

## ON THE EQUATIONS DEFINING A PROJECTIVE CURVE EMBEDDED BY A NON-SPECIAL DIVISOR

By  
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**Introduction.** Let  $C$  be a complete reduced irreducible curve of arithmetic genus  $g$  over an algebraically closed field  $K$ . Let  $L$  be a very ample invertible sheaf of degree  $d$  on  $C$ , and let  $\phi_L: C \hookrightarrow \mathbf{P}^{h^0(L)-1}$  be the projective embedding by means of a basis of  $\Gamma(L)$ . Then the following results are known:

(A) Assume that  $C$  is smooth over  $K$ .

(0) (D. Mumford [5])  $L$  is normally generated, if  $d \geq 2g + 1$ .

(1) (B. Saint-Donat [7]) The largest homogeneous ideal  $I$  defining  $\phi_L(C)$ , i.e.,  $I = \text{Ker}[S\Gamma(L) \rightarrow \bigoplus_{m \geq 0} \Gamma(L^m)]$ , is generated by its elements of degree 2, if  $d \geq 2g + 2$ .

(2) (B. Saint-Donat [7])  $I$  is generated by its elements of degree 2 and 3, if  $d \geq 2g + 1$ .

(B) (T. Fujita [1]) The statements (0) and (1) in (A) are true without the assumption that  $C$  is smooth over  $K$ .

The purposes of the present paper are that we improve the second result (2) of Saint-Donat and that we construct some related examples (corollary 1.4, Example 2.4 and Proposition 3.1).

**Notation and Terminology.** We fix an algebraically closed field  $K$  of characteristic  $p \geq 0$  throughout the paper. We use the word “variety” to mean a reduced irreducible scheme of finite type and proper over  $K$ , and “curve” to mean a variety of dimension 1.

For a finite dimensional vector space  $V$  over  $K$ ,  $S^m V$  means the  $m$ -th symmetric power of  $V$  and  $SV$  means the symmetric algebra of  $V$ , i.e.,  $SV = \bigoplus_{m \geq 0} S^m V$ .

Let  $L$  be an invertible sheaf on a projective variety  $X$ . We denote by  $L^m$  the  $m$ -th tensor product  $L^{\otimes m}$ . For the vector space of global sections  $\Gamma(L)$ , we define  $I$  and  $I_m$  ( $m \geq 1$ ), by

$$I = I(L) = \text{Ker}[S\Gamma(L) \rightarrow \bigoplus_{m \geq 0} \Gamma(L^m)],$$

and

$$I_m = I_m(L) = \text{Ker}[S^m \Gamma(L) \rightarrow \Gamma(L^m)].$$

Let  $L_1, \dots, L_m$  be invertible sheaves on  $X$ . Then  $\mathcal{R}(L_1, \dots, L_m)$  means the kernel of the natural map:

$$\Gamma(L_1) \otimes \dots \otimes \Gamma(L_m) \rightarrow \Gamma(L_1 \otimes \dots \otimes L_m).$$

### §1. Generality.

Let  $X$  be a projective variety, and let  $L$  be an ample invertible sheaf on  $X$ . If the canonical map  $\Gamma(L)^{\otimes m} \rightarrow \Gamma(L^m)$  is surjective for all positive integers  $m$ , then  $L$  is called a normally generated ample invertible sheaf.

We will establish a criterion for surjectivity of the natural map  $I_m(L) \otimes \Gamma(L) \rightarrow I_{m+1}(L)$  for a normally generated ample invertible sheaf  $L$ .

LEMMA 1.1. *Let  $V$  be a finite dimensional vector space, and let  $r$  be a positive integer greater than 1. Then we have*

$$\text{Ker}[V^{\otimes(r+1)} \rightarrow S^{r+1}V] = \text{Ker}[V^{\otimes r} \rightarrow S^r V] \otimes V + V \otimes \text{Ker}[V^{\otimes r} \rightarrow S^r V].$$

A proof of the lemma is easy, so we omit its proof.

PROPOSITION 1.2. *Let  $L$  be a normally generated ample invertible sheaf on a variety  $X$ . If  $m$  is a positive integer greater than 1, then the following conditions are equivalent:*

- (1)  $\Gamma(L) \otimes \mathcal{R}(L^{m-1}, L) \xrightarrow{\xi} \mathcal{R}(L^m, L)$  is surjective,
- (2)  $\mathcal{R}(\overbrace{L, \dots, L}^{m+1}) = \mathcal{R}(\overbrace{L, \dots, L}^m) \otimes \Gamma(L) + \Gamma(L) \otimes \mathcal{R}(\overbrace{L, \dots, L}^m)$ ,
- (3)  $I_m(L) \otimes \Gamma(L) \xrightarrow{\alpha} I_{m+1}(L)$  is surjective.

PROOF<sup>(\*)</sup>. We consider the following exact diagram

$$\begin{array}{ccccccc} 0 & \rightarrow & \mathcal{R}(\overbrace{L, \dots, L}^m) \otimes \Gamma(L) & \xrightarrow{\alpha} & \mathcal{R}(\overbrace{L, \dots, L}^{m+1}) & \longrightarrow & \mathcal{R}(L^m, L) \rightarrow 0 \\ & & & & \uparrow \xi' & & \uparrow \xi \\ & & \Gamma(L) \otimes \mathcal{R}(\overbrace{L, \dots, L}^m) & \rightarrow & \Gamma(L) \otimes \mathcal{R}(L^{m-1}, L) & \rightarrow & 0 \\ & & \uparrow & & \uparrow & & \\ & & 0 & & & & \end{array}$$

It is easy to check that  $\mathcal{R}(\overbrace{L, \dots, L}^{m+1}) = \text{Im}(\alpha) + \text{Im}(\xi')$  if and only if  $\xi$  is surjective.

Next, we will prove the equivalence (2)  $\Leftrightarrow$  (3). Note that the canonical map  $\pi_r: \mathcal{R}(\overbrace{L, \dots, L}^r) \rightarrow I_r(L)$  is surjective for any integers  $r \geq 2$ . For a given  $f \in I_{m+1}(L)$ ,

(\*) The proof of the first part (1)  $\Leftrightarrow$  (2), has been fairly simplified by an idea of Dr. Sekiguchi.

We can find  $s \in \mathcal{R}(\overline{L, \dots, L}^{m+1})$  such that  $\pi_{m+1}(s) = f$ . By (2), we have  $s = \sum_i \beta_i \otimes s_i + \sum_j t_j \otimes \gamma_j$  for suitable elements  $\beta_i, \gamma_j \in \mathcal{R}(\overline{L, \dots, L}^m)$  and  $s_i, t_j \in I(L)$ , so we have  $f = q(\sum_i \pi_m(\beta_i) \otimes s_i + \sum_j \pi_m(\gamma_j) \otimes t_j)$ . Hence (2) implies (3). To prove the implication (3)  $\Rightarrow$  (2), it suffices to show the inclusion relation

$$\mathcal{R}(\overline{L, \dots, L}^{m+1}) \subset \mathcal{R}(\overline{L, \dots, L}^m) \otimes I(L) + I(L) \otimes \mathcal{R}(\overline{L, \dots, L}^m).$$

Let  $s$  be an element of  $\mathcal{R}(\overline{L, \dots, L}^{m+1})$ . Then by (3), there exist  $t_j \in \mathcal{R}(\overline{L, \dots, L}^m)$  and  $s_j \in I(L)$  such that  $\pi_{m+1}(s) = q(\sum_j \pi_m(t_j) \otimes s_j)$ .

Hence

$$s - \sum_j t_j \otimes s_j \in \text{Ker}(I(L)^{\otimes(m+1)} \rightarrow S^{m+1}I(L)).$$

Since by Lemma 1.1,

$$\begin{aligned} & \text{Ker}(I(L)^{\otimes(m+1)} \rightarrow S^{m+1}I(L)) \\ &= \text{Ker}(I(L)^{\otimes m} \rightarrow S^m I(L)) \otimes I(L) + I(L) \otimes \text{Ker}(I(L)^{\otimes m} \rightarrow S^m I(L)) \\ &\subset \mathcal{R}(\overline{L, \dots, L}^m) \otimes I(L) + I(L) \otimes \mathcal{R}(\overline{L, \dots, L}^m), \end{aligned}$$

so we have

$$s \in \mathcal{R}(\overline{L, \dots, L}^m) \otimes I(L) + I(L) \otimes \mathcal{R}(\overline{L, \dots, L}^m). \quad \text{Q.E.D.}$$

**COROLLARY 1.3.** *Let  $L$  be a normally generated ample invertible sheaf on an  $n$ -dimensional variety  $X$ . Assume that  $H^i(X, L^j) = (0)$  for any integers  $i, j \geq 1$ . Then the homogeneous ideal  $I(L)$  is generated by  $I_2, \dots, I_{n+3}$ .*

**PROOF.** By Proposition 1.2, it suffices to prove that the natural map  $\Gamma(L) \otimes \mathcal{R}(L^{m-1}, L) \rightarrow \mathcal{R}(L^m, L)$  is surjective for any integer  $m \geq n+3$ . It is just the theorem of Mumford [5, Theorem 5].

**COROLLARY 1.4.** *Let  $L$  be a normally generated ample invertible sheaf on a curve  $C$ . Assume that  $H^1(C, L) = (0)$ . Then  $I(L)$  is generated by  $I_2$  and  $I_3$ .*

**PROOF.** By Proposition 1.2 and Corollary 1.3, it suffices to show that the natural map  $\Gamma(L) \otimes \mathcal{R}(L^2, L) \rightarrow \mathcal{R}(L^3, L)$  is surjective. It is a direct consequence of the following lemma.

**LEMMA 1.5.** (T. Fujita [1, Lemma 1.8]) *Let  $L, M$  and  $N$  be invertible sheaves on a curve  $C$ . Assume that  $H^1(C, M \otimes L^{-1}) = (0)$  and that  $\Gamma(L)$  is base point free and that the natural map  $\Gamma(M \otimes L^{-1}) \otimes \Gamma(N) \rightarrow \Gamma(M \otimes N \otimes L^{-1})$  is surjective. Then the natural map  $\Gamma(L) \otimes \mathcal{R}(M, N) \rightarrow \mathcal{R}(L \otimes M, N)$  is surjective.*

REMARK 1.6. Let  $L$  be an invertible sheaf of degree  $d$  on a curve  $C$ . If  $d \geq 2g+1$ , then  $L$  is a normally generated ample invertible sheaf with  $H^i(C, L) = 0$ . Therefore by Corollary 1.4,  $I(L)$  is generated by  $I_2$  and  $I_3$ ,

This is another proof of the second result of Saint-Donat (c.f Introduction (2)).

## §2. Example I.

In this section we use the word "curve" to mean a smooth curve over  $K$ . We assume that the characteristic of the ground field  $K$  is not 2. The purpose of this section is to show that the first result of Saint-Donat (see Introduction (1)) is the best possible for each genus  $g \geq 1$ , namely, there exists a curve  $C$  of genus  $g$  with invertible sheaf  $L$  on  $C$  of degree  $2g+1$  such that the homogeneous ideal  $I(L)$  is not generated by  $I_2(L)$ .

REMARK 2.1. Let  $C$  be a curve of genus 1 or 2, and let  $L$  be an invertible sheaf of degree  $2g+1$  on  $C$ . Then the homogeneous ideal  $I(L)$  is not generated by  $I_2(L)$ .

Indeed,  $C$  is embedded by  $\Gamma(L)$  to  $\mathbf{P}^2$  if the genus is 1 (resp. to  $\mathbf{P}^3$  if the genus is 2), but the dimension of  $I_2(L)$  is 0 if the genus is 1 (resp. is 1 if the genus is 2).

From now on, we fix a hyper-elliptic curve  $C$  of genus  $g \geq 3$ . Let  $K(C)$  be the function field of  $C$ . Since the characteristic of the ground field  $K$  is not 2, there exist functions  $x, y \in K(C)$  such that  $K(C) = K(x, y)$  with a relation

$$y^2 = (x-a_1) \cdot (x-a_2) \cdots (x-a_{2g+1}).$$

Let  $P_\infty$  be the closed point on  $C$  such that  $x(P_\infty) = \infty$ , and let  $L = \mathcal{O}_C((2g+1)P_\infty)$ . For any divisor  $D$  on  $C$ , we regard  $\mathcal{O}_C(D)$  as a subsheaf of  $K(C)$  in the canonical way. Then we have that the  $g+2$  functions

$$\{1, x, \dots, x^g, y\}$$

forms a basis of  $\Gamma(L)$  and that the  $\frac{1}{2}(g+2)(g+3)$  elements

$$\left\{ \begin{array}{cccccc} 1 \odot 1 & & & & & \\ 1 \odot x & x \odot x & & & & \\ 1 \odot x^2 & x \odot x^2 & x^2 \odot x^2 & & & \\ \vdots & \vdots & \vdots & & & \\ 1 \odot x^g & x \odot x^g & x^2 \odot x^g & \cdots & x^g \odot x^g & \\ 1 \odot y & x \odot y & x^2 \odot y & \cdots & x^g \odot y & y \odot y \end{array} \right\}$$

forms a basis of  $S^2\Gamma(L)$ , where the symbol  $\odot$  means a symmetric product.

PROPOSITION 2.2. *The vector space*

$$I_2(L) = \text{Ker}[S^2\Gamma(L) \rightarrow \Gamma(L^2)]$$

is generated by  $\{x^i \odot x^j - x^{i-1} \odot x^{j+1} \mid 1 \leq i \leq j \leq g-1\}$  over  $K$ .

PROOF. It is easy to show that the above set is included in  $I_2$ . Let  $V$  be a subspace of  $I_2$  generated by the above set, and let  $W$  be a subspace of  $S^2\Gamma(L)$  generated by the following elements:

$$\left\{ \begin{array}{cccc} 1 \odot 1 & 1 \odot x, & \dots, & 1 \odot x^g \\ x \odot x^g & x^2 \odot x^g, & \dots, & x^g \odot x^g \\ 1 \odot y & x \odot y, & \dots, & x^g \odot y \\ y \odot y \end{array} \right\}.$$

Then the natural map  $W \rightarrow S^2\Gamma(L)/V$  is surjective. Indeed, if  $i \leq g-j$ , then

$$x^i \odot x^j \equiv x^{i-1} \odot x^{j+1} \equiv \dots \equiv 1 \odot x^{j+i} \pmod{V},$$

and if  $i > g-j$ , then

$$x^i \odot x^j \equiv x^{i+1} \odot x^{j-1} \equiv \dots \equiv x^{i+j-g} \odot x^g \pmod{V}.$$

Hence we have  $\dim[S^2\Gamma(L)] - \dim(V) \leq \dim(W)$ , so we have

$$\dim(V) \geq \frac{1}{2}g(g-1) = \dim(I_2).$$

Since  $I_2 \supset V$ , we have  $I_2 = V$ . Q.E.D.

COROLLARY 2.3. *Let  $\{X_0, X_1, \dots, X_g, Y\}$  be a homogeneous coordinate of the projective space  $\mathbf{P}^{g+1}$  corresponding to a basis  $\{1, x, \dots, x^g, y\}$  of  $\Gamma(L)$ . Then the vector space of quadrics vanishing on  $\phi_L(C)$  is generated by the quadrics*

$$\{X_i X_j - X_{i-1} X_{j+1} \mid 1 \leq i \leq j \leq g-1\}$$

over  $K$ .

EXAMPLE 2.4. Let  $(C, L)$  be the above curve with invertible sheaf. Then the degree  $L$  is  $2g+1$ , but the homogeneous ideal  $I(L)$  is not generated by  $I_2(L)$ .

In fact, if the homogeneous ideal  $I(L)$  is generated by  $I_2$ , then

$$\phi_L(C) = \bigcap_{1 \leq i \leq j \leq g-1} V(X_i X_j - X_{i-1} X_{j+1})$$

by Corollary 2.3, where  $V(X_i X_j - X_{i-1} X_{j+1})$  is the set of zeros of  $X_i X_j - X_{i-1} X_{j+1}$  in  $\mathbf{P}^{g+1}$ . Let  $H$  be the linear subvariety of  $\mathbf{P}^{g+1}$  defined by the equations:

$$X_0 - X_1, X_1 - X_2, \dots, X_{g-1} - X_g.$$

Then  $H \cong \mathbf{P}^1$ , and

$$H \subset \bigcap_{1 \leq i \leq j \leq g-1} V(X_i X_j - X_{i-1} X_{j+1}).$$

Hence we have  $H = \phi_L(C)$ , because  $H \subset \phi_L(C)$  and  $\phi_L(C)$  is irreducible. This contradicts  $g \geq 1$ .

### § 3. Example II.

We continue assuming that the characteristic of the ground field  $K$  is not 2 and that a “curve” means a smooth curve over  $K$ .

In this section we will show that there are many examples of curves of genus  $g$  with invertible sheaf of degree  $2g$  on which Corollary 1.4 works effectively. Note that since the degree of  $L$  is  $2g$ , the condition  $H^1(C, L) = (0)$  in Corollary 1.4 is automatically satisfied. Therefore our problem is reduced to constructing many curves of genus  $g$  which have a normally generated ample invertible sheaf of degree  $2g$ .

**PROPOSITION 3.1.** *Let  $C$  be a curve of genus  $g \geq 5$ . Suppose that there exists an invertible sheaf  $M$  of degree  $g-1$  on  $C$  such that  $\Gamma(M)$  is a base point free pencil. Then almost all invertible sheaves of degree  $2g$  on  $C$  are ample with normal generation.*

The following lemma, B. Saint-Donat [8] called it “base point free pencil trick”, plays an important role in the proof of our proposition.

**LEMMA 3.2.** (Mumford [5, p. 57], Saint-Donat [8, Lemma 2.6]) *Let  $M$  and  $N$  be invertible sheaves on a curve. Suppose that  $\Gamma(M)$  is a base point free pencil. Then we have an isomorphism*

$$\text{Ker}[\Gamma(M) \otimes \Gamma(N) \rightarrow \Gamma(M \otimes N)] \cong \Gamma(N \otimes M^{-1}).$$

We will use the following notation.

$\text{Pic}^d(C)$ : the connected component of the Picard scheme of  $C$  whose member represents an invertible sheaf of degree  $d$ ,

$G_d^r$ : the closed subvariety of  $\text{Pic}^d(C)$  representing the set of invertible sheaves of degree  $d$  and of projective dimension  $\geq r$ ,

$F_d^r$ : the closed subvariety of  $\text{Pic}^d(C)$  defined by the image of the morphism

$$G_{d-1}^r \times C \ni (L, P) \longrightarrow L(P) \in \text{Pic}^d(C)$$

(if  $G_{d-1}^r = \emptyset$ , then  $F_d^r$  means the void subset).

Note that  $F_d^r \subset G_d^r$  and that if  $r \geq 1$ , the set  $G_d^r - F_d^r$  represents the set of in-

vertible sheaves free from base points, of degree  $d$  and of proejctive dimension  $r$ .

PROOF OF PROPOSITION 3.1. There exists an invertible sheaf  $M_0$  of degree  $g-1$  such that  $\Gamma(M_0)$  is a base point free pencil and  $M_0^2 \neq \omega$ , where  $\omega$  is the canonical sheaf on  $C$ . Indeed, since  $G_{g-1}^1 - F_{g-1}^1$  is non-empty open in  $G_{g-1}^1$  by our assumption and since

$$\dim G_{g-1}^1 \geq g-4 \geq 1 \quad [4, \text{Theorem 1}],$$

$G_{g-1}^1 - F_{g-1}^1$  has infinitely many elements. So there exists such an invertible sheaf. We put

$$V = G_{g+1}^1 - F_{g+1}^1 = \text{Pic}^{g+1}(C) - F_{g+1}^1, \text{ and} \\ U = \{N \otimes M_0 \mid N \in V\} \subset \text{Pic}^{2g}(C).$$

Obviously,  $V$  is non-empty open in  $\text{Pic}^{g+1}(C)$ . Hence  $U$  is non-empty open in  $\text{Pic}^{2g}(C)$ . We will show that any invertible sheaf in  $U$  is ample with normal generation. Let  $L$  be an invertible sheaf in  $U$ . By the generalized lemma of Castelnuovo [5, Theorem 2], we have natural map  $\Gamma(L^m) \otimes \Gamma(L) \rightarrow \Gamma(L^{m+1})$  is surjective for  $m \geq 2$ . Therefore it suffices to show that the natural map  $\Gamma(L) \otimes \Gamma(L) \rightarrow \Gamma(L^2)$  is surjective. Consider the commutative diagram

$$\begin{array}{ccc} \Gamma(L \otimes M_0^{-1}) \otimes \Gamma(M_0) \otimes \Gamma(L) & \xrightarrow{1 \otimes \phi_1} & \Gamma(L \otimes M_0^{-1}) \otimes \Gamma(M_0 \otimes L) \\ \downarrow & & \downarrow \phi_2 \\ \Gamma(L) \otimes \Gamma(L) & \longrightarrow & \Gamma(L^2), \end{array}$$

where  $\phi_1$  is the natural map  $\Gamma(M_0) \otimes \Gamma(L) \rightarrow \Gamma(M_0 \otimes L)$ . By Lemma 3.2, we have  $\text{Ker } \phi_1 \cong \Gamma(L \otimes M_0^{-1})$  and  $\text{Ker } \phi_2 \cong \Gamma(M_0^2)$ . Therefore we have

$$\begin{aligned} \dim(\text{Ker } \phi_1) &= \dim[\Gamma(L \otimes M_0^{-1})] = 2, \\ \dim[\Gamma(M_0) \otimes \Gamma(L)] &= 2(g+1), \\ \dim[\Gamma(M_0 \otimes L)] &= 2g, \\ \dim(\text{Ker } \phi_2) &= \dim[\Gamma(M_0^2)] = g-1 \quad (\text{Note that } M_0^2 \neq \omega), \\ \dim[\Gamma(L \otimes M_0^{-1}) \otimes \Gamma(M_0 \otimes L)] &= 4g \quad \text{and} \\ \dim[\Gamma(L^2)] &= 3g+1. \end{aligned}$$

Hence  $\phi_1$  and  $\phi_2$  are surjective, and hence the natural map  $\Gamma(L) \otimes \Gamma(L) \rightarrow \Gamma(L^2)$  is surjective. Q.E.D.

Next, we will give a sufficient condition for a curve to have an invertible sheaf  $M$  of degree  $g-1$  such that  $\Gamma(M)$  is a base point free pencil. Our result on it is a direct consequence of the following theorem of Martens and Mumford.

THEOREM OF MARTENS AND MUMFORD [6, Appendix]. *Let  $C$  be a curve of genus  $g \geq 5$ . Then there exists integer  $d$ ,  $3 \leq d \leq g-2$ , such that  $\dim G_d^1 \geq d-3$  if*

and only if  $C$  is hyperelliptic, or trigonal, or double covering of an elliptic curve ( $g \geq 6$ ), or non-singular plane quintic.

PROPOSITION 3.3. *Let  $C$  be a curve of genus  $g \geq 5$  neither hyperelliptic, nor trigonal, nor double covering of an elliptic curve ( $g \geq 6$ ), nor non-singular plane quintic. Then there exists an invertible sheaf  $M$  of degree  $g-1$  on  $C$  such that  $\Gamma(M)$  is a base point free pencil.*

PROOF. We must prove that  $G_{g-1}^1 - F_{g-1}^1 \neq \phi$  in our case. For this, it suffices to show that  $\dim G_{g-1}^1 > \dim F_{g-1}^1$ . By the results of Martens, Kleiman and Laksov [4, Theorem 1 and 3, Theorem 5], we have

$$\begin{aligned} g-3 &\geq \dim G_{g-1}^1 \geq g-4, \text{ and} \\ g-4 &\geq \dim G_{g-2}^1 \geq g-6. \end{aligned}$$

Note that if  $G_{g-2}^1 \neq \phi$ , then

$$\dim F_{g-1}^1 = \dim G_{g-2}^1 + 1 \quad [4, \text{p. 115}]$$

and that if  $G_{g-2}^1 = \phi$ , then  $F_{g-1}^1 = \phi$ . Suppose that  $\dim G_{g-1}^1 = \dim F_{g-1}^1$ . Then  $\dim G_{g-2}^1 \geq g-5$ . This contradicts the theorem of Martens and Mumford. Q.E.D.

Finally, we state an elementary remark relative to our topic.

REMARK 3.4. If  $C$  is a curve of genus  $g \geq 4$ , then there exists a non-special very ample invertible sheaf on  $C$  which is not normally generated.

Indeed, for a non-special normally generated ample invertible sheaf  $L$ , we have

$$\deg L \geq g + \frac{1}{2} + \sqrt{2g + \frac{1}{4}}$$

because  $\dim S^2 \Gamma(L) \geq \dim \Gamma(L^2)$ . On the other hand, by the theorem of Halphen [2, Theorem 1.2], there exists a non-special very ample invertible sheaf of degree  $d$ , if  $d \geq g+3$ .

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