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M_0-SPACES ARE μ-SPACES

By

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1. Introduction. The μ-spaces were introduced by K. Nagami [N]. A space X is said to be a μ-space if X is embedded in the countable product of F_σ-metrizable paracompact spaces. The class of M_3-μ-spaces is a harmonious class in dimension theory and is a subclass of hereditary M_1-spaces (see [M] and [T]). Especially every 0-dimensional M_3-μ-space has a σ-closure preserving clopen base. Heath and Junnila [HJ] called such a space an *M_0-space.* Then, what spaces are M_3-spaces to be μ-spaces? There was no result on this question yet. In this paper we shed some light on this question.

Throughout this paper all spaces are assumed to be regular T_1 and all maps are assumed to be continuous. The letter N denotes the positive integers.

2. Results.

THEOREM 2.1. *Let X be an M_3-space with a peripherally compact σ-closure preserving quasi-base. Then X is embedded in the countable product of F_σ-metrizable M_3-spaces and is therefore a μ-space.*

PROOF. Let $\mathcal{B}=\cup\{\mathcal{B}_n : n\in N\}$ be a peripherally compact σ-closure preserving closed quasi-base of X. Let $n\in N$. To construct a space M_n, let us fix n. Let $V(x)=X-\cup\{B\in\mathcal{B}_n : x\notin B\}$ and $\hat{x}=\{y\in X: V(x)=V(y)\}$. Then by [J, Theorem 4.8], there exists a σ-discrete closed refinement $H=\cup\{H_m : m\in N\}$ of $\{\hat{x} : x\in X\}$. By [O, Lemma 3.2], there exist a metrizable space Z and a one-to-one onto map $f: X\to Z$ such that every $f(H_m)$ is a discrete closed family and $f(\mathcal{B}_n)$ is a closure preserving closed family. For $B\in\mathcal{B}_n$, there exists a map $\Psi'_B: f(B)\to I$ such that $\Psi'^{-1}_B(0)=f(\partial B)$, because ∂B is compact, where ∂B denotes the boundary of B. Let $\Psi_B: X\to I$ such that

$$\Psi_B(x)=\Psi'_B\circ f(x) \text{ if } x\in B;\text{ and}$$
$$\Psi_B(x)=0 \text{ if } x\notin B.$$

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Then Ψ_B is continuous. We define

$$g_B : X \to Z \times I \text{ by } g_B(x) = (f(x), \Psi_B(x));$$
$$h_B : g_B(X) \to Z \text{ by } h_B(f(x), \Psi_B(x)) = f(x);$$
$$g_n : X \to \prod_{B \in \mathcal{B}_n} g_B(x) \text{ by } g_n(x) = (g_B(x));$$
$$M_n = g_n(x); \text{ and}$$
$$\pi_B : M_n \to g_B(X) \text{ by } \pi_B((x_B)) = x_B.$$

Note that $h_B|_{g_B(B)}$ and $h_B|_{g_B(X-B)}$ are homeomorphisms.

Now, we show that M_n is an F_σ-metrizable M_3-space. It is obvious that M_n is regular T_1. To prove that M_n is F_σ-metrizable, let $H \in \mathcal{H}$. Since $\mathcal{H}$ is a σ-discrete closed cover of X, it is enough to show that $f \circ g_n^{-1}|_{g_n(H)} : g_n(H) \to f(H)$ is homeomorphic. Obviously $f \circ g_n^{-1}|_{g_n(H)}$ is a continuous one-to-one onto map. Let $B \in \mathcal{B}_n$, U an open set of $g_B(H)$ and $W = \pi_B^{-1}(U)$. Note that $f \circ g_n^{-1}|_{g_n(H)}(W) = h_B|_{g_B(H)}(U)$. If $H \cap B = \phi$, then $f \circ g_n^{-1}|_{g_n(H)}(W)$ is open in $f(H)$. Because $h_B|_{g_B(X-B)}$ is a homeomorphism and $H \subset X - B$. Let $H \cap B \neq \phi$. There exists $x \in X$ such that $H \subset \hat{x}$. Then $\hat{x} \cap B \neq \phi$. From the definition of $\hat{x}$, $\hat{x} \subset B$. Hence $H \subset B$. Since $h_B|_{g_B(B)}$ is a homeomorphism, $f \circ g_n^{-1}|_{g_n(H)}(W)$ is open in $f(H)$. Therefore $f \circ g_n^{-1}|_{g_n(H)}$ is an open map and is a homeomorphism. To show that M_n is an M_3-space, let $\mathcal{W}$ be a σ-discrete closed quasi-base of Z. Then $g_n(\mathcal{B}_n) \cup g_n \circ f^{-1}(\mathcal{W})$ is a quasi-subbase of M_n, because $\{g_B(B)\} \cup g_B \circ f^{-1}(\mathcal{W})$ is a quasi-subbase of $g_B(X)$. Obviously $g_n(\mathcal{B}_n) \cup g_n \circ f^{-1}(\mathcal{W})$ is a σ-closure preserving closed family. Therefore M_n is an M_3-space.

Let $g : X \to \prod_{n \in N} M_n$ such that $g(x) = (g_n(x))$. Then g is clearly an embedding and the proof is completed.

COROLLARY 2.2. *Let X be an M_0-space. Then X is embedded in the countable product of F_σ-metrizable M_0-spaces and is therefore a μ-space.*

PROOF. In the above proof, replace I with $\{0, 1\} \subset I$, and Z with a 0-dimensional one (see [P, Theorem 2]).

COROLLARY 2.3. *Let X be a closed image of an F_σ-metrizable M_3-space with* $\dim X = 0$. *Then X is a μ-space.*

PROOF. Let $X = \cup \{X_n : n \in N\}$, where each X_n is a closed Lašnev subspace. Then each X_n has an M-structure, so by [M, Theorem 3.15], X has an M-structure. By [M, Theorem 2.1], X is an M_0-space. Therefore by the above corollary, X is a μ-space.

We do not know whether every perfect image of a μ-space is a μ-space. This problem has already been posed by K. Nagami [N]. Perhaps the following two problems are some approach to this problem in the class of M_3-spaces.

PROBLEM 2.4. *Is every closed image of an F_σ-metrizable M_3-space a μ-space?*

PROBLEM 2.5. *Is every M_3-μ-space embedded in the countable product of F_σ-metrizable M_3-spaces?*

COROLLARY 2.6. *Every M_3-μ-space is a perfect image of a 0-dimensional M_3-μ-space.*

PROOF. T. Mizokami [M] proved that every M_3-μ-space is a perfect image of an M_0-space. But by Corollary 2.2, every M_0-space is a μ-space.

An inner characterization of M_3-μ-spaces is not obtained yet. So many proofs on M_3-μ-spaces have returned to the definition and have been therefore complicated. But for 0-dimensional spaces, we have the following characterizations.

THEOREM 2.7. *For a 0-dimensional space X, the following statements are mutually equivalent.*

(1) *X is an M_3-μ-space.*

(2) *X is an M_0-space.*

(3) *X is an M_3-space with an M-structure (for the definition, see* [M]).

(4) *X is a regularly stratifiable space (for the definition, see* [T]).

(5) *X is a strongly regularly stratifiable space (for the definition, see* [T]).

PROOF. (1)$\Longrightarrow$(3) follows from [M, Theorem 4.5]. (3)$\Longrightarrow$(2) follows from [M, Theorem 2.1]. (2)$\Longrightarrow$(1) follows from Corollary 2.2. (1)$\Longrightarrow$(5) follows from [T, Theorem 5.4]. (5)$\Longrightarrow$(4) is trivial. (4)$\Longrightarrow$(2) follows from [T, Corollary 6.3].

PROBLEM 2.8. *Find an inner characterization of M_3-μ-spaces. Are M_3-spaces with an M-structure or regularly stratifiable spaces μ-spaces?*

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