case where $c < 0$, solving the differential equation (3.1) we have

(1) $F(\alpha(t)) \equiv k$,
(2) $F(\alpha(t)) = k \tanh(k(t + d))$,

(3) $F(\alpha(t))=k\coth(k(t+d))$,

where $k=\sqrt{-c}$ and d is real number. Therefore we can not apply the above arguments to this case.

Open Question.

Does there exist a complete real hypersurface M in $P_n(C)$ such that $t(p)=n$ for a point p on M?

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