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THE GROTHENDIECK RING OF VECTOR SPACES WITH TWO IDEMPOTENT ENDOMORPHISMS

By

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Introduction.

In this paper we are concerned with a particular bialgebra Λ over a field k, which is generated as an algebra by e_1, e_2 with defining relations $e_1^2=e_1$, $e_2^2=e_2$, and whose comultiplication $\Delta: \Lambda\rightarrow\Lambda\otimes\Lambda$ and counit $\varepsilon: \Lambda\rightarrow k$ are given by the formulas

$$\Delta(e_1)=e_1\otimes e_1+(1-e_1)\otimes(1-e_2)$$

$$\Delta(e_2)=(1-e_2)\otimes(1-e_1)+e_2\otimes e_2$$

$$\varepsilon(e_1)=\varepsilon(e_2)=1.$$

The purpose of this paper is to compute the representation ring of Λ, namely the Grothendieck ring of finite dimensional Λ-modules with respect to $\oplus$ and $\otimes$, when k is an algebraically closed field of characteristic zero. The classification of indecomposable Λ-modules is known and our main task is to decompose tensor product of indecomposable Λ-modules.

The results are summarized at the end of Section 1. Our computations involve the decomposition of tensor product of $\boldsymbol{Z}_2$-graded $k[x]$-modules. More generally we do this for $\boldsymbol{Z}_e(=\boldsymbol{Z}/e\boldsymbol{Z})$-graded $k[x]$-modules for any integer $e\geqq 2$. Here, for $\boldsymbol{Z}_e$-graded $k[x]$-modules A, B, we give $A\otimes B$ the standard grading and let x act on it by

$$x(a\otimes b)=xa\otimes b+\omega^i a\otimes xb \qquad \deg a=i,$$

where ω is a fixed primitive e^{th} root of 1.

The bialgebra Λ comes from a certain universal construction. In general, for k-algebras A, B such that $\dim A<\infty$, there is a k-algebra $a(A, B)$ equipped with a k-algebra map $\rho: B\rightarrow A\otimes a(A, B)$ having the following property: For any k-algebra C, the map $\mathrm{Hom}_{k\text{-alg}}(a(A, B), C)\rightarrow\mathrm{Hom}_{k\text{-alg}}(B, A\otimes C)$ induced by ρ is a bijection. The algebra $a(A, A)$ becomes naturally a bialgebra. The bialgebra $a(A, A)^\circ$ in the dual space $a(A, A)^*$ is the universal measuring bialgebra

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of A in the terminology of Sweedler [3]. Our bialgebra Λ is isomorphic to $a(A, A)$ with $A=k\times k$. General theory of such bialgebras will appear elsewhere.

1. Main results.

Throughout this paper k is an algebraically closed field of characteristic zero, $\otimes$ is over k and all modules are finite dimensional over k. Let Λ be a k-algebra generated by e_{ij}, $i, j=1, 2$, with defining relations

$$1=\sum_{j} e_{ij}, \qquad i=1, 2$$

$$e_{ij}e_{ik}=\delta_{jk}e_{ij}, \qquad i, j, k=1, 2.$$

We make Λ a bialgebra, defining comultiplication $\Delta: \Lambda\to\Lambda\otimes\Lambda$ and counit $\varepsilon: \Lambda\to k$ by the formulas

$$\Delta(e_{ik})=\sum_{j} e_{ij}\otimes e_{jk}$$

$$\varepsilon(e_{ij})=\delta_{ij}.$$

This bialgebra is identified with the one in Introduction by $e_{ii}=e_i$. For right Λ-modules V, W, we always regard $V\otimes W$ as a right Λ-module through the map Δ. Our object is to decompose Λ-modules $V\otimes W$ for all indecomposable Λ-modules V, W.

We begin with a parametrization of indecomposable Λ-modules. Since a Λ-module structure on V is determined by the subspaces Ve_{ij} of V, the classification of Λ-modules is a special case of that of quadruples of subspaces in vector spaces, which was done by Gelfand and Ponomarev, and by Nazarova.

For vector spaces V_{ij}, $i, j=1, 2$, and an isomorphism $\alpha: V_{11}\oplus V_{12}\to V_{21}\oplus V_{22}$, define a Λ-module $M(\alpha)$ as the vector space $V_{11}\oplus V_{12}$ on which e_{11}, e_{12} act as the projections to V_{11}, V_{12}, and e_{21}, e_{22} act as the projections to $\alpha^{-1}(V_{21}), \alpha^{-1}(V_{22})$ respectively. We write the isomorphism α in a matrix form

$$\alpha=\begin{pmatrix}\alpha_{11} & \alpha_{12}\\ \alpha_{21} & \alpha_{22}\end{pmatrix}, \qquad \alpha_{ij}: V_{1j}\to V_{2i}.$$

Let $\mathcal{E}$ be the category of $k[x]$-modules on which x acts nilpotently. Indecomposable objects of $\mathcal{E}$ are $V_n:=k[x]/(x^{n+1})$, $n\geqq 0$. By a $\boldsymbol{Z}_2$-graded $k[x]$-module we mean a $k[x]$-module A equipped with a $\boldsymbol{Z}_2(=\boldsymbol{Z}/2\boldsymbol{Z})$-grading $A=A_0\oplus A_1$ such that $x(A_i)\subset A_{i+1}$ for $i\in\boldsymbol{Z}_2$. A homomorphism of $\boldsymbol{Z}_2$-graded $k[x]$-modules is a $k[x]$-linear map preserving grading. Let $\mathcal{D}$ be the category of $\boldsymbol{Z}_2$-graded $k[x]$-modules on which x acts nilpotently. For each $n\geqq 0$ and $j=0, 1$, let V_n^j be a $\boldsymbol{Z}_2$-graded $k[x]$-module which has a basis $v, xv, \cdots, x^n v$ such that $\deg v = j$ and $x^{n+1}v=0$. The modules V_n^j for $n \geqq 0$, $j=0, 1$ form a complete list of indecomposable objects in $\mathcal{D}$.

For an object A of $\mathcal{D}$, define Λ-modules $L_1(A)$, $L_0(A)$ by

$$L_1(A) = M\begin{pmatrix} f_0 & 1_{A_1} \\ 1_{A_0} & f_1 \end{pmatrix}$$

$$L_0(A) = M\begin{pmatrix} 1_{A_0} & f_1 \\ f_0 & 1_{A_1} \end{pmatrix}$$

where $f_0 : A_0 \to A_1$, $f_1 : A_1 \to A_0$ are multiplication by x. For an object A of $\mathcal{E}$ and $\lambda \in k - \{0, 1\}$, define a Λ-module $L_\lambda(A)$ by

$$L_\lambda(A) = M\begin{pmatrix} 1_A & 1_A \\ 1_A & f \end{pmatrix}$$

where $f : A \to A$ is the map $a \mapsto (1-\lambda)a + xa$. From the table of indecomposable representations of the $\tilde{D}_4$-graph in Dlab and Ringel [1], we see the following.

PROPOSITION 1.1. *The Λ-modules*

$$L_1(V_n^j),\ L_0(V_n^j) \qquad n \geqq 0,\ j=0, 1$$

$$L_\lambda(V_n) \qquad n \geqq 0,\ \lambda \in k - \{0, 1\}$$

form a complete list of indecomposable Λ-modules.

Obviously $L_1(V_0^0) \cong k$, the trivial Λ-module. We define functors

$$\otimes \ : \mathcal{E} \times \mathcal{E} \longrightarrow \mathcal{E}$$

$$\otimes \ : \mathcal{D} \times \mathcal{D} \longrightarrow \mathcal{D}$$

$$\otimes' : \mathcal{D} \times \mathcal{D} \longrightarrow \mathcal{D}$$

$$p^* : \mathcal{D} \longrightarrow \mathcal{E}$$

$$p_* : \mathcal{E} \longrightarrow \mathcal{D}$$

$$(\overline{\ \ }) : \mathcal{D} \longrightarrow \mathcal{D}$$

in the following way. If A, B are $k[x]$-modules, the $k[x]$-module $A \otimes B$ is defined to be the vector space $A \otimes B$ on which x acts as

$$x(a \otimes b) = xa \otimes b + a \otimes xb\,.$$

If A, B are $\boldsymbol{Z}_2$-graded $k[x]$-modules, the $\boldsymbol{Z}_2$-graded $k[x]$-modules $A \otimes B$ and

$A\otimes' B$ have the underlying space $A\otimes B$, and the grading and the action of x are defined as

$$A\otimes B: (A\otimes B)_k = \bigoplus_{k=i+j} A_i\otimes B_j$$

$$x(a\otimes b) = xa\otimes b + (-1)^i a\otimes xb,\ a\in A_i,\ b\in B$$

$$A\otimes' B: (A\otimes' B)_k = A\otimes B_k$$

$$x(a\otimes b) = xa\otimes xb\,.$$

If we exhibit a $\boldsymbol{Z}_2$-graded $k[x]$-module $A=A_0\oplus A_1$ and a $k[x]$-module B by the diagrams

$$A_0 \underset{f_1}{\overset{f_0}{\rightleftarrows}} A_1 \qquad B\circlearrowleft g$$

where f_0, f_1, g are multiplication by x, the functors p_*, p^*, $\overline{(\ )}$ are defined as

$$p^*: A_0 \underset{f_1}{\overset{f_0}{\rightleftarrows}} A_1 \longmapsto A_0\circlearrowleft f_1 f_0 \oplus A_1\circlearrowleft f_0 f_1$$

$$p_*: \qquad B\circlearrowleft g \longmapsto B \underset{g}{\overset{1}{\rightleftarrows}} B \oplus B \underset{1}{\overset{g}{\rightleftarrows}} B$$

$$\overline{(\ )}: A_0 \underset{f_1}{\overset{f_0}{\rightleftarrows}} A_1 \longmapsto A_1 \underset{f_0}{\overset{f_1}{\rightleftarrows}} A_0\,.$$

THEOREM 1.2. *Let λ, $\mu\in k$ and let A, B be objects of $\mathscr{D}$ or $\mathscr{E}$. Then we have an isomorphism of Λ-modules*

$$L_\lambda(A)\otimes L_\mu(B)\cong L_{\lambda\mu}(C)$$

where C is an object of $\mathscr{D}$ or $\mathscr{E}$ defined as follows.

(i)	$C=A\otimes B$	*if* $\lambda=\mu=1$
(ii)	$C=p^*A\otimes B$	*if* $\lambda=1,\ \mu\neq 0,\ 1$
(iii)	$C=A\otimes p^*B$	*if* $\lambda\neq 0,\ 1,\ \mu=1$
(iv)	$C=A\otimes B\oplus A\otimes B$	*if* $\lambda,\ \mu\neq 0,\ 1,\ \lambda\mu\neq 1$
(v)	$C=p_*(A\otimes B)$	*if* $\lambda,\ \mu\neq 0,\ 1,\ \lambda\mu=1$
(vi)	$C=B^{\oplus\dim A}$	*if* $\lambda=1,\ \mu=0$
(vii)	$C=B^{\oplus 2\dim A}$	*if* $\lambda\neq 0,\ 1,\ \mu=0$
(viii)	$C=A^{\oplus\dim B_0}\oplus\overline{A}^{\oplus\dim B_1}$	*if* $\lambda=0,\ \mu=1$

(ix) $C=A^{\oplus\dim B}\oplus\bar{A}^{\oplus\dim B}$ *if* $\lambda=0,\ \mu\neq 0,\ 1$

(x) $C=A\otimes' B$ *if* $\lambda=\mu=0$.

Proof will be given in Section 2.

We next describe the effect of the functors $\otimes$, $\otimes'$, p^*, p_*, $\overline{(\ )}$ on indecomposable modules in $\mathcal{D}$ and $\mathcal{E}$.

PROPOSITION 1.3. (i) *We have isomorphisms in* $\mathcal{E}$

$$V_m\otimes V_n\cong\bigoplus_{l=0}^{\min(m,n)}V_{m+n-2l}$$

for all $m,\ n\geqq 0$.

(ii) *The Grothendieck ring* S *of* $(\mathcal{E},\oplus,\otimes)$ *is the polynomial ring on one generator* $[V_1]$.

This is well-known and an immediate consequence of the Clebsch-Gordan rule for tensor product of simple $\mathfrak{sl}_2$-modules. See also Littlewood [2, p. 195].

PROPOSITION 1.4. (i) *We have isomorphisms in* $\mathcal{D}$

$$V_m^i\otimes V_n^j\cong\begin{cases}\bigoplus_{l=0}^{\min(m,n)}V_{m+n-2l}^{i+j+l} & \text{if } mn \text{ is even}\\ \bigoplus_{\substack{l=0\\ l:\ \text{even}}}^{\min(m,n)-1}(V_{m+n-1-2l}^{i+j+l}\oplus V_{m+n-1-2l}^{i+j+l+1}) & \text{if } mn \text{ is odd}\end{cases}$$

for all $m,\ n\geqq 0,\ i,\ j\in\boldsymbol{Z}_2$.

(ii) *The Grothendieck ring* R *of* $(\mathcal{D},\oplus,\otimes)$ *is a commutative ring generated by the classes* $[V_0^1]$, $[V_1^0]$, $[V_2^0]$ *with defining relations*

$$[V_0^1]^2=1(=[V_0^0])$$

$$[V_1^0]^2=[V_2^0](1+[V_0^1]).$$

We shall prove this in Section 3. In fact we shall determine decomposition of tensor product of $\boldsymbol{Z}_e$-graded $k[x]$-modules for any $e\geqq 2$.

PROPOSITION 1.5. (i) *We have isomorphisms in* $\mathcal{D}$

$$V_m^i\otimes' V_n^j\cong\begin{cases}\bigoplus_{l=0}^{m-1}V_l^j\oplus\bigoplus_{l=0}^{n-m}V_m^{j+l}\oplus\bigoplus_{l=0}^{m-1}V_l^{j+n-l} & \text{if } m\leqq n\\ \bigoplus_{l=0}^{n-1}V_l^j\oplus\bigoplus_{l=0}^{m-n}V_n^j\oplus\bigoplus_{l=0}^{n-1}V_l^{j+n-l} & \text{if } m>n\end{cases}$$

for all $m,\ n\geqq 0,\ i,\ j\in\boldsymbol{Z}_2$.

(ii) *The Grothendieck ring* T *(without* 1) *of* $(\mathcal{D},\oplus,\otimes')$ *has a* $\boldsymbol{Z}$*-basis* $\{e_n^j : n\geqq 0,\ j\in\boldsymbol{Z}_2\}$, *where*

$$e_n^j=[V_n^j]-[V_{n-1}^j]-[V_{n-1}^{j+1}]+[V_{n-2}^{j+1}]$$

with the convention $V^j_{-1}=V^j_{-2}=0$ *and we have*

$$e^i_m e^j_n=\begin{cases} e^j_n & \text{if } m=n \\ 0 & \text{if } m\neq n. \end{cases}$$

PROPOSITION 1.6. (i) *We have isomorphisms*

$$p^*V^j_n\cong\begin{cases} V_{n/2}\oplus V_{n/2-1} & \text{if } n \text{ is even} \\ V_{(n-1)/2}\oplus V_{(n-1)/2} & \text{if } n \text{ is odd} \end{cases}$$

$$p_*V_n\cong V^0_{2n+1}\oplus V^1_{2n+1}$$

$$\bar{V}^j_n\cong V^{j+1}_n$$

for all $n\geqq 0$, $j\in \boldsymbol{Z}_2$.

(ii) *The functor* $p^*: \mathcal{D}\to\mathcal{E}$ *induces a surjective ring homorphism* $p^*: R\to S$ *such that*

$$p^*[V^1_0]=1, \qquad p^*[V^0_1]=2, \qquad p^*[V^0_2]=1+[V_1]$$

and the functor $p_*: \mathcal{E}\to\mathcal{D}$ *induces an injective homomorphism* $p_*: S\to R$ *such that*

$$p_*p^*(a)=(1+[V^1_0])[V^0_1]a$$

for all $a\in R$.

Proofs of Propositions 1.5, 1.6 are easy and omitted.

Combining these results, we see that the representation ring of Λ is isomorphic to the ring K defined as follows. The additive group of K is the direct sum

$$K=\bigoplus_{\lambda\in k} K_\lambda$$

where

$$K_\lambda=\begin{cases} R & \text{if } \lambda=1 \\ S & \text{if } \lambda\neq 0, 1 \\ T & \text{if } \lambda=0 \end{cases}$$

and

$R=\boldsymbol{Z}[\varepsilon, \phi_1, \phi_2]$ a commutative ring with defining relations

$\varepsilon^2=1$, $\phi_1(\phi_1-1-\varepsilon)=0$,

$S=\boldsymbol{Z}[\phi]$ a polynomial ring,

$T=\bigoplus_{n\geq 0, j=0,1}\boldsymbol{Z}e^j_n$ is a ring without 1 such that $e^i_m e^j_n=\delta_{mn}e^j_n$.

$1\in R$ is the identity element of K. For $a\in K_\lambda$, $b\in K_\mu$, the product $a\cdot b$ lies in

$K_{\lambda\mu}$ and

$$\begin{array}{lll}
\lambda=\mu=1 & \Longrightarrow a\cdot b=ab & \\
\lambda=1,\ \mu\neq 0,\ 1 & \Longrightarrow a\cdot b=p^*(a)b & \\
\lambda\neq 0,\ 1,\ \mu=1 & \Longrightarrow a\cdot b=ap^*(b) & \\
\lambda,\ \mu\neq 0,\ 1,\ \lambda\mu\neq 1 & \Longrightarrow a\cdot b=2ab & \\
\lambda,\ \mu\neq 0,\ 1,\ \lambda\mu=1 & \Longrightarrow a\cdot b=p_*(ab) & \\
\lambda=\mu=0 & \Longrightarrow a\cdot b=ab & \\
\lambda=0 & \Longrightarrow \varepsilon\cdot a=a\,, & a\cdot\varepsilon=\bar{a} \\
 & \quad\ \phi_1\cdot a=2a\,, & a\cdot\phi_1=a+\bar{a} \\
 & \quad\ \phi_2\cdot a=3a\,, & a\cdot\phi_2=2a+\bar{a} \\
 & \quad\ \psi^i\cdot a=2^{1+i}a\,, & a\cdot\psi^i=2^i(a+\bar{a})
\end{array}$$

where the multiplications in the right hand sides are those of the rings R, S or T, and

$p^*: R\to S$ is a ring homomorphism such that $\varepsilon\mapsto 1$, $\phi_1\mapsto 2$, $\phi_2\mapsto 1+\psi$

$p_*: S\to R$ is an R-linear map such that $1\mapsto(1+\varepsilon)\phi_1$

$\overline{(\ )}: T\to T$ is an additive map interchanging e_n^0 and e_n^1 for all $n\geqq 0$.

2. Proof of Theorem 1.2.

Let λ, $\mu\in k-\{0\}$ and let

$$A=\Big(A_0 \underset{f_1}{\overset{f_0}{\rightleftarrows}} A_1\Big),\qquad B=\Big(B_0 \underset{g_1}{\overset{g_0}{\rightleftarrows}} B_1\Big)$$

be $\boldsymbol{Z}_2$-graded $k[x]$-modules with the notation in Section 1 and suppose that $1-\lambda-f_0f_1$, $1-\lambda-f_1f_0$, $1-\mu-g_0g_1$, $1-\mu-g_1g_0$ are nilpotent.

We restate Theorem 1.2 in terms of the functor M as follows:

(2.1) If $\lambda=\mu=1$, then

$$M\begin{pmatrix} f_0 & 1 \\ 1 & f_1\end{pmatrix}\otimes M\begin{pmatrix} g_0 & 1 \\ 1 & g_1\end{pmatrix}\cong M\begin{pmatrix} l_0 & 1 \\ 1 & l_1\end{pmatrix}$$

where

$$A_0\otimes B_0\oplus A_1\otimes B_1 \underset{l_1}{\overset{l_0}{\rightleftarrows}} A_0\otimes B_1\oplus A_1\otimes B_0$$

$$l_0=\begin{pmatrix}1\otimes g_0 & f_1\otimes 1\\ f_0\otimes 1 & -1\otimes g_1\end{pmatrix}$$

$$l_1=\begin{pmatrix}1\otimes g_1 & f_1\otimes 1\\ f_0\otimes 1 & -1\otimes g_0\end{pmatrix}.$$

(2.2) If $\lambda\mu\neq 1$, then

$$M\begin{pmatrix}f_0 & 1\\ 1 & f_1\end{pmatrix}\otimes M\begin{pmatrix}g_0 & 1\\ 1 & g_1\end{pmatrix}\cong M\begin{pmatrix}1 & 1\\ 1 & l_0\end{pmatrix}\oplus M\begin{pmatrix}1 & 1\\ 1 & l_1\end{pmatrix}$$

where

$$1-\lambda\mu-l_0=(1-\lambda-f_1f_0)\otimes 1+1\otimes(1-\mu-g_0g_1)\in\mathrm{End}(A_0\otimes B_1)$$

$$1-\lambda\mu-l_1=(1-\lambda-f_0f_1)\otimes 1+1\otimes(1-\mu-g_1g_0)\in\mathrm{End}(A_1\otimes B_0).$$

(2.3) If λ, $\mu\neq 1$, $\lambda\mu=1$, $A_0=A_1$, $B_0=B_1$, $f_0=1$, $g_0=1$, then

$$M\begin{pmatrix}1 & 1\\ 1 & f_1\end{pmatrix}\otimes M\begin{pmatrix}1 & 1\\ 1 & g_1\end{pmatrix}\cong M\begin{pmatrix}1 & 1\\ 1 & l\end{pmatrix}\oplus M\begin{pmatrix}l & 1\\ 1 & 1\end{pmatrix}$$

where

$$-l=(1-\lambda-f_1)\otimes 1+1\otimes(1-\mu-g_1)\in\mathrm{End}(A_1\otimes B_1).$$

(2.4) If $\mu=1$, then

$$M\begin{pmatrix}f_0 & 1\\ 1 & f_1\end{pmatrix}\otimes M\begin{pmatrix}1 & g_1\\ g_0 & 1\end{pmatrix}\cong M\begin{pmatrix}1\otimes 1 & 1\otimes g_1\\ 1\otimes g_0 & 1\otimes 1\end{pmatrix}$$

where the left factor 1 in $1\otimes 1$, $1\otimes g_0$, $1\otimes g_1$ is the identity map on $A_0\oplus A_1$.

(2.5) If $\lambda=1$, then

$$M\begin{pmatrix}1 & f_1\\ f_0 & 1\end{pmatrix}\otimes M\begin{pmatrix}g_0 & 1\\ 1 & g_1\end{pmatrix}\cong M\begin{pmatrix}1\otimes 1_{B_0} & f_1\otimes 1_{B_0}\\ f_0\otimes 1_{B_0} & 1\otimes 1_{B_0}\end{pmatrix}\oplus M\begin{pmatrix}1\otimes 1_{B_1} & f_0\otimes 1_{B_1}\\ f_1\otimes 1_{B_1} & 1\otimes 1_{B_1}\end{pmatrix}.$$

(2.6) If $\lambda=\mu=1$, then

$$M\begin{pmatrix}1 & f_1\\ f_0 & 1\end{pmatrix}\otimes M\begin{pmatrix}1 & g_1\\ g_0 & 1\end{pmatrix}\cong M\begin{pmatrix}1 & f_1\otimes g_1\\ f_0\otimes g_0 & 1\end{pmatrix}\oplus M\begin{pmatrix}1 & f_0\otimes g_1\\ f_1\otimes g_0 & 1\end{pmatrix}.$$

Indeed, cases (2.1)-(2.6) correspond to cases (i)-(x) in Theorem 1.2 in the following way

(2.1) $\Longleftrightarrow$ (i)

(2.2) $\Longleftrightarrow$ (ii), (iii), (iv)

(2.3) $\Longleftrightarrow$ (v)

(2.4) $\Longleftrightarrow$ (vi), (vii)

$$(2.5) \Longleftrightarrow \text{(viii), (ix)}$$

$$(2.6) \Longleftrightarrow \text{(x)}$$

Note that in some cases the present A, B, λ, μ are different from A, B, λ, μ in Theorem 1.2.

LEMMA 2.7. *Given isomorphisms*

$$\alpha=\begin{pmatrix}\alpha_{11} & \alpha_{12}\\ \alpha_{21} & \alpha_{22}\end{pmatrix}: V_{11}\oplus V_{12} \longrightarrow V_{21}\oplus V_{22}$$

$$\beta=\begin{pmatrix}\beta_{11} & \beta_{12}\\ \beta_{21} & \beta_{22}\end{pmatrix}: W_{11}\oplus W_{12} \longrightarrow W_{21}\oplus W_{22}$$

$$\beta^{-1}=\begin{pmatrix}\beta'_{11} & \beta'_{12}\\ \beta'_{21} & \beta'_{22}\end{pmatrix}: W_{21}\oplus W_{22} \longrightarrow W_{11}\oplus W_{12}$$

with $\alpha_{ij}: V_{1j}\to V_{2i}$, $\beta_{ij}: W_{1j}\to W_{2i}$, $\beta'_{ij}: W_{2j}\to W_{1i}$, *we have an isomorphism of* Λ*-modules*

$$M(\alpha)\otimes M(\beta)\cong M(\gamma)$$

where

$$\gamma: Z_{11}\oplus Z_{12} \longrightarrow Z_{21}\oplus Z_{22}$$

$$Z_{ik}=\bigoplus_j V_{ij}\otimes W_{jk}$$

$$\gamma=\begin{pmatrix}\alpha_{11}\otimes 1 & \alpha_{12}\otimes\beta'_{11} & 0 & \alpha_{12}\otimes\beta'_{12}\\ \alpha_{21}\otimes\beta_{11} & \alpha_{22}\otimes 1 & \alpha_{21}\otimes\beta_{12} & 0\\ 0 & \alpha_{12}\otimes\beta'_{21} & \alpha_{11}\otimes 1 & \alpha_{12}\otimes\beta'_{22}\\ \alpha_{21}\otimes\beta_{21} & \otimes & \alpha_{21}\otimes\beta_{22} & \alpha_{22}\otimes 1\end{pmatrix}$$

The columns of this matrix correspond to $V_{11}\otimes W_{11}$, $V_{12}\otimes W_{21}$, $V_{11}\otimes W_{12}$, $V_{12}\otimes W_{22}$, *and the rows correspond to* $V_{21}\otimes W_{11}$, $V_{22}\otimes W_{21}$, $V_{21}\otimes W_{12}$, $V_{22}\otimes W_{22}$ *in order.*

Proof is straightforward. Now we shall prove (2.1)-(2.6).

(1) Let

$$\alpha=\begin{pmatrix}f_0 & 1\\ 1 & f_1\end{pmatrix}, \qquad \beta=\begin{pmatrix}g_0 & 1\\ 1 & g_1\end{pmatrix}.$$

Then

$$\beta^{-1}=\begin{pmatrix}(g_1g_0-1)^{-1}g_1 & -(g_1g_0-1)^{-1}\\ -(g_0g_1-1)^{-1} & (g_0g_1-1)^{-1}g_0\end{pmatrix}$$

so $M(\alpha)\otimes M(\beta)\cong M(\gamma)$ with

$$\gamma=\begin{pmatrix} f_0\otimes 1 & 1\otimes(g_1g_0-1)^{-1}g_1 & 0 & -1\otimes(g_1g_0-1)^{-1} \\ 1\otimes g_0 & f_1\otimes 1 & 1\otimes 1 & 0 \\ 0 & -1\otimes(g_0g_1-1)^{-1} & f_0\otimes 1 & 1\otimes(g_0g_1-1)^{-1}g_0 \\ 1\otimes 1 & 0 & 1\otimes g_1 & f_1\otimes 1 \end{pmatrix}$$

Multiplying an invertible matrix with γ on the left, we have

$$\gamma\cong\begin{pmatrix} 1\otimes g_0 & f_1\otimes 1 & 1\otimes 1 & 0 \\ f_0\otimes(1-g_1g_0) & -1\otimes g_1 & 0 & 1\otimes 1 \\ 1\otimes 1 & 0 & 1\otimes g_1 & f_1\otimes 1 \\ 0 & 1\otimes 1 & f_0\otimes(1-g_0g_1) & -1\otimes g_0 \end{pmatrix}=\begin{pmatrix} h_0 & 1 \\ 1 & h_1 \end{pmatrix},$$

where

$$h_0\cong\begin{pmatrix} 1\otimes g_0 & f_1\otimes 1 \\ f_0\otimes(1-g_1g_0) & -1\otimes g_1 \end{pmatrix}, \qquad h_1=\begin{pmatrix} 1\otimes g_1 & f_1\otimes 1 \\ f_0\otimes(1-g_0g_1) & -1\otimes g_0 \end{pmatrix}.$$

(1a) We shall prove (2.1). Let $\lambda=\mu=1$. Then $A, B\in\mathcal{D}$. Let l_0, l_1 be as in (2.1).

LEMMA 2.8. *The $\boldsymbol{Z}_2$-graded $k[x]$-modules*

$$A_0\otimes B_0\oplus A_1\otimes B_1 \underset{l_1}{\overset{l_0}{\rightleftarrows}} A_0\otimes B_1\oplus A_1\otimes B_0$$

$$A_0\otimes B_0\oplus A_1\otimes B_1 \underset{h_1}{\overset{h_0}{\rightleftarrows}} A_0\otimes B_1\oplus A_1\otimes B_0.$$

are isomorphic.

From this we have

$$M(\gamma)\cong M\begin{pmatrix} h_0 & 1 \\ 1 & h_1 \end{pmatrix}\cong M\begin{pmatrix} l_0 & 1 \\ 1 & l_1 \end{pmatrix}$$

which proves (2.1).

PROOF OF LEMMA 2.8. The both $\boldsymbol{Z}_2$-graded $k[x]$-modules have the common underlying graded space $A\otimes B$, and x acts on the first module as

$$x(a\otimes b)=xa\otimes b+(-1)^i a\otimes xb \quad a\in A_i$$

and on the second module as

$$x(a\otimes b)=\begin{cases} xa\otimes(1-x^2)b+a\otimes xb & \text{if } a\in A_0 \\ xa\otimes b-a\otimes xb & \text{if } a\in A_1. \end{cases}$$

We may assume that A, B are indecomposable. Let $\dim A=m$, $\dim B=n$, and let $u\in A$, $v\in B$ be homogeneous generators. Let $G=k[s, t]$ be a graded k-algebra

with defining relations $s^m=t^n=0$, $ts=-st$ and $\deg s=\deg t=1$. G acts on the vector space $A\otimes B$ in two different ways.
The first action:

$$s(a\otimes b)=xa\otimes b$$

$$t(a\otimes b)=(-1)^i a\otimes xb, \qquad a\in A_i.$$

The second action:

$$s(a\otimes b)=\begin{cases} xa\otimes(1-x^2)b & \text{if } a\in A_0 \\ xa\otimes b & \text{if } a\in A_1 \end{cases}$$

$$t(a\otimes b)=(-1)^i a\otimes xb, \qquad a\in A_i.$$

To prove the lemma, it is enough to show that these two Z_2-graded G-modules $A\otimes B$ are isomorphic. With respect to either action, $s^i t^j(u\otimes v)$ $(0\leqq i<m, 0\leqq j<n)$ form a basis of $A\otimes B$. Hence the both G-modules are free on the generator $u\otimes v$. This proves the lemma.

(1b) Suppose next that $\lambda\mu\neq 1$. We shall prove (2.2). Putting

$$k_0=f_1f_0\otimes(1-g_0g_1)+1\otimes g_0g_1$$

$$k_1=f_0f_1\otimes(1-g_1g_0)+1\otimes g_1g_0,$$

we have

$$h_0h_1=\begin{pmatrix} k_0 & 0 \\ 0 & k_1 \end{pmatrix}.$$

Since $1-k_0$, $1-k_1$ have the unique eigenvalue $\lambda\mu$, h_0h_1 is an isomorphism. Similarly h_1h_0 is an isomorphism. Therefore

$$\gamma\cong\begin{pmatrix} 1 & 1 \\ 1 & h_0h_1 \end{pmatrix}\cong\begin{pmatrix} 1 & 1 \\ 1 & k_0 \end{pmatrix}\oplus\begin{pmatrix} 1 & 1 \\ 1 & k_1 \end{pmatrix}.$$

LEMMA 2.9. *Let $s\in\operatorname{End}V$, $t\in\operatorname{End}W$ be nilpotent endomorphisms and $\lambda, \mu\in k-\{0\}$. Then $(\lambda+s)\otimes(\mu+t)-\lambda\mu$, $s\otimes 1+1\otimes t\in\operatorname{End}(V\otimes W)$ are conjugate.*

The proof of the lemma is similar to that of Lemma 2.8. Let l_0, l_1 be as in (2.2). Applying the lemma to $s=1-\lambda-f_1f_0$, $t=1-\mu-g_0g_1$, we see that k_0 and l_0 are conjugate. Similarly k_1 and l_1 are conjugate. Thus

$$\gamma\cong\begin{pmatrix} 1 & 1 \\ 1 & l_0 \end{pmatrix}\oplus\begin{pmatrix} 1 & 1 \\ 1 & l_1 \end{pmatrix}$$

which proves (2.2).

(1c) Suppose λ, $\mu\neq 1$, $\lambda\mu=1$. Let $A_0=A_1$, $f_0=1$, $B_0=B_1$, $g_0=1$. Then

$$h_0=P\begin{pmatrix}1 & 0\\ 0 & -k\end{pmatrix}Q, \qquad h_1=Q^{-1}\begin{pmatrix}k & 0\\ 0 & -1\end{pmatrix}P^{-1},$$

where P, Q are some invertible matrices and $k=f_1\otimes(1-g_1)+1\otimes g_1$. Let l_0 be as in (2.3). Using Lemma 2.9 with $s=1-\lambda-f_1$, $t=1-\mu-g_1$, we see that k and l are conjugate. Hence

$$\gamma\cong\begin{pmatrix}1 & 1\\ 1 & k\end{pmatrix}\oplus\begin{pmatrix}-k & 1\\ 1 & -1\end{pmatrix}\cong\begin{pmatrix}1 & 1\\ 1 & l\end{pmatrix}\oplus\begin{pmatrix}l & 1\\ 1 & 1\end{pmatrix}.$$

This proves (2.3).

(2) We shall prove (2.4). Let

$$\alpha=\begin{pmatrix}f_0 & 1\\ 1 & f_1\end{pmatrix}, \qquad \beta=\begin{pmatrix}1 & g_1\\ g_0 & 1\end{pmatrix}, \qquad \mu=1.$$

Then

$$\beta^{-1}=\begin{pmatrix}(1-g_1g_0)^{-1} & -(1-g_1g_0)^{-1}g_1\\ -(1-g_0g_1)^{-1}g_0 & (1-g_0g_1)^{-1}\end{pmatrix}.$$

So

$$\gamma=\begin{pmatrix}f_0\otimes 1 & 1\otimes(1-g_1g_0)^{-1} & 0 & -1\otimes(1-g_1g_0)^{-1}g_1\\ 1\otimes 1 & f_1\otimes 1 & 1\otimes g_1 & 0\\ 0 & -1\otimes(1-g_0g_1)^{-1}g_0 & f_0\otimes 1 & 1\otimes(1-g_0g_1)^{-1}\\ 1\otimes g_0 & 0 & 1\otimes 1 & f_1\otimes 1\end{pmatrix}$$

$$\cong\begin{pmatrix}1\otimes 1 & f_1\otimes 1 & 1\otimes g_1 & 0\\ f_0\otimes(g_1g_0-1) & -1\otimes 1 & 0 & 1\otimes g_1\\ 1\otimes g_0 & 0 & 1\otimes 1 & f_1\otimes 1\\ 0 & 1\otimes g_0 & f_0\otimes(g_0g_1-1) & -1\otimes 1\end{pmatrix}$$

Put

$$h_0=\begin{pmatrix}1\otimes 1 & f_1\otimes 1\\ f_0\otimes(g_1g_0-1) & -1\otimes 1\end{pmatrix}\in\mathrm{End}(A_0\otimes B_0\oplus A_1\otimes B_0)$$

$$h_1=\begin{pmatrix}1\otimes 1 & f_1\otimes 1\\ f_0\otimes(g_0g_1-1) & -1\otimes 1\end{pmatrix}\in\mathrm{End}(A_0\otimes B_1\oplus A_1\otimes B_1).$$

These are isomorphisms, so

$$\gamma\cong\begin{pmatrix}1_A\otimes 1_{B_0} & (1_A\otimes g_1)h_1^{-1}\\ (1_A\otimes g_0)h_0^{-1} & 1_A\otimes 1_{B_1}\end{pmatrix},$$

where $A=A_0\oplus A_1$. We claim that the following two objects of $\mathcal{D}$ are isomorphic.

$$A\otimes B_0 \underset{(1\otimes g_1)h_1^{-1}}{\overset{(1\otimes g_0)h_0^{-1}}{\rightleftarrows}} A\otimes B_1$$

$$A\otimes B_0 \underset{1\otimes g_1}{\overset{1\otimes g_0}{\rightleftarrows}} A\otimes B_1 .$$

Note that the isomorphism class of an object $C=C_0\oplus C_1$ of $\mathscr{D}$ is determined by the integers $\dim \mathrm{Ker}(x^n : C_i \to C_{i+n})$ for $n>0$, $i=0, 1$. Since

$$(1\otimes g_0)h_0 = h_1(1\otimes g_0), \qquad (1\otimes g_1)h_1 = h_0(1\otimes g_1),$$

we have

$$\dim \mathrm{Ker}(1\otimes g_i)h_i^{-1} \cdots (1\otimes g_{i+n})h_{i+n}^{-1} = \dim \mathrm{Ker}(1\otimes g_i) \cdots (1\otimes g_{i+n})h_{i+n}^{-n-1}$$
$$= \dim \mathrm{Ker}(1\otimes g_i) \cdots (1\otimes g_{i+n}),$$

where indices are taken modulo 2. Thus the above two objects are isomorphic. It follows that

$$\begin{pmatrix} 1 & (1\otimes g_1)h_1^{-1} \\ (1\otimes g_0)h_0^{-1} & 1 \end{pmatrix} \cong \begin{pmatrix} 1 & 1\otimes g_1 \\ 1\otimes g_0 & 1 \end{pmatrix}.$$

This proves (2.4).

(3) Let

$$\alpha = \begin{pmatrix} 1 & f_1 \\ f_0 & 1 \end{pmatrix}, \qquad \lambda = 1$$

$$\beta = \begin{pmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{pmatrix}$$

$$\beta^{-1} = \begin{pmatrix} \beta'_{11} & \beta'_{12} \\ \beta'_{21} & \beta'_{22} \end{pmatrix}.$$

Then $M(\alpha)\otimes M(\beta) \cong M(\gamma)$, where

$$\gamma = \begin{pmatrix} 1\otimes 1 & f_1\otimes\beta'_{11} & 0 & f_1\otimes\beta'_{12} \\ f_0\otimes\beta_{11} & 1\otimes 1 & f_0\otimes\beta_{12} & 0 \\ 0 & f_1\otimes\beta'_{21} & 1\otimes 1 & f_1\otimes\beta'_{22} \\ f_0\otimes\beta_{21} & 0 & f_0\otimes\beta_{22} & 1\otimes 1 \end{pmatrix}$$

$$\cong \begin{pmatrix} 1\otimes 1 - f_1 f_0\otimes\beta'_{11}\beta_{11} & 0 & 0 & f_1\otimes\beta'_{12} \\ 0 & 1\otimes 1 & f_0\otimes\beta_{12} & 0 \\ 0 & f_1\otimes\beta'_{21} & 1\otimes 1 - f_1 f_0\otimes\beta'_{22}\beta_{22} & 0 \\ f_0\otimes\beta_{21} & 0 & 0 & 1\otimes 1 \end{pmatrix}$$

$$\cong \begin{pmatrix} h_0 & f_1\otimes\beta'_{12} \\ f_0\otimes\beta_{21} & 1\otimes 1 \end{pmatrix} \oplus \begin{pmatrix} 1\otimes 1 & f_0\otimes\beta_{12} \\ f_1\otimes\beta'_{21} & h_1 \end{pmatrix}$$

with

$$h_0 = 1\otimes 1 - f_1 f_0\otimes\beta'_{11}\beta_{11}$$

$$h_1=1\otimes 1-f_1f_0\otimes\beta'_{22}\beta_{22}.$$

Since f_1f_0 is nilpotent, h_0, h_1 are isomorphisms. Hence

$$\gamma\cong\begin{pmatrix}1 & h_0^{-1}(f_1\otimes\beta'_{12})\\ f_0\otimes\beta_{21} & 1\end{pmatrix}\oplus\begin{pmatrix}1 & f_0\otimes\beta_{12}\\ h_1^{-1}(f_1\otimes\beta'_{21}) & 1\end{pmatrix}.$$

(3a) To prove (2.5) we let

$$\beta=\begin{pmatrix}g_0 & 1\\ 1 & g_1\end{pmatrix}.$$

Then

$$\gamma\cong\begin{pmatrix}1 & k_0^{-1}(f_1\otimes 1)\\ f_0\otimes 1 & 1\end{pmatrix}\oplus\begin{pmatrix}1 & f_0\otimes 1\\ k_1^{-1}(f_1\otimes 1) & 1\end{pmatrix},$$

where

$$k_0=f_1f_0\otimes g_1g_0-1\otimes g_1g_0+1\otimes 1$$
$$k_1=f_1f_0\otimes g_0g_1-1\otimes g_0g_1+1\otimes 1.$$

Put

$$k'_0=f_0f_1\otimes g_1g_0-1\otimes g_1g_0+1\otimes 1$$
$$k'_1=f_0f_1\otimes g_0g_1-1\otimes g_0g_1+1\otimes 1.$$

These are isomorphisms and we have

$$\begin{cases}(f_0\otimes 1)k_0=k'_0(f_0\otimes 1)\\ (f_1\otimes 1)k'_0=k_0(f_1\otimes 1)\end{cases}\qquad\begin{cases}(f_0\otimes 1)k_1=k'_1(f_0\otimes 1)\\ (f_1\otimes 1)k'_1=k_1(f_1\otimes 1).\end{cases}$$

Then, by the same argument as in (2), we know that there are isomorphisms in $\mathscr{D}$

$$\begin{array}{cc} A_0\otimes B_0 \underset{k_0^{-1}(f_1\otimes 1)}{\overset{f_0\otimes 1}{\rightleftarrows}} A_1\otimes B_0 & A_1\otimes B_1 \underset{f_0\otimes 1}{\overset{k_1^{-1}(f_1\otimes 1)}{\rightleftarrows}} A_0\otimes B_1\\ \wr\| & \wr\| \\ A_0\otimes B_0 \underset{f_1\otimes 1}{\overset{f_0\otimes 1}{\rightleftarrows}} A_1\otimes B_0 & A_1\otimes B_1 \underset{f_0\otimes 1}{\overset{f_1\otimes 1}{\rightleftarrows}} A_0\otimes B_1.\end{array}$$

Thus

$$\gamma\cong\begin{pmatrix}1 & f_1\otimes 1\\ f_0\otimes 1 & 1\end{pmatrix}\oplus\begin{pmatrix}1 & f_0\otimes 1\\ f_1\otimes 1 & 1\end{pmatrix}$$

which proves (2.5).

(3b) Finally we prove (2.6). Let

$$\beta=\begin{pmatrix}1 & g_1\\ g_0 & 1\end{pmatrix},\qquad \mu=1.$$

Then

$$\gamma \cong \begin{pmatrix} 1 & k_0^{-1}(f_1 \otimes g_1) \\ f_0 \otimes g_0 & 1 \end{pmatrix} \oplus \begin{pmatrix} 1 & f_0 \otimes g_1 \\ k_1^{-1}(f_1 \otimes g_0) & 1 \end{pmatrix},$$

where

$$k_0 = f_1 f_0 \otimes 1 + 1 \otimes g_1 g_0 - 1 \otimes 1$$
$$k_1 = f_1 f_0 \otimes 1 + 1 \otimes g_0 g_1 - 1 \otimes 1.$$

Put

$$k_0' = f_0 f_1 \otimes 1 + 1 \otimes g_0 g_1 - 1 \otimes 1$$
$$k_1' = f_0 f_1 \otimes 1 + 1 \otimes g_1 g_0 - 1 \otimes 1.$$

Then

$$\begin{cases} (f_0 \otimes g_0) k_0 = k_0'(f_0 \otimes g_0) \\ (f_1 \otimes g_1) k_0' = k_0 (f_1 \otimes g_1) \end{cases} \qquad \begin{cases} (f_0 \otimes g_1) k_1 = k_1'(f_0 \otimes g_1) \\ (f_1 \otimes g_0) k_1' = k_1 (f_1 \otimes g_0). \end{cases}$$

As in (2) there are isomorphisms in $\mathcal{D}$

$$\begin{array}{ccc} A_0 \otimes B_0 \underset{k_0^{-1}(f_1 \otimes g_1)}{\overset{f_0 \otimes g_0}{\rightleftarrows}} A_1 \otimes B_1 & & A_1 \otimes B_0 \underset{f_0 \otimes g_1}{\overset{k_1^{-1}(f_1 \otimes g_0)}{\rightleftarrows}} A_0 \otimes B_1 \\ \wr\| & & \wr\| \\ A_0 \otimes B_0 \underset{f_1 \otimes g_1}{\overset{f_0 \otimes g_0}{\rightleftarrows}} A_1 \otimes B_1 & & A_1 \otimes B_0 \underset{f_0 \otimes g_1}{\overset{f_1 \otimes g_0}{\rightleftarrows}} A_0 \otimes B_1. \end{array}$$

Thus

$$\gamma \cong \begin{pmatrix} 1 & f_1 \otimes g_1 \\ f_0 \otimes g_0 & 1 \end{pmatrix} \oplus \begin{pmatrix} 1 & f_0 \otimes g_1 \\ f_1 \otimes g_0 & 1 \end{pmatrix}.$$

This proves (2.6).

3. Tensor product of graded $k[x]$-modules.

Througout this section we fix $\omega \in k$ a primitive e^{th} root of unity with $e \geqq 2$. By a graded $k[x]$-module we mean a $k[x]$-module $M = \oplus_{i \in \boldsymbol{Z}} M_i$ such that $\dim M < \infty$, $xM_i \subset M_{i+1}$ for all $i \in \boldsymbol{Z}$. If M, N are graded $k[x]$-modules we make the vector space $M \otimes N$ a graded $k[x]$-module in the following way.

$$(M \otimes N)_k = \bigoplus_{i+j=k} M_i \otimes N_j$$

$$x(a \otimes b) = xa \otimes b + \omega^i a \otimes xb \quad a \in M_i,\ b \in N.$$

This operation $\otimes$ on graded $k[x]$-modules is associative. For each $m \geqq 0$ and $i \in \boldsymbol{Z}$, let V_m^i be a graded $k[x]$-module of dimension $m+1$ generated by an element of degree i. The modules V_m^i for $m \geqq 0$, $i \in \boldsymbol{Z}$ furnish a complete list of indecomposable graded $k[x]$-modules. The main result of this section is the following.

THEOREM 3.1. *For any $m, n \geqq 0$ we have an isomorphism of graded $k[x]$-modules*

$$V_m^0 \otimes V_n^0 \cong \bigoplus_{l=0}^{\min(m,n)} V_{l*}^l,$$

where $l \mapsto l_*$ is defined in the following way. Write $m=re+i$, $n=se+j$, $l=qe+h$ with $r, s, q \in \boldsymbol{N}$, $0 \leqq i, j, h < e$.

$l_*=m+n-2l$	*if*	$\max(i+j-e+2, 0) \leqq h \leqq \min(i, j)$
	or if	$\max(i, j)+1 \leqq h \leqq \min(i+j+1, e-1)$
$l_*=(r+s-2q+1)e-1$	*if*	$0 \leqq h \leqq i+j-e+1$
$l_*=(r+s-2q)e-1$	*if*	$\min(i, j)+1 \leqq h \leqq \max(i, j)$
$l_*=(r+s-2q-1)e-1$	*if*	$i+j+2 \leqq h \leqq e-1$.

Here we understand $V_{-1}^l=0$.

Proposition 1.4 (i) follows from this, by letting $e=2$ and reducing the grading modulo 2. See also Lemma 3.5 and the end of this section.

The proof of Theorem 3.1 goes as follows. We first decompose $V_m^i \otimes V_1^0$, $V_1^0 \otimes V_m^i$, $V_{re}^0 \otimes V_e^0$ directly. In the Grothendieck ring we can express all $[V_m^i]$ as polynomials of $[V_0^j]$, $[V_1^0]$, $[V_e^0]$. Then a straightforward computation gives the desired formula.

We begin with preliminary observation. Let $m, n \geqq 0$ and let $G=k[s, t]$ be a graded k-algebra with defining relations $ts=\omega st$, $s^{m+1}=t^{n+1}=0$ and $\deg s= \det t=1$. Let G_k be the degree k part of G for each $k \geqq 0$. Put $x=s+t$. Since

$$x \cdot s^i t^j = s^{i+1} t^j + \omega^i s^i t^{j+1},$$

when G is viewed as a graded $k[x]$-module by left multiplication, G is isomorphic to $V_m^0 \otimes V_n^0$. Since $tx=\omega xt+(1-\omega)t^2$ and

$$0=s^{m+1}=(x-t)^{m+1}=x^{m+1}+c_1 x^m t+\cdots+c_{m+1} t^{m+1}$$

for some $c_1, \cdots, c_{m+1} \in k$, G has a basis $x^i t^j$, $0 \leqq i \leqq m$, $0 \leqq j \leqq n$. Assume $m \geqq n$ and put

$$z=x^m+c_1 x^{m-1} t+\cdots+c_m t^m.$$

Then the following hold.

(i) The left multiplication $x: G_k \to G_{k+1}$ is injective for $k<n$, bijective for $n \leqq k<m$, and surjective for $m \leqq k$.

(ii) G/xG has a basis $t^j \bmod xG$, $0 \leqq j \leqq n$.

(iii) $\mathrm{Ker}(x : G \to G)$ has a basis zt^j, $0 \leqq j \leqq n$.

(iv) For each $0 \leqq j \leqq n$, put

$$l_j = \sup\{l : zt^j \in x^l G_{m+j-l}\} .$$

Then

$$G \cong \bigoplus_{j=0}^{n} V_{l_j}^{m+j-l_j}$$

as graded $k[x]$-modules.

(i) is clear and (ii), (iii) follow from (i). To see (iv), decompose $G = \bigoplus_i k[x]u_i$ with u_i homogeneous elements such that $x^{m_i}u_i \neq 0$, $x^{m_i+1}u_i = 0$. Then the elements $x^{m_i}u_i$ form a basis of $\mathrm{Ker}(x : G \to G)$. Since zt^j, $0 \leqq j \leqq n$, have mutually different degrees $m+j$, the bases $\{zt^j\}$ and $\{x^{m_i}u_i\}$ of $\mathrm{Ker}(x : G \to G)$ are equal up to a permutation and scalar multiples. Hence $\{l_j\}$ is a permutation of $\{m_i\}$. This proves (iv).

LEMMA 3.2. *For any $m \geqq 0$ we have*

$$V_m^0 \otimes V_1^0 \cong \begin{cases} V_{m+1}^0 \oplus V_{m-1}^1 & \text{if } m+1 \not\equiv 0 \pmod e \\ V_m^0 \oplus V_m^1 & \text{if } m+1 \equiv 0 \pmod e). \end{cases}$$

PROOF. We may assume $m > 0$. In the above observation we specialize (m, n) to $(m, 1)$. Then $t^2 = 0$, $tx = \omega xt$ and

$$0 = (x-t)^{m+1} = x^{m+1} - \frac{\omega^{m+1}-1}{\omega-1} x^m t ,$$

so

$$z = x^m - \frac{\omega^{m+1}-1}{\omega-1} x^{m-1} t$$

$$zt = x^m t .$$

If $m+1 \not\equiv 0$, then $(\omega^{m+1}-1)/(\omega-1) \neq 0$, so

$$z \in x^{m-1}G_1 , \qquad z \notin x^m G_0$$

$$zt = \frac{\omega-1}{\omega^{m+1}-1} x^{m+1} \in x^{m+1} G_0 .$$

Thus, by (iv) of the observation, $G \cong V_{m-1}^1 \oplus V_{m+1}^0$ as graded $k[x]$-modules.

If $m+1 \equiv 0$, then $z = x^m$, $x^{m+1} = 0$. So $zt \notin x^{m+1}G_0$. Thus $G \cong V_m^0 \oplus V_m^1$.

LEMMA 3.3. *For any $r > 0$ we have*

$$V_{re}^0 \otimes V_e^0 \cong V_{(r+1)e}^0 \oplus V_{(r+1)e-2}^1 \oplus V_{re-1}^2 \oplus \cdots \oplus V_{re-1}^{e-1} \oplus V_{(r-1)e}^e .$$

PROOF. We specialize (m, n) in the previous observation to (re, e) Then $t^{e+1}=0$, $x^{e}=s^{e}+t^{e}$ and s^{e}, t^{e} are central elements in G. We have

$$0=(x-t)^{re+1}=(x^{e}-t^{e})^{r}(x-t)=x^{re+1}-x^{re}t-rx^{(r-1)e+1}t^{e},$$

so

$$z=x^{re}-x^{re-1}t-rx^{(r-1)e}t^{e}$$

and

$$zt^{j}=x^{re}t^{j}-x^{re-1}t^{j+1}, \qquad 1\leqq j\leqq e-1$$

$$zt^{e}=x^{re}t^{e}.$$

Let us determine the integers $l_{j}:=\sup\{l: zt^{j}\in x^{l}G_{re+j-l}\}$ for $0\leqq j\leqq e$. Clearly $l_{0}=(r-1)e$. By induction on j, we see easily that

$$x^{re+j}=x^{re}t^{j}+rx^{(r-1)e+j}t^{e}, \qquad j\geqq 1$$

$$x^{re}G_{j}=\langle x^{re}t^{j}, x^{(r-1)e+j}t^{e}\rangle, \qquad j\geqq 1.$$

It follows that $x^{re-1}t^{j+1}\notin x^{re}G_{j}$ for $1\leqq j<e-1$, hence $l_{j}=re-1$. We have

$$x^{re+e-1}-(r+1)x^{re+e-2}t=-rzt^{e-1},$$

and x^{re+e-1}, zt^{e-1} are linearly independent. So $l_{e-1}=re+e-2$. Finally, since $x^{re+e}=(r+1)zt^{e}$, we have $l_{e}=re+e$. Thus

$$G\cong V_{(r+1)e}^{0}\oplus V_{(r+1)e-2}^{1}\oplus V_{re-1}^{2}\oplus\cdots\oplus V_{re-1}^{e-1}\oplus V_{(r-1)e}^{e}$$

as graded $k[x]$-modules.

LEMMA 3.4. *$V_{1}^{0}\otimes V_{m}^{0}\cong V_{m}^{0}\otimes V_{1}^{0}$ for all $m\geqq 0$.*

PROOF. We can decompose $V_{1}^{0}\otimes V_{m}^{0}$ in the same manner as $V_{m}^{0}\otimes V_{1}^{0}$.

LEMMA 3.5. *$V_{0}^{i}\otimes V_{n}^{j}\cong V_{n}^{i+j}\cong V_{n}^{j}\otimes V_{0}^{i}$ for all $n\geqq 0$ and $i, j\in\boldsymbol{Z}$.*

PROOF. Let u, v, w be homogeneous generators of V_{0}^{i}, V_{n}^{j}, V_{n}^{i+j} respectively. The correspondences $\omega^{ki}u\otimes x^{k}v\leftrightarrow x^{k}w\leftrightarrow x^{k}v\otimes u$, $0\leqq k\leqq n$, give the isomorphisms.

Let Q be the Grothendieck ring of the category of graded $k[x]$-modules with respect to $\oplus$, $\otimes$. The classes $[V_{n}^{j}]$ in Q form a basis of Q. We set

$$\varepsilon=[V_{0}^{1}]$$

$$\phi_{n}=[V_{n}^{0}] \qquad n\geqq 0$$

$$\phi_{-1}=0.$$

Then $\phi_{0}=1$ and by Lemma 3.5 ε is a central invertible element in Q and

$$[V_n^j]=\varepsilon^j\phi_n \qquad n\geqq 0,\ j\in\boldsymbol{Z}.$$

By Lemma 3.4 ϕ_1 is also central and by Lemma 3.2

$$\phi_m\phi_1=\begin{cases}\phi_{m+1}+\varepsilon\phi_{m-1} & \text{if } m+1\not\equiv 0 \pmod e\\ (1+\varepsilon)\phi_m & \text{if } m+1\equiv 0 \pmod e\end{cases} \tag{3.6}$$

for $m\geqq 0$ and by Lemma 3.3

$$\phi_{re}\phi_e=\phi_{(r+1)e}+\varepsilon\phi_{(r+1)e-2}+(\varepsilon^2+\cdots+\varepsilon^{e-1})\phi_{re-1}+\varepsilon^e\phi_{(r-1)e} \tag{3.7}$$

for $r>0$. It follows that Q is generated by ε, ε^{-1}, ϕ_1, ϕ_e and in particular Q is commutative.

For each integer $n\geqq -1$, define a polynomial $H_n(s, t)$ with integral coefficients by

$$H_n(x+y,\ xy)=\frac{x^{n+1}-y^{n+1}}{x-y}$$

with x, y indeterminates. Then $H_{-1}=0$, $H_0=1$ and we have a formula

$$H_m(s,\ t)H_n(s,\ t)=\sum_{l=0}^{\min(m,n)} t^l H_{m+n-2l}(s,\ t)$$

for m, $n\geqq -1$. Put

$$\theta_n=H_n(\phi_e-\varepsilon\phi_{e-2},\ \varepsilon^e)\in Q$$

$$\sigma_n=H_n(\phi_1,\ \varepsilon)\in Q$$

for $n\geqq -1$. Then

$$\theta_m\theta_n=\sum_{l=0}^{\min(m,n)}\varepsilon^{le}\theta_{m+n-2l} \tag{3.8}$$

$$\sigma_m\sigma_n=\sum_{l=0}^{\min(m,n)}\varepsilon^{l}\sigma_{m+n-2l}\,. \tag{3.9}$$

By an easy induction it follows from (3.6) and (3.9) that

$$\sigma_i=\phi_i \qquad 0\leqq i\leqq e-1 \tag{3.10}$$

$$\sigma_{e-1+i}=(1+\varepsilon^i)\phi_{e-1}-\varepsilon^i\phi_{e-1-i} \qquad 0\leqq i\leqq e-1. \tag{3.11}$$

LEMMA 3.12. *We have*

$$\phi_i\phi_j=\sum_{h=\max(i+j-e+2,0)}^{\min(i,j)}\varepsilon^h\phi_{i+j-2h}+\sum_{h=0}^{i+j-e+1}\varepsilon^h\phi_{e-1}$$

for $-1\leqq i,\ j\leqq e-1$.

PROOF. We may assume $i\geqq j\geqq 0$. When $i+j\leqq e-2$, the formula results from (3.9), (3.10). Let $i+j=e-1+l$ with $0\leqq l\leqq e-1$. Then by (3.9) and (3.11) we have

$$\begin{aligned}\phi_i\phi_j&=\sigma_i\sigma_j\\&=\sum_{h=0}^{j}\varepsilon^h\sigma_{i+j-2h}\\&=\sum_{0\leq h\leq l/2}\varepsilon^h\{(1+\varepsilon^{l-2h})\phi_{e-1}-\varepsilon^{l-2h}\phi_{e-1-l+2h}\}+\sum_{l/2<h\leq j}\varepsilon^h\phi_{e-1+l-2h}\\&=\sum_{0\leq h\leq l/2}(\varepsilon^h+\varepsilon^{l-h})\phi_{e-1}-\sum_{0\leq h\leq l/2}\varepsilon^{l-h}\phi_{e-1-l+2h}\\&\quad+\sum_{l/2<h\leq l}\varepsilon^h\phi_{e-1+l-2h}+\sum_{l<h\leq j}\varepsilon^h\phi_{e-1+l-2h}\\&=\sum_{h=0}^{l}\varepsilon^h\phi_{e-1}+\sum_{h=l+1}^{j}\varepsilon^h\phi_{i+j-2h},\end{aligned}$$

which proves the lemma.

LEMMA 3.13. *$\phi_{re+i}=\theta_r\phi_i+\varepsilon^{i+1}\theta_{r-1}\phi_{e-2-i}$ for $r\geqq 0$, $0\leqq i\leqq e-1$.*

PROOF. Denoting by ϕ'_{re+i} the right hand side, it is enough to show that

$$\phi'_0=1$$

$$\phi'_e=\phi_e$$

$$\phi'_{re+i}\phi_1=\phi'_{re+i+1}+\varepsilon\phi'_{re+i-1}\qquad 0\leqq i\leqq e-2,\ r\geqq 0$$

$$\phi'_{re}\phi_e=\phi'_{(r+1)e}+\varepsilon\phi'_{(r+1)e-2}+(\varepsilon^2+\cdots+\varepsilon^{e-1})\phi'_{re-1}+\varepsilon^e\phi'_{(r-1)e}\qquad r>0.$$

The second equality follows from the definition of θ_1 and the third follows from (3.6) without difficulty. For the last, using (3.8) and Lemma 3.12, we have

$$\begin{aligned}\phi'_{re}\phi_e&=(\theta_r+\varepsilon\theta_{r-1}\phi_{e-2})(\theta_1+\varepsilon\phi_{e-2})\\&=\theta_r\theta_1+\varepsilon\theta_{r-1}\theta_1\phi_{e-2}+\varepsilon\theta_r\phi_{e-2}+\varepsilon^2\theta_{r-1}\phi_{e-2}^2\\&=\theta_{r+1}+\varepsilon^e\theta_{r-1}+\varepsilon(\theta_r+\varepsilon^e\theta_{r-2})\phi_{e-2}\\&\quad+\varepsilon\theta_r\phi_{e-2}+\varepsilon^2\theta_{r-1}(\varepsilon^{e-2}\phi_0+(1+\varepsilon+\cdots+\varepsilon^{e-3})\phi_{e-1})\\&=\theta_{r+1}+\varepsilon\theta_r\phi_{e-2}+\varepsilon(\theta_r\phi_{e-2}+\varepsilon^{e-1}\theta_{r-1}\phi_0)\\&\quad+(\varepsilon^2+\cdots+\varepsilon^{e-1})\theta_{r-1}\phi_{e-1}+\varepsilon^e(\theta_{r-1}+\varepsilon\theta_{r-2}\phi_{e-2}),\end{aligned}$$

as required.

PROOF OF THEOREM 3.1. From Lemmas 3.12 and 3.13 we can deduce easily that

$$\phi_{re+i}\phi_j=\sum_{h=\max(i+j-e+2,0)}^{\min(i,j)}\varepsilon^h\phi_{re+i+j-2h}+\sum_{h=0}^{i+j-e+1}\varepsilon^h\phi_{(r+1)e-1}+\sum_{h=i+1}^{j}\varepsilon^h\phi_{re-1}$$

for $r \geqq 0$, $0 \leqq i \leqq e-1$, $-1 \leqq j \leqq e-1$. Replacing j by $e-2-j$ and multiplying ε^{j+1}, we have

$$\phi_{re+i}\varepsilon^{j+1}\phi_{e-2-j}=\sum_{h=\max(i,j)+1}^{\min(i+j+1,e-1)}\varepsilon^h\phi_{re+i+j+e-2h}+\sum_{h=j+1}^{i}\varepsilon^h\phi_{(r+1)e-1}+\sum_{h=i+j+2}^{e-1}\varepsilon^h\phi_{re-1}$$

for $r \geqq 0$, $0 \leqq i$, $j \leqq e-1$. Using (3.8) and Lemma 3.13, we can also see

$$\phi_{re+k}\theta_s=\sum_{q=0}^{\min(r,s)}\varepsilon^{qe}\phi_{(r+s-2q)e+k}$$

if $r \geqq 0$, $r \geqq s \geqq -1$, $0 \leqq k \leqq e-1$ or if r, $s \geqq -1$, $k=e-1$.

Now let $m=re+i$, $n=se+j$ with r, $s \geqq 0$, $0 \leqq i$, $j \leqq e-1$. The formula to prove is symmetric in m, n, so we may assume $r \geqq s$. By the above three formulas, we have

$$\begin{aligned}\phi_{re+i}\phi_{se+j}&=\phi_{re+i}\phi_j\theta_s+\phi_{re+i}\varepsilon^{1+j}\phi_{e-2-j}\phi_{s-1}\\&=\sum_{(1)}\varepsilon^{qe+h}\phi_{(r+s-2q)e+i+j-2h}+\sum_{(2)}\varepsilon^{qe+h}\phi_{(r+s-1-2q)e+i+j+e-2h}\\&\quad+\sum_{(3)}\varepsilon^{qe+h}\phi_{(r+s-2q)e+e-1}+\sum_{(4)}\varepsilon^{qe+h}\phi_{(r-1+s-2q)e+e-1}\\&\quad+\sum_{(5)}\varepsilon^{qe+h}\phi_{(r+s-1-2q)e+e-1}+\sum_{(6)}\varepsilon^{qe+h}\phi_{(r-1+s-1-2q)e+e-1},\end{aligned}$$

where the k^{th} summation $\sum_{(k)}$ is over the elements (q, h) in the set I_k defined below.

$I_1: 0 \leqq q \leqq \min(r, s)$, $\max(i+j-e+2, 0) \leqq h \leqq \min(i, j)$

$I_2: 0 \leqq q \leqq \min(r, s-1)$, $\max(i, j)+1 \leqq h \leqq \min(i+j+1, e-1)$

$I_3: 0 \leqq q \leqq \min(r, s)$, $0 \leqq h \leqq i+j-e+1$

$I_4: 0 \leqq q \leqq \min(r-1, s)$, $i+1 \leqq h \leqq j$

$I_5: 0 \leqq q \leqq \min(r, s-1)$, $j+1 \leqq h \leqq i$

$I_6: 0 \leqq q \leqq \min(r-1, s-1)$, $i+j+2 \leqq h \leqq e-1$.

As observed earlier, $(V_m^0 \otimes V_n^0)/x(V_m^0 \otimes V_n^0)$ has a basis consisting of homogeneous elements of degrees $0, 1, \cdots, \min(m, n)$. Therefore the map $I_1 \amalg \cdots \amalg I_6 \to [0, \min(m, n)]$ taking (q, h) to $qe+h$ must be a bijection. Since the ranges of h in $I_1, \cdots, I_6$ give a partition of $[0, e-1]$, putting $l=qe+h$, we have

$$\phi_m\phi_n=\sum_{l=0}^{\min(m,n)}\varepsilon^l\phi_{l*}$$

with l_* as described in Theorem 3.1. This proves the theorem.

PROPOSITOIN 3.14. *The ring Q is a commutative ring generated by ε, ε^{-1}, ϕ_1, ϕ_e with a defining relation*

$$H_{e-1}(\phi_1, \varepsilon)(\phi_1-1-\varepsilon)=0.$$

PROOF. This follows from (3.6) and the fact that $\{\varepsilon^k\phi_1^i\phi_e^r : k\in\boldsymbol{Z}, 0\leqq i\leqq e-1, r\geqq 0\}$ is a basis of Q. Details are omitted.

Finally we pass from the $\boldsymbol{Z}$-graded case to the $\boldsymbol{Z}_e$-graded case. We consider only $\boldsymbol{Z}_e(=\boldsymbol{Z}/e\boldsymbol{Z})$-graded $k[x]$-modules $M=\bigoplus_{i\in\boldsymbol{Z}_e}M_i$ such that $xM_i\subset M_{i+1}$ for all $i\in\boldsymbol{Z}_e$ and x acts on M nilpotently. For such modules M, N, we make the space $M\otimes N$ a $\boldsymbol{Z}_e$-graded $k[x]$-module in the same manner as in the beginning of this section. For a graded $k[x]$-module M, let π_*M be the $\boldsymbol{Z}_e$-graded $k[x]$-module such that $\pi_*M=M$ as $k[x]$-modules and $(\pi_*M)_j=\bigoplus_{\pi(i)=j}M_i$ for $j\in\boldsymbol{Z}_e$, where $\pi:\boldsymbol{Z}\to\boldsymbol{Z}_e$ is the natural projection. Then the assignment $M\mapsto\pi_*M$ commutes with $\otimes$, and the objects $\pi_*V_n^j$, $n\geqq 0$, $0\leqq j\leqq e-1$, form a complete list of indecomposable $\boldsymbol{Z}_e$-graded $k[x]$-modules. Therefore the Grothendieck ring of the category of $\boldsymbol{Z}_e$-graded $k[x]$-modules is isomorphic to $Q/(\varepsilon^e-1)$. When $e=2$, we obtain Proposition 1.4 (ii) from Proposition 3.14.

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