

ON THE EQUATIONS DEFINING A PROJECTIVE CURVE EMBEDDED BY A NON-SPECIAL DIVISOR

By
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Introduction. Let C be a complete reduced irreducible curve of arithmetic genus g over an algebraically closed field K . Let L be a very ample invertible sheaf of degree d on C , and let $\phi_L: C \hookrightarrow \mathbf{P}^{h^0(L)-1}$ be the projective embedding by means of a basis of $\Gamma(L)$. Then the following results are known:

(A) Assume that C is smooth over K .

(0) (D. Mumford [5]) L is normally generated, if $d \geq 2g + 1$.

(1) (B. Saint-Donat [7]) The largest homogeneous ideal I defining $\phi_L(C)$, i.e., $I = \text{Ker}[S\Gamma(L) \rightarrow \bigoplus_{m \geq 0} \Gamma(L^m)]$, is generated by its elements of degree 2, if $d \geq 2g + 2$.

(2) (B. Saint-Donat [7]) I is generated by its elements of degree 2 and 3, if $d \geq 2g + 1$.

(B) (T. Fujita [1]) The statements (0) and (1) in (A) are true without the assumption that C is smooth over K .

The purposes of the present paper are that we improve the second result (2) of Saint-Donat and that we construct some related examples (corollary 1.4, Example 2.4 and Proposition 3.1).

Notation and Terminology. We fix an algebraically closed field K of characteristic $p \geq 0$ throughout the paper. We use the word “variety” to mean a reduced irreducible scheme of finite type and proper over K , and “curve” to mean a variety of dimension 1.

For a finite dimensional vector space V over K , $S^m V$ means the m -th symmetric power of V and SV means the symmetric algebra of V , i.e., $SV = \bigoplus_{m \geq 0} S^m V$.

Let L be an invertible sheaf on a projective variety X . We denote by L^m the m -th tensor product $L^{\otimes m}$. For the vector space of global sections $\Gamma(L)$, we define I and I_m ($m \geq 1$), by

$$I = I(L) = \text{Ker}[S\Gamma(L) \rightarrow \bigoplus_{m \geq 0} \Gamma(L^m)],$$

and

$$I_m = I_m(L) = \text{Ker}[S^m \Gamma(L) \rightarrow \Gamma(L^m)].$$

Let L_1, \dots, L_m be invertible sheaves on X . Then $\mathcal{R}(L_1, \dots, L_m)$ means the kernel of the natural map:

$$\Gamma(L_1) \otimes \dots \otimes \Gamma(L_m) \rightarrow \Gamma(L_1 \otimes \dots \otimes L_m).$$

§1. Generality.

Let X be a projective variety, and let L be an ample invertible sheaf on X . If the canonical map $\Gamma(L)^{\otimes m} \rightarrow \Gamma(L^m)$ is surjective for all positive integers m , then L is called a normally generated ample invertible sheaf.

We will establish a criterion for surjectivity of the natural map $I_m(L) \otimes \Gamma(L) \rightarrow I_{m+1}(L)$ for a normally generated ample invertible sheaf L .

LEMMA 1.1. *Let V be a finite dimensional vector space, and let r be a positive integer greater than 1. Then we have*

$$\text{Ker}[V^{\otimes(r+1)} \rightarrow S^{r+1}V] = \text{Ker}[V^{\otimes r} \rightarrow S^r V] \otimes V + V \otimes \text{Ker}[V^{\otimes r} \rightarrow S^r V].$$

A proof of the lemma is easy, so we omit its proof.

PROPOSITION 1.2. *Let L be a normally generated ample invertible sheaf on a variety X . If m is a positive integer greater than 1, then the following conditions are equivalent:*

- (1) $\Gamma(L) \otimes \mathcal{R}(L^{m-1}, L) \xrightarrow{\xi} \mathcal{R}(L^m, L)$ is surjective,
- (2) $\mathcal{R}(\overbrace{L, \dots, L}^{m+1}) = \mathcal{R}(\overbrace{L, \dots, L}^m) \otimes \Gamma(L) + \Gamma(L) \otimes \mathcal{R}(\overbrace{L, \dots, L}^m)$,
- (3) $I_m(L) \otimes \Gamma(L) \xrightarrow{\alpha} I_{m+1}(L)$ is surjective.

PROOF^(*). We consider the following exact diagram

$$\begin{array}{ccccccc} 0 & \rightarrow & \mathcal{R}(\overbrace{L, \dots, L}^m) \otimes \Gamma(L) & \xrightarrow{\alpha} & \mathcal{R}(\overbrace{L, \dots, L}^{m+1}) & \longrightarrow & \mathcal{R}(L^m, L) \rightarrow 0 \\ & & & & \uparrow \xi' & & \uparrow \xi \\ & & \Gamma(L) \otimes \mathcal{R}(\overbrace{L, \dots, L}^m) & \rightarrow & \Gamma(L) \otimes \mathcal{R}(L^{m-1}, L) & \rightarrow & 0 \\ & & \uparrow & & \uparrow & & \\ & & 0 & & 0 & & \end{array}$$

It is easy to check that $\mathcal{R}(\overbrace{L, \dots, L}^{m+1}) = \text{Im}(\alpha) + \text{Im}(\xi')$ if and only if ξ is surjective.

Next, we will prove the equivalence (2) \Leftrightarrow (3). Note that the canonical map $\pi_r: \mathcal{R}(\overbrace{L, \dots, L}^r) \rightarrow I_r(L)$ is surjective for any integers $r \geq 2$. For a given $f \in I_{m+1}(L)$,

(*) The proof of the first part (1) \Leftrightarrow (2), has been fairly simplified by an idea of Dr. Sekiguchi.

We can find $s \in \mathcal{R}(\overline{L, \dots, L}^{m+1})$ such that $\pi_{m+1}(s) = f$. By (2), we have $s = \sum_i \beta_i \otimes s_i + \sum_j t_j \otimes \gamma_j$ for suitable elements $\beta_i, \gamma_j \in \mathcal{R}(\overline{L, \dots, L}^m)$ and $s_i, t_j \in I(L)$, so we have $f = q(\sum_i \pi_m(\beta_i) \otimes s_i + \sum_j \pi_m(\gamma_j) \otimes t_j)$. Hence (2) implies (3). To prove the implication (3) \Rightarrow (2), it suffices to show the inclusion relation

$$\mathcal{R}(\overline{L, \dots, L}^{m+1}) \subset \mathcal{R}(\overline{L, \dots, L}^m) \otimes I(L) + I(L) \otimes \mathcal{R}(\overline{L, \dots, L}^m).$$

Let s be an element of $\mathcal{R}(\overline{L, \dots, L}^{m+1})$. Then by (3), there exist $t_j \in \mathcal{R}(\overline{L, \dots, L}^m)$ and $s_j \in I(L)$ such that $\pi_{m+1}(s) = q(\sum_j \pi_m(t_j) \otimes s_j)$.

Hence

$$s - \sum_j t_j \otimes s_j \in \text{Ker}(I(L)^{\otimes(m+1)} \rightarrow S^{m+1}I(L)).$$

Since by Lemma 1.1,

$$\begin{aligned} & \text{Ker}(I(L)^{\otimes(m+1)} \rightarrow S^{m+1}I(L)) \\ &= \text{Ker}(I(L)^{\otimes m} \rightarrow S^m I(L)) \otimes I(L) + I(L) \otimes \text{Ker}(I(L)^{\otimes m} \rightarrow S^m I(L)) \\ &\subset \mathcal{R}(\overline{L, \dots, L}^m) \otimes I(L) + I(L) \otimes \mathcal{R}(\overline{L, \dots, L}^m), \end{aligned}$$

so we have

$$s \in \mathcal{R}(\overline{L, \dots, L}^m) \otimes I(L) + I(L) \otimes \mathcal{R}(\overline{L, \dots, L}^m). \quad \text{Q.E.D.}$$

COROLLARY 1.3. *Let L be a normally generated ample invertible sheaf on an n -dimensional variety X . Assume that $H^i(X, L^j) = (0)$ for any integers $i, j \geq 1$. Then the homogeneous ideal $I(L)$ is generated by I_2, \dots, I_{n+3} .*

PROOF. By Proposition 1.2, it suffices to prove that the natural map $\Gamma(L) \otimes \mathcal{R}(L^{m-1}, L) \rightarrow \mathcal{R}(L^m, L)$ is surjective for any integer $m \geq n+3$. It is just the theorem of Mumford [5, Theorem 5].

COROLLARY 1.4. *Let L be a normally generated ample invertible sheaf on a curve C . Assume that $H^1(C, L) = (0)$. Then $I(L)$ is generated by I_2 and I_3 .*

PROOF. By Proposition 1.2 and Corollary 1.3, it suffices to show that the natural map $\Gamma(L) \otimes \mathcal{R}(L^2, L) \rightarrow \mathcal{R}(L^3, L)$ is surjective. It is a direct consequence of the following lemma.

LEMMA 1.5. (T. Fujita [1, Lemma 1.8]) *Let L, M and N be invertible sheaves on a curve C . Assume that $H^1(C, M \otimes L^{-1}) = (0)$ and that $\Gamma(L)$ is base point free and that the natural map $\Gamma(M \otimes L^{-1}) \otimes \Gamma(N) \rightarrow \Gamma(M \otimes N \otimes L^{-1})$ is surjective. Then the natural map $\Gamma(L) \otimes \mathcal{R}(M, N) \rightarrow \mathcal{R}(L \otimes M, N)$ is surjective.*

REMARK 1.6. Let L be an invertible sheaf of degree d on a curve C . If $d \geq 2g+1$, then L is a normally generated ample invertible sheaf with $H^i(C, L) = 0$. Therefore by Corollary 1.4, $I(L)$ is generated by I_2 and I_3 ,

This is another proof of the second result of Saint-Donat (c.f Introduction (2)).

§2. Example I.

In this section we use the word "curve" to mean a smooth curve over K . We assume that the characteristic of the ground field K is not 2. The purpose of this section is to show that the first result of Saint-Donat (see Introduction (1)) is the best possible for each genus $g \geq 1$, namely, there exists a curve C of genus g with invertible sheaf L on C of degree $2g+1$ such that the homogeneous ideal $I(L)$ is not generated by $I_2(L)$.

REMARK 2.1. Let C be a curve of genus 1 or 2, and let L be an invertible sheaf of degree $2g+1$ on C . Then the homogeneous ideal $I(L)$ is not generated by $I_2(L)$.

Indeed, C is embedded by $\Gamma(L)$ to \mathbf{P}^2 if the genus is 1 (resp. to \mathbf{P}^3 if the genus is 2), but the dimension of $I_2(L)$ is 0 if the genus is 1 (resp. is 1 if the genus is 2).

From now on, we fix a hyper-elliptic curve C of genus $g \geq 3$. Let $K(C)$ be the function field of C . Since the characteristic of the ground field K is not 2, there exist functions $x, y \in K(C)$ such that $K(C) = K(x, y)$ with a relation

$$y^2 = (x-a_1) \cdot (x-a_2) \cdots (x-a_{2g+1}).$$

Let P_∞ be the closed point on C such that $x(P_\infty) = \infty$, and let $L = \mathcal{O}_C((2g+1)P_\infty)$. For any divisor D on C , we regard $\mathcal{O}_C(D)$ as a subsheaf of $K(C)$ in the canonical way. Then we have that the $g+2$ functions

$$\{1, x, \dots, x^g, y\}$$

forms a basis of $\Gamma(L)$ and that the $\frac{1}{2}(g+2)(g+3)$ elements

$$\left\{ \begin{array}{cccccc} 1 \odot 1 & & & & & \\ 1 \odot x & x \odot x & & & & \\ 1 \odot x^2 & x \odot x^2 & x^2 \odot x^2 & & & \\ \vdots & \vdots & \vdots & & & \\ 1 \odot x^g & x \odot x^g & x^2 \odot x^g & \cdots & x^g \odot x^g & \\ 1 \odot y & x \odot y & x^2 \odot y & \cdots & x^g \odot y & y \odot y \end{array} \right\}$$

forms a basis of $S^2\Gamma(L)$, where the symbol \odot means a symmetric product.

PROPOSITION 2.2. *The vector space*

$$I_2(L) = \text{Ker}[S^2\Gamma(L) \rightarrow \Gamma(L^2)]$$

is generated by $\{x^i \odot x^j - x^{i-1} \odot x^{j+1} \mid 1 \leq i \leq j \leq g-1\}$ over K .

PROOF. It is easy to show that the above set is included in I_2 . Let V be a subspace of I_2 generated by the above set, and let W be a subspace of $S^2\Gamma(L)$ generated by the following elements:

$$\left\{ \begin{array}{cccc} 1 \odot 1 & 1 \odot x, & \cdots, & 1 \odot x^g \\ x \odot x^g & x^2 \odot x^g, & \cdots, & x^g \odot x^g \\ 1 \odot y & x \odot y, & \cdots, & x^g \odot y \\ y \odot y \end{array} \right\}.$$

Then the natural map $W \rightarrow S^2\Gamma(L)/V$ is surjective. Indeed, if $i \leq g-j$, then

$$x^i \odot x^j \equiv x^{i-1} \odot x^{j+1} \equiv \cdots \equiv 1 \odot x^{j+i} \pmod{V},$$

and if $i > g-j$, then

$$x^i \odot x^j \equiv x^{i+1} \odot x^{j-1} \equiv \cdots \equiv x^{i+j-g} \odot x^g \pmod{V}.$$

Hence we have $\dim[S^2\Gamma(L)] - \dim(V) \leq \dim(W)$, so we have

$$\dim(V) \geq \frac{1}{2}g(g-1) = \dim(I_2).$$

Since $I_2 \supset V$, we have $I_2 = V$. Q.E.D.

COROLLARY 2.3. *Let $\{X_0, X_1, \dots, X_g, Y\}$ be a homogeneous coordinate of the projective space \mathbf{P}^{g+1} corresponding to a basis $\{1, x, \dots, x^g, y\}$ of $\Gamma(L)$. Then the vector space of quadrics vanishing on $\phi_L(C)$ is generated by the quadrics*

$$\{X_i X_j - X_{i-1} X_{j+1} \mid 1 \leq i \leq j \leq g-1\}$$

over K .

EXAMPLE 2.4. Let (C, L) be the above curve with invertible sheaf. Then the degree L is $2g+1$, but the homogeneous ideal $I(L)$ is not generated by $I_2(L)$.

In fact, if the homogeneous ideal $I(L)$ is generated by I_2 , then

$$\phi_L(C) = \bigcap_{1 \leq i \leq j \leq g-1} V(X_i X_j - X_{i-1} X_{j+1})$$

by Corollary 2.3, where $V(X_i X_j - X_{i-1} X_{j+1})$ is the set of zeros of $X_i X_j - X_{i-1} X_{j+1}$ in \mathbf{P}^{g+1} . Let H be the linear subvariety of \mathbf{P}^{g+1} defined by the equations:

$$X_0 - X_1, X_1 - X_2, \dots, X_{g-1} - X_g.$$

Then $H \cong \mathbf{P}^1$, and

$$H \subset \bigcap_{1 \leq i \leq j \leq g-1} V(X_i X_j - X_{i-1} X_{j+1}).$$

Hence we have $H = \phi_L(C)$, because $H \subset \phi_L(C)$ and $\phi_L(C)$ is irreducible. This contradicts $g \geq 1$.

§ 3. Example II.

We continue assuming that the characteristic of the ground field K is not 2 and that a “curve” means a smooth curve over K .

In this section we will show that there are many examples of curves of genus g with invertible sheaf of degree $2g$ on which Corollary 1.4 works effectively. Note that since the degree of L is $2g$, the condition $H^1(C, L) = (0)$ in Corollary 1.4 is automatically satisfied. Therefore our problem is reduced to constructing many curves of genus g which have a normally generated ample invertible sheaf of degree $2g$.

PROPOSITION 3.1. *Let C be a curve of genus $g \geq 5$. Suppose that there exists an invertible sheaf M of degree $g-1$ on C such that $\Gamma(M)$ is a base point free pencil. Then almost all invertible sheaves of degree $2g$ on C are ample with normal generation.*

The following lemma, B. Saint-Donat [8] called it “base point free pencil trick”, plays an important role in the proof of our proposition.

LEMMA 3.2. (Mumford [5, p. 57], Saint-Donat [8, Lemma 2.6]) *Let M and N be invertible sheaves on a curve. Suppose that $\Gamma(M)$ is a base point free pencil. Then we have an isomorphism*

$$\text{Ker}[\Gamma(M) \otimes \Gamma(N) \rightarrow \Gamma(M \otimes N)] \cong \Gamma(N \otimes M^{-1}).$$

We will use the following notation.

$\text{Pic}^d(C)$: the connected component of the Picard scheme of C whose member represents an invertible sheaf of degree d ,

G_d^r : the closed subvariety of $\text{Pic}^d(C)$ representing the set of invertible sheaves of degree d and of projective dimension $\geq r$,

F_d^r : the closed subvariety of $\text{Pic}^d(C)$ defined by the image of the morphism

$$G_{d-1}^r \times C \ni (L, P) \longrightarrow L(P) \in \text{Pic}^d(C)$$

(if $G_{d-1}^r = \emptyset$, then F_d^r means the void subset).

Note that $F_d^r \subset G_d^r$ and that if $r \geq 1$, the set $G_d^r - F_d^r$ represents the set of in-

vertible sheaves free from base points, of degree d and of proejective dimension r .

PROOF OF PROPOSITION 3.1. There exists an invertible sheaf M_0 of degree $g-1$ such that $\Gamma(M_0)$ is a base point free pencil and $M_0^2 \neq \omega$, where ω is the canonical sheaf on C . Indeed, since $G_{g-1}^1 - F_{g-1}^1$ is non-empty open in G_{g-1}^1 by our assumption and since

$$\dim G_{g-1}^1 \geq g-4 \geq 1 \quad [4, \text{Theorem 1}],$$

$G_{g-1}^1 - F_{g-1}^1$ has infinitely many elements. So there exists such an invertible sheaf. We put

$$\begin{aligned} V &= G_{g+1}^1 - F_{g+1}^1 = \text{Pic}^{g+1}(C) - F_{g+1}^1, \text{ and} \\ U &= \{N \otimes M_0 \mid N \in V\} \subset \text{Pic}^{2g}(C). \end{aligned}$$

Obviously, V is non-empty open in $\text{Pic}^{g+1}(C)$. Hence U is non-empty open in $\text{Pic}^{2g}(C)$. We will show that any invertible sheaf in U is ample with normal generation. Let L be an invertible sheaf in U . By the generalized lemma of Castelnuovo [5, Theorem 2], we have natural map $\Gamma(L^m) \otimes \Gamma(L) \rightarrow \Gamma(L^{m+1})$ is surjective for $m \geq 2$. Therefore it suffices to show that the natural map $\Gamma(L) \otimes \Gamma(L) \rightarrow \Gamma(L^2)$ is surjective. Consider the commutative diagram

$$\begin{array}{ccc} \Gamma(L \otimes M_0^{-1}) \otimes \Gamma(M_0) \otimes \Gamma(L) & \xrightarrow{1 \otimes \phi_1} & \Gamma(L \otimes M_0^{-1}) \otimes \Gamma(M_0 \otimes L) \\ \downarrow & & \downarrow \phi_2 \\ \Gamma(L) \otimes \Gamma(L) & \longrightarrow & \Gamma(L^2), \end{array}$$

where ϕ_1 is the natural map $\Gamma(M_0) \otimes \Gamma(L) \rightarrow \Gamma(M_0 \otimes L)$. By Lemma 3.2, we have $\text{Ker } \phi_1 \cong \Gamma(L \otimes M_0^{-1})$ and $\text{Ker } \phi_2 \cong \Gamma(M_0^2)$. Therefore we have

$$\begin{aligned} \dim(\text{Ker } \phi_1) &= \dim[\Gamma(L \otimes M_0^{-1})] = 2, \\ \dim[\Gamma(M_0) \otimes \Gamma(L)] &= 2(g+1), \\ \dim[\Gamma(M_0 \otimes L)] &= 2g, \\ \dim(\text{Ker } \phi_2) &= \dim[\Gamma(M_0^2)] = g-1 \quad (\text{Note that } M_0^2 \neq \omega), \\ \dim[\Gamma(L \otimes M_0^{-1}) \otimes \Gamma(M_0 \otimes L)] &= 4g \quad \text{and} \\ \dim[\Gamma(L^2)] &= 3g+1. \end{aligned}$$

Hence ϕ_1 and ϕ_2 are surjective, and hence the natural map $\Gamma(L) \otimes \Gamma(L) \rightarrow \Gamma(L^2)$ is surjective. Q.E.D.

Next, we will give a sufficient condition for a curve to have an invertible sheaf M of degree $g-1$ such that $\Gamma(M)$ is a base point free pencil. Our result on it is a direct consequence of the following theorem of Martens and Mumford.

THEOREM OF MARTENS AND MUMFORD [6, Appendix]. *Let C be a curve of genus $g \geq 5$. Then there exists integer d , $3 \leq d \leq g-2$, such that $\dim G_d^1 \geq d-3$ if*

and only if C is hyperelliptic, or trigonal, or double covering of an elliptic curve ($g \geq 6$), or non-singular plane quintic.

PROPOSITION 3.3. *Let C be a curve of genus $g \geq 5$ neither hyperelliptic, nor trigonal, nor double covering of an elliptic curve ($g \geq 6$), nor non-singular plane quintic. Then there exists an invertible sheaf M of degree $g-1$ on C such that $\Gamma(M)$ is a base point free pencil.*

PROOF. We must prove that $G_{g-1}^1 - F_{g-1}^1 \neq \phi$ in our case. For this, it suffices to show that $\dim G_{g-1}^1 > \dim F_{g-1}^1$. By the results of Martens, Kleiman and Laksov [4, Theorem 1 and 3, Theorem 5], we have

$$\begin{aligned} g-3 &\geq \dim G_{g-1}^1 \geq g-4, \text{ and} \\ g-4 &\geq \dim G_{g-2}^1 \geq g-6. \end{aligned}$$

Note that if $G_{g-2}^1 \neq \phi$, then

$$\dim F_{g-1}^1 = \dim G_{g-2}^1 + 1 \quad [4, \text{p. 115}]$$

and that if $G_{g-2}^1 = \phi$, then $F_{g-1}^1 = \phi$. Suppose that $\dim G_{g-1}^1 = \dim F_{g-1}^1$. Then $\dim G_{g-2}^1 \geq g-5$. This contradicts the theorem of Martens and Mumford. Q.E.D.

Finally, we state an elementary remark relative to our topic.

REMARK 3.4. If C is a curve of genus $g \geq 4$, then there exists a non-special very ample invertible sheaf on C which is not normally generated.

Indeed, for a non-special normally generated ample invertible sheaf L , we have

$$\deg L \geq g + \frac{1}{2} + \sqrt{2g + \frac{1}{4}}$$

because $\dim S^2 \Gamma(L) \geq \dim \Gamma(L^2)$. On the other hand, by the theorem of Halphen [2, Theorem 1.2], there exists a non-special very ample invertible sheaf of degree d , if $d \geq g+3$.

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