TSUKUBA J. MATH.
Vol. 20 No. 2 (1996), 365–378

THE ω-CONSISTENCY OF ELEMENTARY ANALYSIS

By

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§0. Introduction

Let EA be the formal system of elementary analysis. The following is proved by Schütte [7] that we have a proof of the consistency of EA by transfinite induction up to $\varepsilon_{\varepsilon_0}$.

The purpose of this paper is to prove the following two theorems:

THEOREM 1. *The* ω-consistency *of* EA *can be proved by applying transfinite induction up to* $\varepsilon_{\varepsilon_1}$ *for an elementary number theoretical proposition, together with exclusively elementary number theoretical techniques.*

THEOREM 2. *The* ω-consistency *of* EA *cannot be proved by applying transfinite induction to numbers below* $\varepsilon_{\varepsilon_1}$ *for the elementary number theoretical propositions, together with exclusively elementary number theoretical techniques.*

The proof of theorem 1 is carried out as the same line as that of ω-consistency of elementary number theory EN by Hanatani [6], which is an application of Gentzen's consistency proof of EN [3].

The greater part of technical terms and conventions are adopted from English translation [8] of the works of G. Gentzen and Takeuti [9] and Schütte [7].

The author would like to express his hearty gratitude to the late Prof. Shôji Maehara and Prof. Mitio Takano who encouraged him with many valuable advices.

§1. Formal systems EA and EA(M)

We define the formal systems EA and EA(M).

1.1. As primitive symbols we use

1. Denumerably infinitely many free and bound number variables.

Received July 4, 1994
Revised November 7, 1994

2. Denumerably infinitely many free and bound 1-place predicate variables.

3. The individual constant 0.

4. The logical symbols: $\neg$ (not), $\vee$ (or), $\forall$ (for all) and λ (abstruction).

5. Symbols for n-place calculable arithmetic functions and n-place decidable arithmetic predicates $(n \geqq 1)$. Especially $'$ is the successor function symbol.

6. Auxiliary symbols:), (, $'$, $\rightarrow$.

1.2. Terms, formulas and predicators.

Inductive definition of terms:

1. The symbol 0 is a term.
2. Every free number variable is a term.
3. If t is a term then so is t'.
4. If f is a symbol for an n-place calculable arithmetic function $(n \geqq 1)$ and $t_1,\cdots,t_n$ are terms, then $f(t_1,\cdots,t_n)$ is also a term.

Terms built up according to 1 and 3 only are called numerals. A term is said to be numerical if it contains no free number variables.

The prime formulas are formulas of the form $P(t_1,\cdots,t_n)$ where P is a symbol for an n-place decidable arithmetic predicate $(n \geqq 1)$ and $t_1,\cdots,t_n$ are terms.

Inductive definition of formulas and predicators:

1. Every prime formula is a formula.
2. Every free predicate variable is a predicator.
3. If P is a predicator and t is a term, then $P(t)$ is a formula.
4. If A is a formula, then so is $(\neg A)$.
5. If A and B are formulas, then so is $(A \vee B)$.
6. If $F(a)$ is a formula and a is a free number variable and x is a bound number variable which does not occur in $F(a)$, then $\forall x F(x)$ is a formula.
7. If $F(a)$ is a formula and a is a free number variable and x is a bound number variable which does not occur in $F(a)$, then $\lambda x F(x)$ is a predicator.
8. If $F(U)$ is a formula and U is a free predicate variable and X is a bound predicate variable which does not occur in $F(U)$, then $\forall X F(X)$ is a formula.

A predicator is said to be elementary if it contains no bound predicate variables.

1.3. The concept of 'sequent' is defined as in [2].

1.4. As 'basic sequents' we shall admit 'basic logical sequents', 'basic equality sequents' and 'basic mathematical sequents'. A basic logical sequent is a sequent of the form $D \rightarrow D$, where D is an arbitrary formula. A basic equality sequent is a sequent of the form $s=t, F(s) \rightarrow F(t)$ where s and t are arbitrary terms and $F(s)$ is an arbitrary formula containing (at least) one occurrence of the term s and $F(t)$ is a formula which results from $F(s)$ by the replacement of one occurrence of s by t. A basic mathematical sequent is a sequent consisting of prime formulas which becomes true sequent with every arbitrary substitution of numerals for possible occurrences of free variables.

1.5. As 'inference figures' we shall use the ones which result from one of the following inference figure schemata

1.51. Schemata for structural inference figures:

Thinning: in the antecedent $\dfrac{\Gamma \to \Theta}{D, \Gamma \to \Theta}$, in the succedent $\dfrac{\Gamma \to \Theta}{\Gamma \to \Theta, D}$;

Contraction: in the antecedent $\dfrac{D, D, \Gamma \to \Theta}{D, \Gamma \to \Theta}$, in the succedent $\dfrac{\Gamma \to \Theta, D, D}{\Gamma \to \Theta, D}$;

Interchange: in the antecedent $\dfrac{\Gamma, C, D, \Delta \to \Theta}{\Gamma, D, C, \Delta \to \Theta}$, in the succedent $\dfrac{\Gamma \to \Theta, C, D, \Lambda}{\Gamma \to \Theta, D, C, \Lambda}$;

Cut: $$\frac{\Gamma \to \Theta, D \quad D, \Delta \to \Lambda}{\Gamma, \Delta \to \Theta, \Lambda}$$

1.52. Schemata for operational inference figures:

$$\neg\text{-IA}: \frac{\Gamma \to \Theta, A}{\neg A, \Gamma \to \Theta}, \qquad \neg\text{-IS}: \frac{A, \Gamma \to \Theta}{\Gamma \to \Theta, \neg A},$$

$$\vee\text{-IA}: \frac{A, \Gamma \to \Theta \quad B, \Gamma \to \Theta}{A \vee B, \Gamma \to \Theta}, \qquad \vee\text{-IS}: \frac{\Gamma \to \Theta, A}{\Gamma \to \Theta, A \vee B} \quad \frac{\Gamma \to \Theta, B}{\Gamma \to \Theta, A \vee B},$$

$$\forall\text{-IA}: \frac{F(t), \Gamma \to \Theta}{\forall x F(x), \Gamma \to \Theta}, \qquad \forall\text{-IS}: \frac{\Gamma \to \Theta, F(a)}{\Gamma \to \Theta, \forall x F(x)},$$

Restrictions on number variables: The free number variable which is designated by a and is called the eigenvariable of the $\forall$-IS must not occur in the lower sequent of the inference figure.

$$\forall^2\text{-IA}: \frac{F(P), \Gamma \to \Theta}{\forall X F(X), \Gamma \to \Theta}, \qquad \forall^2\text{-IS}: \frac{\Gamma \to \Theta, F(U)}{\Gamma \to \Theta, \forall X F(X)},$$

where P is an arbitrary elementary predicator and X is a bound predicate variable.

Restrictions on predicate variables: The free predicate variable which is designated by U and is called the eigenvariable of the $\forall^2$–IS must not occur in the lower sequent of the inference figure.

$$\lambda - \mathrm{IA}: \frac{F(t), \Gamma \rightarrow \Theta}{\lambda x F(x)(t), \Gamma \rightarrow \Theta}, \qquad \lambda - \mathrm{IS}: \frac{\Gamma \rightarrow \Theta, F(t)}{\Gamma \rightarrow \Theta, \lambda x F(x)(t)}.$$

1.53. Schema for CJ-inference figures

$$\frac{F(a), \Gamma \rightarrow \Theta, F(a')}{F(0), \Gamma \rightarrow \Theta, F(t)}$$

Restrictions on number variables: The free number variable which is designated by a and is called the eigenvariable of the CJ must not occur in the lower sequent of the inference figure.

1.54. Schemata for substitution of terms

$$\frac{\Gamma_1, F(s), \Gamma_2 \rightarrow \Theta}{\Gamma_1, F(t), \Gamma_2 \rightarrow \Theta} \qquad \frac{\Gamma \rightarrow \Theta_1, F(s), \Theta_2}{\Gamma \rightarrow \Theta_1, F(t), \Theta_2},$$

where s and t may replaced by terms without free variables, as long as they designate the same number.

1.6. An EA-derivation is a figure in tree form consisting of a number of sequents (at least one) with one lowest sequent, the endsequent, and certain uppermost sequents which must be basic sequents; the connection between the uppermost sequents and the endsequent is established by inference figures.

A sequent is said to be provable in EA if there exists an EA-derivation whose endsequent is the sequent. A formula A is said to be provable if the sequent $\rightarrow A$ is provable.

A system is called consistent if in it no two formulas A and $\neg A$ are both provable. A system EA is ω-consistent if and only if in it, for no formula $F(a)$ containing only a as free number variable, are all of the formulas $F(\bar{n})$ for $n = 0, 1, 2, \cdots$ and also the formula $\neg \forall x F(x)$ provable.

1.7. Let M be the set

$\{\forall x F(x) : \forall x F(x)$ is an EA-sentence such that $F(\bar{n})$ is EA-provable for each numeral $\bar{n}\}$.

We now define the system EA(M) from EA by adding new axioms $\rightarrow \forall x F(x)$ for all formulas $\forall x F(x)$ in M. We call these new added axioms M-axioms. Then the following proposition holds:

PROPOSITION 1. *The system* EA*(M) is consistent if and only if* EA *is ω-consistent.*

By formalizing the proof of Proposition 1 in EN. We have the following

COROLLARY 1. *The sequent*

$$\omega\text{-consis(EA)} \rightarrow \text{consis(EA}(M))$$

is EN-*provable.*

§2. Proof of Theorem 1.

For the proof of Theorem 1, it suffices by Proposition 1 to prove the consistency of EA(M). We prepare some definitions and propositions.

2.1. Inductive definition of the rank of formula.

1. The rank of every prime formula is 0.
2. If U is a free predicate variable, then the rank of the formula $U(t)$ is 0.
3. The rank of the formula $(\neg A)$ is (the rank of A)+1.
4. The rank of the formula $(A \vee B)$ is max{the rank of A, the rank of B}+1.
5. The rank of the formula $\forall xF(x)$ is (the rank of $F(0)$)+1.
6. The rank of the formula $\lambda xF(x)(t)$ is (the rank of $F(0)$)+1.
7. The rank of the formula $\forall XF(X)$ is the maximum of ω and (the rank of $F(U)$)+1.

LEMMA 1. *The following hold for ranks of formulas:*

1) *The rank of each formula is less than* $\omega+\omega=\omega\cdot 2$.

2) *If* $F(a)$ *is a formula and* t *is a term, then the rank of* $F(a)$ *is the rank of* $F(t)$. *If* $F(U)$ *is a formula and* U *and* V *are free predicate vaiables, then the rank of* $F(U)$ *is the rank of* $F(V)$.

3) *The rank of* A *is less than the rank of* $A \vee B$. *The rank of* B *is less than the rank of* $A \vee B$. *The rank of* A *is less than the rank of* $\neg A$. *The rank of* $F(t)$ *is less than the rank of* $\forall xF(x)$. *The rank of* $F(t)$ *is less than the rank of* $\lambda xF(x)(t)$.

4 *If* $F(U)$ *contains a bound predicate variable, then the rank of* $F(P)$ *is equal to that of* $F(U)$ *for each elementary predicator* P.

5) *If* P *is an elementary predicator the rank of* $F(P)$ *is less than the rank of* $\forall XF(X)$.

2.2. Let ρ and σ be ordinal numbers such that $\rho \leqq \sigma$. Then the ordinal τ such that $\rho+\tau=\sigma$ is uniquely defined and is denoted by $-\rho+\sigma$.

Let ρ and α be ordinals and $\rho<\omega\cdot 2$. Then we define the ordinal function $\omega_\rho(\alpha)$ inductively as follows:

1. $\omega_0(\alpha) := \alpha$.
2. $\omega_{n+1}(\alpha) := \omega^{\omega_n(\alpha)}$ for each natural number n.
3. $\omega_{\omega+n}(\alpha) := \varepsilon_{\omega_n(\alpha)}$ for each natural number n.

Then the following Lemma holds for ordinal function $\omega_\rho(\alpha)$:

LEMMA 2. 1) *If* $\alpha < \beta$, *then* $\omega_\rho(\alpha) < \omega_\rho(\beta)$.

2) *If* $0 < \rho$ *and* $\beta, \gamma < \omega_\rho(\alpha)$, *then* $\beta + 1$, $\max\{\beta, \gamma\} + 1$ *and* $\beta \# \gamma$ *are less than* $\omega_\rho(\alpha)$.

3) *If* $\omega \leqq \rho$ *and* $\beta, \gamma < \omega_\rho(\alpha)$, *then* $\beta + 1$, $\max\{\beta, \gamma\} + 1$, $\beta \# \gamma$, $\beta \cdot \omega$ *and* ω^β *are less than* $\omega_\rho(\alpha)$.

4) *If* $\tau + \rho < \omega \cdot 2$, *then* $\omega_\tau(\omega_\rho(\alpha)) = \omega_{\tau+\rho}(\alpha)$.

2.3. We define the level of the derivational sequent. Let $\boldsymbol{P}$ be a derivation in EA(M) and S be a sequent in $\boldsymbol{P}$. By the level of the derivational sequent S we mean the greatest rank of any cut or of a CJ-inference figure whose lower sequent stands below the sequent S, where the rank of a cut is the rank of the cut formula and the rank of a CJ-inference figure is the rank of the CJ-formula. If there is no such inference figures, then the level is equal to 0.

2.4. We define the assignment of ordinal numbers below $\varepsilon_{\varepsilon_1}$ to EA(M) derivations.

Suppose that an arbitrary derivation $\boldsymbol{P}$ is given.

Each basic sequent in $\boldsymbol{P}$ is assigned the ordinal 1(i.e., ω^0).

Each M-axiom in $\boldsymbol{P}$ is assigned the ordinal $\varepsilon_0 + 1$ if the level $\geqq \omega$, and M-axiom in $\boldsymbol{P}$ is assigned the ordinal $\varepsilon_{\varepsilon_0} + 1$ if the level is less than ω.

Suppose that the ordinal numbers of the upper sequents of an inference figure have already been assigned. The ordinal number of the line of inference is then assigned as follows:

If the inference figure is structural or substitution of terms, then the ordinal number of the upper sequent is assigned unchanged or, in the case of a cut, the natural sum of the ordinal numbers of the two upper sequents is assigned.

If the inference figure is operational, then +1 is adjoined to the ordinal number of the upper sequent; if the figure has two upper sequents, the larger of the two ordinal numbers is selected and +1 is adjoined to it.

If the inference figure is a CJ-inference figures – whose upper sequent has the ordinal number α then $\alpha \cdot \omega$ is assigned as the ordinal number of the line of inference.

From the ordinal number of a line of inference – call it α - the ordinal number of the lower sequent of the inference figure concerned is obtained by $\omega_{\tau}(\alpha)$, where the level of the uppersequents is σ and that of the lower sequent is ρ and $\tau = -\rho + \sigma$.

The ordinal number which is finally obtained for the endsequent of the derivation $\boldsymbol{P}$ is the ordinal number of $\boldsymbol{P}$ and denoted by $o(\boldsymbol{P})$. The ordinal number of a derivational sequent S is also denoted by $o(S)$.

COROLLARY 2. *Let* $\boldsymbol{P}$ *be an* EA*(M)-derivation and S be a derivational sequent in* $\boldsymbol{P}$*. Suppose that there occurs no M-axioms in the subderivation of S. Then the following hold:*

PROOF.1) Since there occurs no M-axiom in $\boldsymbol{P}$, every uppermost sequent has 1. If $\alpha, \beta < \varepsilon_0$, then $\alpha \# \beta, \alpha+1$, $\max\{\alpha, \beta\}+1, \alpha \cdot \omega < \varepsilon_0$. And in this case in the subderivation the difference of the level of uppersequent and that of the lower sequent is a natural number, and if $\alpha < \varepsilon_0$ then $\omega^{\alpha} < \varepsilon_0$, it follows that $o(S) < \varepsilon_0$.

2) Let S_1 be a sequent whose level is less than ω and that of whose upper sequent $\geqq \omega$. Let the difference of the level of uppersequent and that of the lowe sequent S_1 be $\omega + n$, where n is a natural number. By 1), the ordinal α_0 of the line of the inference figure whose lower sequent is S_1 is less than ε_0. Hence it follows that $o(S_1) = \omega_{\omega+n}(\alpha_0) = \varepsilon_{\omega_n(\alpha_0)} < \varepsilon_{\varepsilon_0}$. If $\alpha, \beta < \varepsilon_{\varepsilon_0}$, then $\alpha \# \beta$, $\alpha+1$, $\max\{\alpha, \beta\}+1$, $\alpha \cdot \omega$, $\omega^{\alpha} < \varepsilon_{\varepsilon_0}$. Thus it follows that $o(S) < \varepsilon_{\varepsilon_0}$.

2.5. The proof of consistency of EA(M)

The consistency of the formal system EA(M) is proved in elementary number theory by applying transfinite induction up to $\varepsilon_{\varepsilon_1}$ for an elementary number theoretical proposition, together with exclusively elementary number-theoretical techniques. The proof is carried out as in that of the consistency proof of elementary number theory in Gentzen [3].

Suppose that an EA(M)-derivation of the endsequent $\rightarrow$ is given.

2.51. The reduction step begins with the same preparatory step as in Gentzen [3], 3.2: the replacement of 'redundant' free number variables by 0.

The 'ending' of the given derivation is defined as follows: The ending includes all those derivational sequents which are encountered if we trace each individual path from the endsequent upwards and stop as soon as we reach the line of inference of an operational or CJ-inference figure.

The ending can therefore contain only structural inference figures and substitutions of terms and, after the preparatory step, no free number variables. The uppermost sequents of the ending may be lower sequents of operational or

CJ-inference figures as well as basic sequents of any one of three kinds and M-axioms.

2.52. We now add to the preparatory step an additional step made necessary by the admission of calculable function symbols:

All terms occurring in the ending are to be replaced by the numerical terms which result from the 'evaluation' of the occurring function symbols.

By this replacement all structural inference figures turn into correct inference figures of the same kind, as well as all basic sequents and M-axioms turn into other correct sequents of the same kind. As in Gentzen [3], 3.32, the 'substitutions of terms' and 'basic equality axioms' in the ending are deleted.

2.53. If an uppermost sequent of the ending is the lower sequent of a CJ-inference figure, then a CJ-reduction is carried out.

2.54. If the ending is nowhere bounded above by a CJ-inference figure, then the actual reduction step is preceded by a second preparatory step as in Gentzen [3], 3.34. As in Gentzen [3], 3.42; after the second preparatory step is carried out, we can eliminate of all thinnings and basic logical sequents from the ending.

Thus for the ending the following holds: The uppermost sequents of the ending are basic mathematical sequents, M-axioms or lower sequents of operational inference figures. And the ending can contain only interchanges, contractions and cuts.

Suppose that there occur no lower sequents of operational inference figures in the ending. Then our derivation is identical with the ending itself. This contradicts with our choice of basic mathematical sequents. Thus we can conclude that our derivation contains at least one lower sequent of operational inference figure.

The notion 'a cluster of formulas' is used as the same meaning as that in Gentzen [3], 3.41.

We now assert: there exists at least one cluster of formulae in the ending of our derivation, with at least one uppermost formula on its left side, which is the principal formula of an operational inference figure or the succedent formula of an M-axiom, and with at least one uppermost formula on its right side, which is the principal formula of an operational inference figure.

The cluster is associated a cut. We call this cut a suitable cut.

2.55. Operational reduction

2.551. If a suitable cut is a cut whose both cut formulas belong to the cluster with at least one uppermost formula on its left side which is the principal formula of an operational inference figure, the operational reduction is carried out as in Gentzen [3], 3.5.

2.552. Therefore it remains a case that a suitable cut is a cut whose left cut formula belongs to the cluster with one uppermost formula which is the succedent formula of an M-axiom.

Then, if necessary by using substitution of terms to the auxiliary formula of the operational inference figure in the boundary, we may consider the derivation which looks like this:

$$\vdots$$

$$\frac{F(\bar{n}), \Gamma_1 \rightarrow \Theta_1}{\forall x F(x), \Gamma_1 \rightarrow \Theta_1}$$ The operational inference figure in the boundary

$$\vdots$$

M - axiom

level σ_1 $$\frac{\rightarrow \forall x F(x) \quad \forall x F(x), \Gamma \rightarrow \Theta}{\Gamma \rightarrow \Theta}$$ The cut associated with the cluster

$$\vdots$$

level σ_1

level $\sigma < \sigma_1$ $$\frac{}{\Gamma_0 \rightarrow \Theta_0}$$ 'level line'

$$\vdots$$

$$\frac{}{\rightarrow}$$ The empty sequent.

Let the level of the upper sequents of the suitable cut be σ_1 and that of the lower sequent of the level line be σ. From the definition of the level line it holds that $\sigma_1 > \sigma$. It holds that $\sigma_1 \geqq$ the rank of $\forall x F(x) >$ the rank of $F(\bar{n})$. Let σ_2 =max {the rank of $F(\bar{n})$, σ}. Then it holds that $\sigma_1 > \sigma_2$. Hence $\sigma_1 > \sigma_2 \geqq \sigma$. Now let $\tau = -\sigma + \sigma_2$ and $\rho = -\sigma_2 + \sigma_1$. Then $\tau \geqq 0$, $\rho > 0$ and $\sigma_1 = (\sigma + \tau) + \rho$.

The reduction step consists now of the derivation into the form indicated by the following diagram:

$$\vdots$$

$$\frac{F(\bar{n}), \Gamma_1 \to \Theta_1}{\forall x F(x), \Gamma_1, F(\bar{n}) \to \Theta_1}$$ Interchanges and thinnings

$$\vdots$$

M - axiom

level σ_1 $$\frac{\to \forall x F(x) \quad \forall x F(x), \Gamma, F(\bar{n}) \to \Theta}{\Gamma, F(\bar{n}) \to \Theta}$$

$$\vdots$$

$$\vdots$$

level σ_2 $$\overline{{}_0, F(\bar{n}) \to \Theta_0}$$

level σ $$\frac{\to F(\bar{n}) \quad F(\bar{n}), \Gamma_0 \to \Theta_0}{\Gamma_0 \to \Theta_0}$$ Interchanges, if necessary

New Cut

$$\vdots$$

$$\overline{\quad \to \quad}$$ The endsequent

Above the left upper sequent of new cut we describe an EA-derivation of the sequent $\to F(\bar{n})$.

Let the ordinal number of the level line in the given derivation be α and that of the corresponding line in the new derivation be α_1. Clearly it holds that $\alpha_1 < \alpha$. Our purpose is to prove that the ordinal of new derivation is less than that of the given derivation.

Since $\alpha_1 < \alpha$, it follows that $\omega_\rho(\alpha_1) < \omega_\rho(\alpha)$. Now let γ be the ordinal of the left upper sequent $\to F(\bar{n})$ of the new cut. Then the ordinal of the right upper sequent of the new cut is $\omega_\rho(\alpha_1)$. Thus the ordinal of line of the new cut is $\gamma \# \omega_\rho(\alpha_1)$. Hence the ordinal of the lower sequent is $\omega_\tau(\gamma \# \omega_\rho(\alpha_1))$.

On the other hand, since the ordinal of the level line of the given derivation is α, the ordinal of the lower sequent of the level line is $\omega_{\tau+\rho}(\alpha)$.

Thus it suffices for our purpose to prove that $\omega_\tau(\gamma \# \omega_\rho(\alpha_1)) < \omega_{\tau+\rho}(\alpha)$. Hence it suffices to prove that $\gamma \# \omega_\rho(\alpha_1) < \omega_\rho(\alpha)$, because the ordinal function $\omega_\tau(\beta)$ is strictly monotonic increasing function.

2.552.1. The case when $\sigma_1 = \sigma + (\tau + \rho) < \omega$.

In this case it follows from 2.4, corollary 2, 2) that $\gamma < \varepsilon_{\varepsilon_0}$. From the correlation of the ordinal number of M-axiom, it follows that $\varepsilon_{\varepsilon_0} + 1 < \alpha$. Hence $\gamma < \alpha \leqq \omega_\rho(\alpha)$. Moreover it holds that $\omega_\rho(\alpha_1) < \omega_\rho(\alpha)$ and $\rho > 0$. Therefore it holds that $\gamma \# \omega_\rho(\alpha_1) < \omega_\rho(\alpha)$.

2.552.2. The case when $\sigma_2 = \sigma + \tau < \omega \leqq \sigma + \tau + \rho = \sigma_1$.

In this case it follows that $\rho \geqq \omega$. As before it holds that $\gamma < \varepsilon_{\varepsilon_0}$. From the correlation of the ordinal number of M-axiom, it follows that $\varepsilon_0 + 1 < \alpha$. Hence $\varepsilon_{\varepsilon_0+1} < \varepsilon_\alpha$. But $\rho \geqq \omega$, it follows that $\varepsilon_\alpha \leqq \omega_\rho(\alpha)$. Thus it holds that $\gamma \leqq \omega_\rho(\alpha)$. Therefore it holds that

$$\gamma \# \omega_\rho(\alpha_1) < \omega_\rho(\alpha).$$

2.552.3. The case when $\sigma_2 = \sigma + \tau \geqq \omega$.

In this case it follows from 2.4, corollary 2, 1) that $\gamma < \varepsilon_0$. Clearly it holds that $\varepsilon_0 + 1 < \alpha$. Thus it holds that $\gamma < \alpha$. Therefore it holds that $\gamma \# \omega_\rho(\alpha_1) < \omega_\rho(\alpha)$.

§3. Proof of Theorem 2

3.1. We use $\prec$ as primitive symbol of EN expressing 2-place decidable arithmetic predicate which corresponds an order relation of ordinals $< \Gamma_0$.

Let P be a unary predicator. Then we write

P^* for $\lambda x \forall y (y \prec x \supset P(y))$,

$Pr(P)$ for $\forall x (P^*(x) \supset P(x))$,

P' for $\lambda x \forall y (P^*(y) \supset P^*(y \hat{+} \hat{\omega}^x))$,

$I(P,a)$ for $Pr(P) \supset P^*(a)$,

$I(a)$ for $\forall X I(X,a)$.

As stated in Schütte [7], we obtain the following

LEMMA 3. *Let U be a unary free predicate variable. Then the followings hold:*

1) *The sequent* $Pr(U) \to Pr(U')$ *is* EN-*provable.*

2) *The sequent* $I(U,a) \to I(U,a \hat{+} 1)$ *is* EN-*provable.*

3) *The sequent* $I(U',a \hat{+} 1) \to I(U,\hat{\omega}^a)$ *is* EN-*provable.*

We use the notation $P^{(K)}$ for a unary predicator P and each natural number k.

We write
$P^{(0)}$ for P,
$P^{(k+1)}$ for $(P^{(k)})'$
Then we obtain the following

LEMMA 4. *Let P be a unary predicator. Then the sequent*

$$I(P^{(K)},\hat{\varepsilon}_0) \to I(P,\tilde{\omega}_{\bar{K}}(\hat{\varepsilon}_0 \hat{+} 1))$$

is EA-*provable for each natural number k.*

PROOF. From Lemma 3, 2) and 3), we can show that the sequent

$$I(P^{(k)},a) \to I(P,\tilde{\omega}_{\bar{k}}(a))$$

is EA-provable. Then from that sequent we see that

$$I(P^{(k)},a \hat{+} 1) \to I(P,\tilde{\omega}_{\bar{k}}(a \hat{+} 1))$$

is EA-provable. As the sequent

$$I(P^{(k)},a) \to I(P^{(k)}a \hat{+} 1)$$

is EA-provable, we see that

$$I(P^{(k)},a) \to I(P,\tilde{\omega}_{\bar{k}}(a \hat{+} 1))$$

is EA-provable.

LEMMA 5. *The formula* $Pr(\lambda x I(\hat{\varepsilon}_x))$ is EA-*provable.*

Using Lemma 3, 2), 3) and the fact that the following sequents are basic mathematical sequents, we have

LEMMA 5.

$$a \prec \hat{\varepsilon}_0 \to a \prec \tilde{\omega}_{f(a)}(0)$$

$$a \prec \hat{\varepsilon}_0 \to f(a) \prec \omega$$

$$a \prec \hat{\varepsilon}_b,\ 0 \prec b \to a \prec \tilde{\omega}_{g(a,b)}(\hat{\varepsilon}_{h(a,b)} + 1)$$

$$a \prec \hat{\varepsilon}_b,\ 0 \prec b \to g(a,b) \prec \omega$$

$$a \prec \hat{\varepsilon}_b, 0 \prec b \to h(a,b) \prec b$$

where f, g, h are function symbols for some calculable arithmetic functions.

3.2. Let T be the formal system obtained from EA by adding new axioms $\to I((\lambda x I(\hat{\varepsilon}_x))^{(k)}, \hat{\varepsilon}_0)$, whee k is any natural number.
From Gentzen's result [4], the formula $Pr(U) \supset (\bar{i} \prec \hat{\varepsilon}_0 \supset U(\bar{i}))$ is EN-provable for each natural number i, so T is a subsystem of EA(M). Then the following Proposition 3 holds.

PROPOSITION 3. *For each natural number k the transfinite induction*

$$I(\hat{\varepsilon}_{\tilde{\omega}_{\bar{k}_0}(\hat{\varepsilon}_0 \hat{+} 1)}) \textit{ is provable in } \mathrm{T}.$$

PROOF. $I((\lambda x I(\hat{\varepsilon}_x))^{(k)}, \hat{\varepsilon}_0)$ is T-provable, hence from Lemma 4 the formula $I(\lambda x I(\hat{\varepsilon}_x), \tilde{\omega}_{\bar{k}}(\hat{\varepsilon}_0 \hat{+} 1))$ is T-provable.

$$I(\hat{\varepsilon}_{\tilde{\omega}_{\bar{k}_0}(\hat{\varepsilon}_0 \hat{+} 1)}) \text{ is T-provable.}$$

By Gödel [5], translated in English by [1], we obtain the following

PROPOSITION 4. (Gödel's 2nd incompleteness theorem for T) *If* T *is consistent, consis*(T) *is not provable in* T.

3.3. Proof of Theorem 2

Suppose that ω-consistency of EA is proved by applying transfinite induction to numbers below $\varepsilon_{\varepsilon_1}$ for some elementary number theoretical propositions, together with exclusively elementary number theoretical techniques.

Then by formalizing this proof we can know that for some $\alpha_1, \cdots, \alpha_n < \varepsilon_{\varepsilon_1}$ and EN-formulas $F_1(a), \cdots, F_n(a)$ the sequent

$$\forall \bar{x}_1 I(\lambda x F_1(x), \alpha_i), \cdots, \forall \bar{x}_n I(\lambda x F_n(x), \alpha_n) \to \omega\text{-consis(EA)}$$

is EN-provable, where $\forall \bar{x}_i I(\lambda x F_i(x), \alpha_i)$ is the universal closure of the formula $I(\lambda x F_i(x), \alpha_i)$.

Then, since $F_i(a)$ is an EN-formula, the sequent $I(\alpha_i) \to \forall \bar{x}_i I(\lambda x F_i(x), \alpha_i)$ is EA-provable. From Proposition 3 the sequent $\to I(\alpha_i)$ is provable in T. Hence ω-consis(EA) is provable in T.

On the other hand T is a subsystem of EA(M), so by Corollary 1, the sequent ω-consis(EA) $\to$ consis(T) is EN-provable. Hence it follows that consis(T) is provable in T. Therefore by Proposition 4 we see that T is inconsistent. Hence EA(M) is inconsistent. From this is follows that EA is ω-inconsistent. This contradicts to our assumption.

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