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EQUIVARIANT POINT THEOREMS

(Dedicated to Professor A. Komatu on his 70th birthday)

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1. Introduction.

This paper is a continuation of my previous paper [13], and is concerned with generalizations of the following two classical theorems on a continuous map f of an n-sphere S^n to itself.

THEOREM 1.1. *If the degree of f is even then there exists $x \in S^n$ such that $f(-x)=f(x)$.*

THEOREM 1.2. *If the degree of f is odd then there exists $x \in S^n$ such that $f(-x)=-f(x)$.*

Throughout this paper, a prime p is fixed, and $G=\{1, T, \cdots, T^{p-1}\}$ will denote a cyclic group of order p.

Generalizing the situation in the above theorems, we shall consider the following problems.

PROBLEM 1. *Let $f: N \to M$ be a continuous map between G-spaces. Under what conditions does f have an equivariant point, i.e., a point $x \in N$ such that*

$$(1.1) \qquad f(T^i x)=T^i f(x)$$

for $i=1, 2, \cdots, p-1$?

PROBLEM 2. *Let $f: L \to M$ and $g: L \to N$ be continuous maps of a space L to G-spaces M and N. Under what conditions do there exist p points $x_1, \cdots, x_p \in L$ such that*

$$(1.2) \qquad f(x_{i+1})=T^i f(x_1), \ g(x_{i+1})=T^i g(x_1)$$

for $i=1, 2, \cdots, p-1$?

We shall denote by $A(f)$ the set of points $x \in N$ satisfying (1.1), and by $A(f, g)$ the set of points $(x_1, \cdots, x_p) \in L^p$ satisfying (1.2).

If $L=N$ in Problem 2, then $A(f, \mathrm{id})$ may be identified with $A(f)$. Therefore

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Problem 2 is more general than Problem 1; still Problem 2 can be reduced to Problem 1. In fact, if we define $h : L^p \to M \times N$ by

$$(1.3) \qquad h(x_1, \cdots, x_p) = (f(x_1), g(x_1)) \quad (x_i \in L),$$

and regard L^p and $M \times N$ as G-spaces by cyclic permutations and the diagonal action respectively, then we have $A(h) = A(f, g)$.

Throughout this paper, a manifold will always mean a compact connected topological manifold which is assumed to be oriented if p is odd. The dimension of manifolds $M, N, \cdots$ will be denoted by $m, n, \cdots$. By a G-manifold is meant a manifold on which G acts topologically.

In this paper we shall consider Problems 1 and 2 in case M and N are G-manifolds. Some answers have been obtained by Conner-Floyd [3], Munkholm [10], Fenn [5], Lusk [8] and others with respect to generalizations of Theorem 1.1, and by Milnor [9] and the author [13] with respect to generalizations of Theorem 1.2. By pushing the line of [13] we shall prove in this paper more general results.

Throughout this paper the cohomology stands for the Čech cohomology and it takes coefficients from Z_p, the group of integers mod p.

2. Theorems

In this section we shall state our main theorems answering to Problem 2 and then corollaries answering to Problem 1. The main theorems will be proved in § 5 and § 6.

Let $\omega_k \in H^k(BG)$ $(k=0, 1, \cdots)$ denote the usual generators, where BG is the classifying space for G. If X is a paracompact space on which G acts freely, $H^*(X/G)$ can be regarded as an $H^*(BG)$-module via the homomorphism induced by a classifying map of X; in particular we have $\omega_k = \omega_k \cdot 1 \in H^k(X/G)$.

The first main theorem is stated as follows, and it generalizes Theorem 1.1 (see Remark 1 below).

THEOREM A. *Let $f : L \to M$ and $g : L \to N$ be continuous maps of a compact space L to G-manifolds M and N. Suppose that*

i) *the action on M is trivial;*
ii) *the action on N is free and $\omega_n \in H^n(N/G)$ is not zero;*
iii) *$n \geqq (p-1)m$;*
iv) *$f^* : H^q(M) \to H^q(L)$ $(q>0)$ is trivial;*
v) *$g^* : H^n(N) \to H^n(L)$ is not trivial.*

Then we have $A(f, g) \neq \phi$; if L is moreover a manifold, we have

$$\dim A(f, g) \geqq pl - (p-1)(m+n) \geqq 0,$$

where $\dim A$ *denotes the covering dimension of* A.

Putting $L=N$ and $g=\mathrm{id}$, we get

COROLLARY. *Let* $f: N \to M$ *be a continuous map of a* G*-manifold* N *to a manifold* M. *Suppose that*

i) *the action on* N *is free and* $\omega_n \in H^n(N/G)$ *is not zero*;

ii) $f^*: H^q(M) \to H^q(N)$ $(q>0)$ *is trivial.*

Then we have

$$\dim A(f) \geqq n-(p-1)m,$$

where M *is regarded as a* G*-manifold by the trivial action.*

REMARK 1. Taking

$$N = \text{a mod } p \text{ homology } n\text{-sphere}$$

in the above corollary, we have the results due to Conner-Floyd [3], Munkholm [10] and the author [12], which are direct generalizations of Theorem 1.1.

REMARK 2. Taking

$$L=N=\text{a mod } p \text{ homology } n\text{-sphere},$$

$$M=S^m,\ \deg f=0,\ \deg g \not\equiv 0 \bmod p$$

in Theorem A, we have the results due the to Fenn [5] and Lusk [8].

To state the second main theorem and its corollaries, we shall make some preparations.

For any indexing set I, consider the complement $I_0^p = I^p - dI$ of the diagonal in I^p, and define $(i_1, \cdots, i_p)$, $(i_1', \cdots, i_p') \in I_0^p$ to be equivalent if $(i_1', \cdots, i_p')$ is a cyclic permutation of $(i_1, \cdots, i_p)$. We denote by $R(I_0^p)$ a set of representatives of the equivalent classes.

Let $f: L \to M$ and $g: L \to N$ be continuous maps of a manifold L to G-manifolds M and N. Given homogeneous bases $\{\alpha_i\}_{i \in I}$, $\{\beta_j\}_{j \in J}$ of $H^*(M)$, $H^*(N)$ and sets $R(I_0^p)$, $R(J_0^p)$, we define $\lambda(f, g)$, $\lambda'(f, g) \in \boldsymbol{Z}_p$ as follows.

Define $\Delta: M \to M^p$ by

$$\Delta(x) = (x, Tx, \cdots, T^{p-1}x) \quad (x \in M), \tag{2.1}$$

and put

$$\Delta_!(1) = \sum_{(i_1, \cdots, i_p) \in I^p} c_{i_1 \cdots i_p} \alpha_{i_1} \times \cdots \times \alpha_{i_p} \quad (c_{i_1 \cdots i_p} \in \boldsymbol{Z}_p) \tag{2.2}$$

for the Gysin homomorphism $\Delta_!: H^*(M) \to H^*(M^p)$.

Similarly, put

$$\Delta_!(1)=\sum_{(j_1,\cdots,j_p)\in J^p} d_{j_1\cdots j_p}\beta_{j_1}\times\cdots\times\beta_{j_p} \qquad (d_{j_1\cdots j_p}\in \boldsymbol{Z}_p)$$

for the homomorphism $\Delta_! : H^*(N)\to H^*(N^p)$.

We define

$$\lambda(f, g)=\langle (f^{*p}\sum_{(i_1,\cdots,i_p)\in R(I_0^p)} c_{i_1\cdots i_p}\alpha_{i_1}\times\cdots\times\alpha_{i_p})(g^{*p}\Delta_!(1)),\ [L]^p\rangle,$$

$$\lambda'(f, g)=\langle (f^{*p}\Delta_!(1))(g^{*p}\sum_{(j_1,\cdots,j_p)\in R(J_0^p)} d_{j_1\cdots j_p}\beta_{j_1}\times\cdots\times\beta_{j_p}),\ [L]^p\rangle.$$

Obviously we have $\lambda(f, g)=\lambda'(g, f)$.

If $L=N$ and $g^*=\mathrm{id}$, we write $\lambda(f)=\lambda(f, g)$. It follows that

$$\lambda(f)=\sum_{(i_1,\cdots,i_p)\in R(I_0^p)} c_{i_1\cdots i_p}\langle (f^*\alpha_{i_1})(T^*f^*\alpha_{i_2})\cdots(T^{*p-1}f^*\alpha_{i_p}),\ [N]\rangle.$$

REMARK 3. By the definition of $\Delta_!$ we have

$$\langle \Delta^*(\alpha_{k_1}\times\cdots\times\alpha_{k_p}),\ [M]\rangle=\langle (\alpha_{k_1}\times\cdots\times\alpha_{k_p})\Delta_!(1),\ [M]^p\rangle.$$

From this we get

$$y_{k_1\cdots k_p}=\sum_{(i_1,\cdots,i_p)\in I^p}(-1)^{\varepsilon(i_1,\cdots,i_p,k_1,\cdots,k_p)}c_{i_1\cdots i_p}z_{k_1i_1}\cdots z_{k_pi_p}, \tag{2.3}$$

where

$$y_{k_1\cdots k_p}=\langle \alpha_{k_1}(T^*\alpha_{k_2})\cdots(T^{*p-1}\alpha_{k_p}),\ [M]\rangle,$$

$$z_{ki}=\langle \alpha_k\alpha_i,\ [M]\rangle,$$

$$\varepsilon(i_1, \cdots, i_p, k_1, \cdots, k_p)=\sum_{s=1}^{p-1}|\alpha_{i_s}|(|\alpha_{k_{s+1}}|+\cdots+|\alpha_{k_p}|),$$

being $|\alpha|=\deg\alpha$. The relations (2.3) for $(k_1, \cdots, k_p)\in I^p$ characterize the coefficients $c_{i_1\cdots i_p}$ ([6]). In particular, if $p=2$ we see that the matrix (c_{ij}) is the inverse of the matrix (y_{ij}).

Now the second main theorem is stated as follows, and it generalizes Theorem 1.2 (see Remark 5 below).

THEOREM B. *Let $f : L\to M$ and $g : L\to N$ be continuous maps of a manifold L to G-manifolds M and N. Suppose that*

i) *$i^* : H^q(M)\to H^q(M^G)$ is trivial for $q\geqq m/p$, where M^G is the fixed point set of M, and i is the inclusion;*

ii) *the action on N is free;*

iii) *$pl=(p-1)(m+n)$.*

Then $\lambda(f, g)$ and $\lambda'(f, g)$ are independent of the choices of $\{\alpha_i\}_{i\in I}$, $\{\beta_j\}_{j\in J}$, $R(I_0^p)$,

$R(J_0^p)$, and we have $\lambda(f, g)=\lambda'(f, g)$; If $\lambda(f, g)\neq 0$ we have $A(f, g)\neq\phi$.

Putting $L=N$ and $g=\mathrm{id}$ in Theorem B we have

COROLLARY 1. *Let $f: N\to M$ be a continuous map between G-manifolds, and suppose that*

i) *$i^*: H^q(M)\to H^q(M^G)$ is trivial for $q\geqq m/p$;*

ii) *the action on N is free;*

iii) *$n=(p-1)m$.*

Then $\lambda(f)$ is independent of the choices of $\{\alpha_i\}_{i\in I}$ and $R(I_0^p)$, and if $\lambda(f)\neq 0$ we have $A(f)\neq\phi$.

Put $L=M$ and $f=\mathrm{id}$ in Theorem B, and replace the notations M, N, g by N, M, f respectively. Then we get

COROLLARY 2. *Let $f: N\to M$ be a continuous map between G-manifolds, and suppose that*

i) *$i^*: H^q(N)\to H^q(N^G)$ is trivial for $q\geqq n/p$;*

ii) *the action on M is free;*

iii) *$n=(p-1)m$.*

Then the same conclusions as in Corollary 1 hold.

REMARK 4. The above two corollaries for $p=2$ have been obtained in [13].

The following proposition will be proved in §4 (see p. 407 of [2] for $p=2$).

PROPOSITION 2.1. *If M is a G-manifold such that $i^*: H^{m/p}(M)\to H^{m/p}(M^G)$ is trivial, then*

$$\langle\alpha(T^*\alpha)\cdots(T^{*p-1}\alpha), [M]\rangle=0 \quad (\alpha\in H^*(M)).$$

Let M be the one in Proposition 2.1 for $p=2$. Then, the proposition and the Poincaré duality show that $H^*(M)$ has a homogeneous basis $\{\mu_1, \cdots, \mu_r, \mu'_1, \cdots, \mu'_r\}$ such that

$$\langle\mu_i(T^*\mu_k), [M]\rangle=0,\ \langle\mu'_i(T^*\mu'_k), [M]\rangle=0,\ \langle\mu_i(T^*\mu'_k), [M]\rangle=\delta_{ik}.$$

In terms of this basis we see that

$$\lambda(f)=\sum_{i=1}^{r}\langle(f^*\mu_i)(T^*f^*\mu'_i), [N]\rangle$$

if $p=2$. In particular, if $M=N$ and $f^*=\mathrm{id}$ then $\lambda(f)$ equals the *semi-characteristic*

$$\chi_{1/2}(M)=\dim H^*(M)/2 \bmod 2.$$

Thus, for $p=2$ we have the following

COROLLARY 3. *Let M be a manifold with a free involution T, and assume $\chi_{1/2}(M)\neq 0$. Let $f, g: M\to M$ be continuous maps such that $f^*=g^*=\mathrm{id}: H^*(M)\to H^*(M)$. Then there exist $x, x'\in M$ such that $f(x')=Tf(x)$ and $g(x')=Tg(x)$. In particular, there exists a point $x\in M$ such that $fT(x)=Tf(x)$.*

REMARK 5. Taking

$$M=\text{a mod 2 homology } m\text{-sphere}$$

in Corollary 3, we have the result due to Milnor [9], which is a direct generalization of Theorem 1.2.

3. Method.

In this section we shall explain how to prove Theorems A and B.

Let M be a G-manifold. If we regard M^p as a G-manifold by cyclic permutations, the map $\Delta: M\to M^p$ in (2.1) is an equivariant embedding. Regard S^{2k+1} as a G-manifold by the standard free action. Then we have a pair $(S^{2k+1}\times_G M^p, S^{2k+1}\times_G \Delta M)$ of manifolds, and hence the Thom isomorphism

$$\theta_k: H^q(S^{2k+1}\times_G \Delta M)\cong H^{q+(p-1)m}(S^{2k+1}\times_G (M^p, M^p-\Delta M))$$

which is the composite of the duality isomorphisms for $S^{2k+1}\times_G \Delta M$ and for $(S^{2k+1}\times_G M^p, S^{2k+1}\times_G \Delta M)$ (see p. 353 of [14]). We denote the Thom class $\theta_k(1)$ by $\hat{U}_M^{(k)}$.

The isomorphisms θ_k for sufficiently large k define the Thom isomorphism

$$\theta: H_G^q(\Delta M)\cong H_G^{q+(p-1)m}(M^p, M^p-\Delta M)$$

of the equivariant cohomology. The element $\theta(1)$ is denoted by $\hat{U}_M$, and is called the *equivariant fundamental cohomology class of M.*

The image of $\hat{U}_M$ in $H_G{}^{m(p-1)}(M^p)$ is denoted by $\hat{U}'_M$, and is called the *equivariant diagonal cohomology class* of M.

If the action of G on M is free, the diagonal set dM is in $M^p-\Delta M$. In this case the image of $\hat{U}_M$ in $H_G{}^{m(p-1)}(M^p, dM)$ is denoted by $\hat{U}''_M$, and is called the *modified equivariant diagonal cohomology class of M.*

LEMMA 3.1. *Let M and N be G-manifolds, and regard $M\times N$ as a G-manifold by the diagonal action. If the action on N is free, we have*

$$\hat{U}''_{M\times N}=\pm(q_1^{p*}\hat{U}'_M)(q_2^{p*}\hat{U}''_N),$$

where $q_1^{p*}: H_G^*(M^p)\to H_G^*((M\times N)^p)$ and $q_2^{p*}: H_G^*(N^p, dN)\to H_G^*((M\times N)^p, d(M\times N))$ are induced by the projections $q_1: M\times N\to M$, $q_2: M\times N\to N$.

PROOF. There are the following natural inclusions of manifolds:

$$\begin{array}{ccc} (S^{2k+1}\underset{G}{\times}\Delta M)\times S^{2k+1}\underset{G}{\times}\Delta N) & \subset & (S^{2k+1}\underset{G}{\times}M^p)\times(S^{2k+1}\underset{G}{\times}N^p) \\ \cup & & \cup \\ S^{2k+1}\underset{G}{\times}\Delta(M\times N) & \subset & S^{2k+1}\underset{G}{\times}(M\times N)^p \end{array}$$

From properties of the Thom class (see 325 of [4]), it follows that the Thom class for the pair in the upper line equals $\pm\hat{U}_M^{(k)}\times\hat{U}_N^{(k)}$, and that it is sent to $\pm\hat{U}_{M\times N}^{(k)}$ by the homomorphism i^* induced by the natural inclusion of the lower line to the upper. Therefore we have

$$\hat{U}_{M\times N}^{(k)}=\pm i^*(\hat{U}_M^{(k)}\times\hat{U}_N^{(k)})=\pm i^*(p_1^*\hat{U}_M^{(k)}\cdot p_2^*\hat{U}_N^{(k)})=\pm(q_1^{p*}\hat{U}_M^{(k)})(q_2^{p*}\hat{U}_N^{(k)}),$$

where p_1, p_2 are the projections of $(S^{2k+1}\underset{G}{\times}M^p)\times(S^{2k+1}\underset{G}{\times}N^p)$ to $S^{2k+1}\underset{G}{\times}M^p$, $S^{2k+1}\underset{G}{\times}N^p$. This fact proves immediately the desired result.

LEMMA 3.2. *Let $f: N\to M$ be a continuous map of a G-space N to a G-manifold M, and define an equivariant map $\hat{f}: N\to M^p$ by*

$$\hat{f}(x)=(f(x), fT(x), \cdots, fT^{p-1}(x)) \quad (x\in N).$$

If the action on M is free, and if $\hat{f}^*(\hat{U}_M'')\neq 0$ for the homomorphism $\hat{f}^*: H_G^*(M^p, dM)\to H_G^*(N, N^G)$, then we have $A(f)\neq\phi$. If N is moreover a G-manifold, we have

$$\dim A(f)\geqq n-(p-1)m\geqq 0.$$

PROOF. In virtue of a commutative diagram

$$\begin{array}{ccc} H_G^*(M^p, M^p-\Delta M) & \xrightarrow{i^*} & H_G^*(M^p, dM) \\ \big\downarrow \hat{f}^* & & \big\downarrow \hat{f}^* \\ H_G^*(N, N-A(f)) & \xrightarrow{i^*} & H_G^*(N, N^G), \end{array}$$

$\hat{f}^*(\hat{U}_M'')\neq 0$ implies $H_G^{m(p-1)}(N, N-A(f))\neq 0$. Therefore $A(f)\neq\phi$. If N is a G-manifold, we have isomorphisms

$$\begin{aligned} H^{n-m(p-1)}&(A(f)/G)\cong H_{m(p-1)}(N'/G, (N'-A(f))/G) \\ &\cong H^{m(p-1)}(N'/G, (N'-A(f))/G)\cong H_G^{m(p-1)}(N', N'-A(f)) \\ &\cong H_G^{m(p-1)}(N, N-A(f)), \end{aligned}$$

where $N'=N-N^G$. Therefore $H^{n-m(p-1)}(A(f)/G)\neq 0$, and so $\dim A(f)\geqq n-m(p-1)\geqq 0$. This completes the proof.

PROPOSITION 3.3. *Let $f: L\to M$ and $g: L\to N$ be continuous maps of a space*

L to G-manifolds M and N. Suppose that the action on N is free. Then if

$$(f^{p*}\hat{U}'_M)(g^{p*}\hat{U}''_N)\in H_G^{(m+n)(p-1)}(L^p, dL)$$

is not zero, we have $A(f, g)\neq\phi$. If L is moreover a manifold, we have

$$\dim A(f, g)\geqq pl-(p-1)(m+n)\geqq 0.$$

PROOF. Consider $h: L^p\to M\times N$ defined by (1.3). Then, for the map $\hat{h}: L^p\to (M\times N)^p$ we have $q_1^p\circ\hat{h}=f^p$, $q_2^p\circ\hat{h}=g^p$. Therefore by Lemma 3.1 we have

$$\begin{aligned}\hat{h}^*(\hat{U}''_{M\times N}) &= \pm\hat{h}^*((q_1^{p*}\hat{U}'_M)(q_2{}^{p*}\hat{U}''_N))\\ &= \pm(f^{p*}\hat{U}_M)(g^{p*}\hat{U}''_N).\end{aligned}$$

This proves the desired result by Lemma 3.2.

We shall prove Theorems A and B by making use of Proposition 3.3. For this purpose we are asked to examine the following:

(i) structure of the equivariant cohomologies $H_G^*(X^p)$ and $H_G^*(X^p, dX)$ for a compact space X.

(ii) the equivariant diagonal cohomology class $\hat{U}'_M$ and the modified equivariant diagonal cohomology class $\hat{U}''_M$ for a G-manifold M.

As for (i) we have the results due to Steenrod and Thom, which are stated in §4. Thus Theorems A and B will be proved by examining (ii), as seen in §5 and §6.

4. Preparations

In this section we shall recall some facts needed later.

Let X be a paracompact G-space. Then we have

$$H^*(X)=\varinjlim H^*(K),$$

$$H_G^*(X, X^G)=\varinjlim H^*(K/G, K^G/G),$$

where K ranges over the nerves of G-coverings of X (see Chap III, §6 and Chap VII, §1 of [2]). For each K a cochain map

$$\varphi_K: C^*(K)\longrightarrow C^*(K/G, K^G/G)$$

is defined by

$$\langle\varphi_K(u), \pi(s)\rangle=\sum_{i=0}^{p-1}u(T^is),$$

where $u\in C^*(K)$, s is a simplex of K, and $\pi: K\longrightarrow K/G$ is the projection. Thus we have a homomorphism

(4.1) $$\pi'_! : H^*(X) \longrightarrow H^*_G(X, X^G)$$

defined by the cochain maps φ_K.

We define

(4.2) $$\pi_! : H^*(X) \longrightarrow H^*_G(X)$$

to be the composite $j^* \circ \pi'_!$, where $j^* : H^*_G(X, X^G) \longrightarrow H^*_G(X)$ is induced by the inclusion. It follows that $\pi_!$ is the composite of the usual transfer $H^*(X) \longrightarrow H^*(X/G)$ and the canonical homomorphism $H^*(X/G) \longrightarrow H^*_G(X)$.

We call $\pi_!$ in (4.2) the *transfer*, and $\pi'_!$ in (4.1) the *modified transfer*.

Put

$$\sigma^* = \sum_{i=0}^{p-1} T^{i*} : H^*(X) \longrightarrow H^*(X).$$

Then it is easily seen that

(4.3) $$\pi^* \circ \pi_! = \sigma^*$$

for the canonical homomorphism $\pi^* : H^*_G(X) \longrightarrow H^*(X)$, and that

(4.4) $$\pi_!(\alpha_1) \cdot \pi'_!(\alpha_2) = \pi'_!(\alpha_1 \cdot \sigma^* \alpha_2) = \pi'_!(\sigma^* \alpha_1 \cdot \alpha_2)$$

$(\alpha_1, \alpha_2 \in H^*(X))$. We have also

(4.5) $$\pi_!(\alpha) \cdot \delta^*(\beta) = 0 \qquad (\alpha \in H^*(X),\ \beta \in H^*_G(X^G))$$

for the coboundary homomorphism $\delta^* : H^*_G(X^G) \longrightarrow H^*_G(X, X^G)$.

In fact

$$\begin{aligned}(-1)^{|\alpha|} \pi_!(\alpha) \cdot \delta^*(\beta) &= \delta^*(i^* \pi_!(\alpha) \cdot \beta) \\ &= \delta^*(i^* j^* \pi'_!(\alpha) \cdot \beta) = 0,\end{aligned}$$

where $i^* : H^*_G(X) \longrightarrow H^*_G(X^G)$.

If X is a paracompact G-space, the Smith special cohomology groups $H^*_\rho(X)$ are defined for $\rho = \sigma = \sum_{i=0}^{p-1} T^i$ and $\rho = \tau = 1 - T$, and we have the exact sequences

$$\begin{aligned}\cdots \xrightarrow{\rho^*} H^q(X) \xrightarrow{i^*_\rho} H^q_\rho(X) \oplus H^q(X^G) \\ \xrightarrow{\delta^*_\rho} H^{q+1}_{\bar\rho}(X) \xrightarrow{j^*_\rho} H^{q+1}(X) \longrightarrow \cdots\end{aligned}$$

for $(\rho, \bar\rho) = (\sigma, \tau)$ and (τ, σ). We have also an isomorphism

$$H^*_\sigma(X) \cong H^*_G(X, X^G).$$

(See p. 143 of [2].)

It follows that

$$i_{\tau}^{*}=(\pi_{!}^{\prime}, i^{*}) : H^{*}(X)\longrightarrow H_{G}^{*}(X, X^{G})\oplus H^{*}(X^{G}). \tag{4.6}$$

LEMMA 4.1. *If M is a G-manifold such that the action is not trivial, then it holds*

$$\pi_{!}^{\prime} : H^{m}(M)\cong H_{G}^{m}(M, M^{G}).$$

PROOF. In the exact sequence

$$H^{m}(M)\xrightarrow{i_{\tau}^{*}}H_{\sigma}^{m}(M)\oplus H^{m}(M^{G})\xrightarrow{\delta_{\tau}^{*}}H_{\tau}^{m+1}(M),$$

we have $H_{\tau}^{m+1}(M)=0$, $H^{m}(M)\cong \boldsymbol{Z}_{p}$, $H^{m}(M^{G})\cong H_{0}(M, M-M^{G})=0$, and moreover $H_{\sigma}^{m}(M)\neq 0$ is proved as follows. Therefore we get the desired result by (4.6).

Suppose $H_{\sigma}^{m}(M)=0$. Then, by the Smith cohomology exact sequence, we see that $i_{\sigma}^{*} : H^{m}(M)\cong H_{\tau}^{m}(M)$ and $\tau^{*} : H_{\tau}^{m}(M)\longrightarrow H^{m}(M)$ is onto. This implies that $\tau^{*} : H^{m}(M)\longrightarrow H^{m}(M)$ is onto and so $H^{m}(M)=0$, which is a contradiction.

For a paracompact space X, consider the equivariant cohomology $H_{G}^{*}(X^{p})$, where G acts on X^{p} by cyclic permutations. Then we have the external Steenrod p-th power operation

$$P: H^{q}(X)\longrightarrow H_{G}^{pq}(X^{p}),$$

which is related to the Steenrod square Sq^{i} if $p=2$, and to the reduced p-th power $\mathscr{P}^{i}$ and the Bockstein operation β^{*} if $p\neq 2$ as follows ([15]):

$$d^{*}P(\alpha) = \begin{cases} \sum_{i=0}^{|\alpha|} \omega_{|\alpha|-i}\times Sq^{i}\alpha & \text{if } p=2, \\ h_{q}\sum_{i=0}^{[|\alpha|/2]}(-1)^{i}(\omega_{(|\alpha|-2i)(p-1)}\times \mathscr{P}^{i}\alpha - \omega_{(|\alpha|-2i)(p-1)-1}\times \beta^{*}\mathscr{P}^{i}\alpha) & \text{if } p\neq 2, \end{cases} \tag{4.7}$$

where $d^{*} : H_{G}^{*}(X^{p})\longrightarrow H_{G}^{*}(X)=H^{*}(BG\times X)$ is induced by the diagonal map, and

$$h_{q}=\begin{cases} (-1)^{q/2} & \text{if } q \text{ is even}, \\ (-1)^{(q-1)/2}((p-1)/2)! & \text{if } q \text{ is odd}. \end{cases} \tag{4.8}$$

P is natural, and it satisfies also

$$\pi^{*}P(\alpha)=\alpha^{p} \tag{4.9}$$

for the canonical homomorphism $\pi^{*} : H_{G}^{*}(X^{p})\longrightarrow H^{*}(X^{p})$.

LEMMA 4.2. *Let M be a G-manifold, and let $\alpha \in H^*(M)$ satisfy $i^*(\alpha)=0$ for $i^* : H^*(M) \longrightarrow H^*(M^G)$. Then $\Delta^*P(\alpha)$ is in the image of $j^* : H_G^*(M, M^G) \longrightarrow H_G^*(M)$ induced by the inclusion.*

PROOF. Consider a diagram

$$\begin{tikzcd} & H_G^*(M^p) \arrow[r, "d^*"] \arrow[d, "\Delta^*"] & H^*(BG\times M) \arrow[d, "(id\times i)^*"] \\ H_G^*(M, M^G) \arrow[r, "j^*"] & H_G^*(M) \arrow[r, "i^*"] & H^*(BG\times M^G), \end{tikzcd}$$

in which the rectangle is commutative and the lower sequence is exact. Then it follows from (4.7) that $i^*\Delta^*P(\alpha)=(id\times i)^*d^*P(\alpha)=0$. Therefore $\Delta^*P(\alpha)\in \mathrm{Im} j^*$.

PROOF OF PROPOSITION 2.1. We may assume that the action is non-trivial and $|\alpha|=m/p$. Consider a commutative diagram

$$\begin{tikzcd} H_G^m(M, M^G) \arrow[r, "j^*"] & H_G^m(M) \arrow[d, "\pi^*"] \\ H^m(M) \arrow[u, "\pi_!'"] \arrow[ur, "\pi_!"] \arrow[r, "\sigma^*"] & H^m(M) \end{tikzcd}$$

By Lemmas 4.1 and 4.2, we see

$$\pi^*\Delta^*P(\alpha)\in \mathrm{Im}\,\sigma^*.$$

Since $\sigma^*H^m(M)=0$ and

$$\pi^*\Delta^*P(\alpha)=\Delta^*(\alpha^p)=\alpha(T^*\alpha)\cdots(T^{p-1*}\alpha)$$

by (4.9), the proof completes.

The following theorem is due to Steenrod [15] (see also [12]).

THEOREM 4.3. *Let X be a compact space, and $\{\alpha_i\}_{i\in I}$ be a homogeneous basis of $H^*(X)$. Then the totality of elements*

$$\omega_j P(\alpha_i) \qquad (i\in I,\ j\geqq 0),$$

$$\pi_!(\alpha_{i_1}\times\cdots\times\alpha_{i_p}) \qquad ((i_1, \cdots, i_p)\in R(I_0^p))$$

is a homogeneous basis of $H_G^*(X^p)$.

The following is due to Thom [16] (see also [1], [11], [17]).

THEOREM 4.4. *Let X be a compact space, and $\{\alpha_i\}_{i\in I}$ be a homogeneous basis of $H^*(X)$. Then the totality of elements*

$$\delta^*(\omega_j\times\alpha_i) \qquad (i\in I,\ 0\leqq j<(p-1)|\alpha_i|),$$

$$\pi_!(\alpha_{i_1}\times\cdots\times\alpha_{i_p}) \qquad ((i_1,\cdots,i_p)\in R(I_0^p))$$

is a homogeneous basis of $H_G^*(X^p, dX)$, where $\delta^*: H^*(BG\times X)=H_G^*(dX)\longrightarrow H_G^*(X^p, dX)$ is the coboundary homomorphism. Furthermore we have

$$\pi_!(\alpha\times\alpha)=\sum_{i=0}^{|\alpha|-1}\delta^*(\omega_{|\alpha|-i-1}\times Sq^i\alpha)$$

if $p=2$, and

$$\pi_!(\alpha\times\cdots\times\alpha)=\sum_{i=0}^{[|\alpha|/2]}\varepsilon_i\delta^*(\omega_{(p-1)(|\alpha|-2i)-1}\times\mathscr{P}^i\alpha)$$

with some $\varepsilon_i\not\equiv 0 \bmod p$ if $p\neq 2$.

REMARK. Theorems 4.3 and 4.4 are proved in the literatures for a compact polyhedron. However we can extend them to compact spaces by the device seen in [2].

5. Proof of Theorem A.

The equivariant diagonal cohomology class $\hat{U}'_M$ in case the action on M is trivial has been studied by Haefliger. By Theorem 3.2 in his paper [6] and

(5.1) $$\pi^*(\hat{U}')=\Delta_!(1),$$

we have the following (see the proof of Theorem 9.1 in [13]).

PROPOSITION 5.1. *If the action on M is trivial, then*

$$\hat{U}'_M=\sum_{k=0}^{[m/2]}\omega_{m-2k}P(V_k)+\sum_{i<j}(c_{ij}-c_{ii}c_{jj})\pi_!(\alpha_i\times\alpha_j)$$

if $p=2$, and

$$\hat{U}'_M=h_m\sum_{k=0}^{[m/2p]}(-1)^k\omega_{(p-1)(m-2kp)}P(V_k)$$
$$+\sum_{(i_1,\cdots,i_p)\in R(I_0^p)}(c_{i_1\cdots i_p}-c_{i_1\cdots i_1}\cdots c_{i_p\cdots i_p})\pi_!(\alpha_{i_1}\times\cdots\times\alpha_{i_p})$$

if $p\neq 2$, where $\{\alpha_i\}_{i\in I}$ is a homogeneous basis of $H^*(M)$, $c_{i_1,\cdots,i_p}$, h_m are those in (2.2), (4.8), and $V_k\in H^*(M)$ are the Wu classes given by

$$\langle V_k\cdot\alpha, [M]\rangle=\begin{cases}\langle Sq^k\alpha, [M]\rangle & \text{if } p=2,\\ \langle\mathscr{P}^k\alpha, [M]\rangle & \text{if } p\neq 2.\end{cases}$$

We shall next prove

PROPOSITION 5.2. *If the action on M is free and $\omega_m \in H^*(M/G)$ is not zero, it holds*

$$\omega_m \hat{U}''_M = \delta^*(\omega_{(p-1)m-1} \times \mu),$$

where μ is a generator of $H^m(M)$.

PROOF. Let V be an equivariant open neighbourhood of dM in M^p, and put

$$W = M^p - \Delta M - dM, \qquad C = M^p - \Delta M - V.$$

Then C/G is a closed connected and non-compact subset of W/G, and hence we have $H^{mp}(W/G, W/G - C/G) = 0$ (see p. 260 of [4]). Therefore it follows that

$$H_G^{mp}(M^p - \Delta M, V) \cong H_G^{mp}(W, W - C) \cong H^{mp}(W/G, W/G - C/G) = 0.$$

This shows that $i^* : H_G^{mp}(M^p, M^p - \Delta M) \longrightarrow H_G^{mp}(M^p, dM)$ is onto, and so is

$$i^* \circ \theta : H_G^m(\Delta M) \longrightarrow H_G^{mp}(M^p, dM).$$

It follows from Lemma 4.1 and the assumptions that $H_G^m(\Delta M) \cong \boldsymbol{Z}_p$ is generated by ω_m. By Theorem 4.4, $H_G^{mp}(M^p, dM) \cong \boldsymbol{Z}_p$ is generated by $\delta^*(\omega_{(p-1)m} \times \mu)$. Since $i^* \circ \theta$ is a homomorphism of $H^*(BG)$-modules and it sends 1 to $\hat{U}''_M$, we have the desired result.

We shall now give

PROOF OF THEOREM A. By the assumption ii) and Proposition 5.2, it holds

$$\omega_n \hat{U}''_N = \delta^*(\omega_{(p-1)n-1} \times \nu),$$

where ν is a generator of $H^n(N)$. Therefore we have

$$\omega_n (g^{p*} \hat{U}''_N) = \delta^*(\omega_{(p-1)n-1} \times g^* \nu),$$

and this is not zero by the assumption v) and Theorem 4.4. Since $n \geqq (p-1)m$ by the assumption iii), it holds

$$\omega_{(p-1)m} (g^{p*} \hat{U}''_N) \neq 0.$$

On the other hand, it follows from the assumptions i), iv) and Proposition 5.1 that

$$f^{p*}(\hat{U}'_M) = h_m \omega_{(p-1)m}$$

with $h_m \not\equiv 0 \bmod p$. Consequently we have

$$f^{p*}(\hat{U}'_M) \cdot g^{p*}(\hat{U}''_N) = h_m \omega_{(p-1)m} g^{p*}(\hat{U}''_N) \neq 0,$$

which completes the proof by Proposition 3.3.

6. Proof of Theorem B and an example.

The following proposition has been proved in [13] if $p=2$. By the similar method we shall prove it for any p.

PROPOSITION 6.1. *If $i^*: H^q(M) \longrightarrow H^q(M^G)$ is trivial for $q \geqq m/p$, then we have*

$$\hat{U}'_M = \sum_{(i_1, \cdots, i_p) \in R(I_0^p)} c_{i_1 \cdots i_p} \pi_!(\alpha_{i_1} \times \cdots \times \alpha_{i_p}),$$

$$c_{i \cdots i} = 0 \quad (i \in I),$$

where $\{\alpha_i\}_{i \in I}$ is a homogeneous basis of $H^*(M)$, and $c_{i_1 \cdots i_p}$ are those in (2.2).

Before we proceed to proof we make some preparations.

The equivariant homology group $H_*^G(X^p) = H_*(EG \underset{G}{\times} X^p)$ is canonically identified with $H_*(G; H_*(X)^p)$, the homology group of the group G with coefficients in $H_*(X)^p = H_*(X) \otimes \cdots \otimes H_*(X)$ on which G acts by cyclic permutations. Taking the standard G-free acyclic complex W, we have an element of $H_*(G; H_*(X)^p)$ represented by $w_k \otimes a \otimes \cdots \otimes a$, where $w_k \in W$ is the basis of degree k and $a \in H_*(X)$. The corresponding element in $H_*^G(X^p)$ will be denoted by $P_k(a)$.

LEMMA 6.2. *Suppose that $i^*: H^q(M) \longrightarrow H^q(M^G)$ is trivial for $q \geqq m/p$. Then, for any $k \geqq 0$ and for any $\alpha \in H^*(M)$, we have*

$$\langle \omega_1 \hat{U}'_M, P_{k+1}(\alpha \frown [M]) \rangle = 0$$

if $p=2$, and

$$\langle \hat{U}'_M, P_{2k+1}(\alpha \frown [M]) \rangle = 0,$$

$$\langle \omega_1 \hat{U}'_M, P_{2k+1}(\alpha \frown [M]) \rangle = 0$$

if $p \neq 2$.

PROOF. Similarly to Lemma 4.4 in [13], the result for $p \neq 2$ is proved as follows.

It follows that $P_{2k+1}([M])$ is in the image of

$$i_{k*}: H_{2k+1+pm}(S^{2k+1} \underset{G}{\times} M^p) \longrightarrow H_{2k+1+pm}^G(M^p)$$

induced by the inclusion, and that $i_k^*(\hat{U}'_M)$ is the image of 1 under the homomorphism

$$(\mathrm{id} \underset{G}{\times} \Delta)_!: H^*(S^{2k+1} \underset{G}{\times} M) \longrightarrow H^*(S^{2k+1} \underset{G}{\times} M^p).$$

From these facts we see that $\hat{U}'_M \frown P_{2k+1}([M])$ is in the image of

$$H_{2k+1+m}(S^{2k+1}\underset{G}{\times}M)\xrightarrow{i_{k*}}H^G_{2k+1+m}(M)\xrightarrow{\Delta_*}H^G_{2k+1+m}(M^p).$$

Therefore it follows that

$$\langle\hat{U}'_M,\ P_{2k+1}(\alpha\frown[M])\rangle=\langle\hat{U}'_M,\ P(\alpha)\frown P_{2k+1}([M])\rangle$$

$$=\langle P(\alpha),\ \hat{U}'_M\frown P_{2k+1}([M])\rangle=\varepsilon_k\langle P(\alpha),\ \Delta_* i_{k*}[S^{2k+1}\underset{G}{\times}M]\rangle$$

$$=\varepsilon_k\langle\Delta^*P(\alpha),\ i_{k*}[S^{2k+1}\underset{G}{\times}M]\rangle\qquad(\varepsilon_k\in\boldsymbol{Z}_p),$$

and similarly

$$\langle\omega_1\hat{U}'_M,\ P_{2k+1}(\alpha\frown[M])\rangle=\varepsilon_k\langle\omega_1\Delta^*P(\alpha),\ i_{k*}[S^{2k+1}\underset{G}{\times}M]\rangle.$$

To prove the desired two equalities, we may suppose $p|\alpha|\geqq m+1$ in the first, and $p|\alpha|\geqq m$ in the second. Consequently it suffices to prove that

$$\Delta^*P(\alpha)=0\quad\text{if}\quad p|\alpha|\geqq m+1,$$

$$\omega_1\Delta^*P(\alpha)=0\quad\text{if}\quad p|\alpha|\geqq m.$$

By Lemma 4.2 $\Delta^*P(\alpha)$ and $\omega_1\Delta^*P(\alpha)$ are in the image of $j^*: H^*_G(M, M^G)\longrightarrow H^*_G(M)$, and the Smith cohomology exact sequence implies $H^q_G(M, M^G)=0$ $(q>m)$. Therefore we have the desired results, and the proof completes.

Proof of Proposition 6.1. In virtue of Theorem 4.3 it can be written uniquely that

$$\hat{U}'_M=\sum_{i,j}\xi_{ij}\omega_jP(\alpha_i)+\sum_{(i_1,\cdots,i_p)\in R(I^p_0)}\eta_{i_1\cdots i_p}\pi_!(\alpha_{i_1}\times\cdots\times\alpha_{i_p})$$

with some ξ_{ij}, $\eta_{i_1\cdots i_p}\in\boldsymbol{Z}_p$. Since it is easily seen that

$$\operatorname{Im}\pi_!\frown P_k(a)=0,$$

$$\langle\omega_jP(\alpha),\ P_k(a)\rangle=\delta_{jk}\langle\alpha,\ a\rangle$$

$(\alpha\in H^*(M),\ a\in H_*(M))$, it follows from Lemma 6.2 that $\xi_{ij}=0$. We see from (5.1) that $\eta_{i_1\cdots i_p}=c_{i_1\cdots i_p}$ if $(i_1,\cdots,i_p)\in R(I^p_0)$ and $c_{i\cdots i}=0$ for any $i\in I$. This completes the proof.

Remark 1. Working in the smooth category, Hattori [7] has given formulae for $\hat{U}'_M$ with no assumption on M^G.

The following is immediate from Proposition 6.1 and Theorem 4.5.

Proposition 6.3. *If the action on M is free, then it can be written uniquely that*

$$\hat{U}''_M = \sum_{(i_1, \cdots, i_p) \in R(I_0^p)} c_{i_1 \cdots i_p} \pi'_!(\alpha_{i_1} \times \cdots \times \alpha_{i_p})$$
$$+ \sum_{|\alpha_i| \geqq m - m/p} \varepsilon_i \delta^*(\omega_{(p-1)m - |\alpha_i| - 1} \times \alpha_i)$$

with some $\varepsilon_i \in \boldsymbol{Z}_p$.

REMARK 2. The author does not know how to determine ε_i in the above. If M is a mod p homology sphere, it follows from Propositions 5.2 and 6.3 that

$$\hat{U}''_M = \begin{cases} \pi'_!(1 \times \mu) & \text{if } p=2, \\ \pi'_!(1 \times \mu \times \cdots \times \mu) + \varepsilon \delta^*(\omega_{(p-2)m-1} \times \mu) & \text{if } p \neq 2, \end{cases}$$

where $\varepsilon \not\equiv 0 \bmod p$, and $\mu \in H^m(M)$ is a generator such that $\langle \mu, [M] \rangle = 1$.

We shall now give

PROOF OF THEOREM B. By the assumption i) and Proposition 6.1 we have

$$f^{p*}\hat{U}'_M = \pi_! f^{*p}(\sum_{(i_1, \cdots, i_p) \in R(I_0^p)} c_{i_1 \cdots i_p} \alpha_{i_1} \times \cdots \times \alpha_{i_p}),$$

and by the assumption ii) and Proposition 6.3 we have

$$g^{p*}\hat{U}''_N = \pi'_! g^{*p}(\sum_{(j_1, \cdots, j_p) \in R(J_0^p)} d_{j_1 \cdots j_p} \beta_{j_1} \times \cdots \times \beta_{j_p})$$
$$+ \sum_{|\beta_j| \geqq n - n/p} \varepsilon_j \delta^*(\omega_{(p-)n - |\beta_j| - 1} \times g^*\beta_j).$$

It follows from (5.1) and Proposition 6.1 that

$$\sigma^* \sum_{(i_1, \cdots, i_p) \in R(I_0^p)} c_{i_1 \cdots i_p} \alpha_{i_1} \times \cdots \times \alpha_{i_p} = \Delta_!(1),$$

$$\sigma^* \sum_{(j_1, \cdots, j_p) \in R(J_0^p)} d_{j_1, \cdots, j_p} \beta_{j_1} \times \cdots \times \beta_{j_p} = \Delta_!(1).$$

Thus, by (4.4), (4.5) and the assumption ii), we have

$$(f^p\hat{U}'_M) \cdot (g^{p*}\hat{U}''_N)$$
$$= \pi'_!(f^{*p} \sum_{(i_1, \cdots, i_p) \in R(I_0^p)} c_{i_1 \cdots i_p} \alpha_{i_1} \times \cdots \times \alpha_{i_p})(g^{*p}\Delta_!(1))$$
$$= \pi'_!(f^{*p}\Delta_!(1))(g^{*p} \sum_{(j_1, \cdots, j_p) \in R(J_0^p)} d_{j_1, \cdots, j_p} \beta_{j_1} \times \cdots \times \beta_{j_p})$$

in $H_G^{pl}(L^p, dL)$.

It follows from Theorem 4.4 that $H_G^{pl}(L^p, dL) \cong \boldsymbol{Z}_p$ is generated by $\delta^*(\omega_{(p-1)l-1} \times \rho)$ or $\pi'_!(\rho \times \cdots \times \rho)$, where $\rho \in H^l(L)$ is a generator such that $\langle \rho, [L] \rangle = 1$.

Consequently we have

$$(f^{p*}\hat{U}'_M)(g^{p*}\hat{U}''_N) = \lambda(f, g)\pi'_!(\rho \times \cdots \times \rho)$$
$$= \lambda'(f, g)\pi'_!(\rho \times \cdots \times \rho),$$

which completes the proof by Proposition 3.3.

Theorem B for $p=2$, particularly corollary 3 in § 2, has interesting applications as is seen in [13]. The author does not know so interesting applications of Theorems B for $p\neq 2$. However there is the following example for which Theorem B for $p=3$ is applicable.

Let $n=1, 3$ or 7, and take in Theorem B

$$L=S^n\times S^n,\ M=S^n\times S^n,\ N=S^n,$$

where the action on N is any free G-action, and action on M is given as follows:

$$T(x, y)=(y, y^{-1}x^{-1}),$$

x, y being complex numbers, quaternions or Cayley numbers according as $n=1, 3$ or 7. It follows that the fixed point set of M is homeomorphic to $S^{n-1}+$point. Thus the assumptions i), ii), iii) in Theorem B are satisfied.

Let $\nu\in H^n(S^n)$ denote a generator, and put $\nu_1=\nu\times 1$, $\nu_2=1\times\nu\in H^n(S^n\times S^n)$. Then, by Remark 3 in § 2, it can be seen that

$$\Delta_!(1)=\sigma^*(1\times\nu_1\nu_2\times\nu_1\nu_2-\nu_1\times\nu_1\times\nu_1\nu_2-\nu_2\times\nu_2\times\nu_1\nu_2-\nu_2\times\nu_1\times\nu_1\nu_2)$$

for the homomorphism $\Delta_!: H^*(M)\longrightarrow H^*(M^3)$, and

$$\Delta_!(1)=\sigma^*(1\times\nu\times\nu)$$

for the homomorphism $\Delta_!: H^*(N)\longrightarrow H^*(N^3)$. Therefore, if continuous maps $f: L\longrightarrow M$, $g: L\longrightarrow N$ satisfy

$$f^*(\nu_i)=a_{i1}\nu_1+a_{i2}\nu_2,\quad g^*(\nu)=b_1\nu_1+b_2\nu_2$$

$(a_{ij}, b_i\in\boldsymbol{Z}_3)$, simple calculation shows

$$\lambda(f, g)=\begin{vmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{vmatrix}\left(\begin{vmatrix} a_{11} & a_{12}\\ b_1 & b_2\end{vmatrix}-\begin{vmatrix} a_{21} & a_{22}\\ b_1 & b_2\end{vmatrix}\right).$$

This yields by Theorem B the following

THEOREM 6.4. *Let $n=1, 3$ or 7, and let $f_1, f_2, g: S^n\times S^n\longrightarrow S^n$ be continuous maps of type (a_{11}, a_{12}), (a_{21}, a_{22}), (b_1, b_2) respectively. Let $T: S^n\longrightarrow S^n$ be a homomorphism of period 3 without fixed points. Then, if*

$$\begin{vmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{vmatrix}\left(\begin{vmatrix} a_{11} & a_{12}\\ b_1 & b_2\end{vmatrix}-\begin{vmatrix} a_{21} & a_{22}\\ b_1 & b_2\end{vmatrix}\right)\not\equiv 0 \bmod 3,$$

there exist $x, y, z\in S^n\times S^n$ such that

$$(f_2(x), f_2(y), f_2(z))=(f_1(y), f_1(z), f_1(x)),$$

$$(Tg(x), Tg(y), Tg(z)) = (g(y), g(z), g(x)),$$

$$f_1(x)f_1(y)f_1(z) = 1.$$

In particular, taking f_i=projection to the i-th factor, we have

COROLLARY. *If $b_1+b_2\neq 0$ then there exist $x, y, z\in S^n$ such that*

$$Tg(x, y) = g(y, z),\ Tg(y, z) = g(z, x),\ xyz = 1,$$

where n, g and T are those in Theorem 6.4.

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