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UNIQUENESS AND EXISTENCE OF DUALITIES OVER COMPACT RINGS

Dedicated to Santuzza Ghezzo Baldassarri

By

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0. Introduction

0.1. In relatively recent times it has been proved that some classical dualities between additive categories are unique. For example, Roeder [13] proved that any duality on the category of locally compact abelian groups coincides, up to natural equivalences, with the Pontryagin duality.

Inspired by this fact, I. Prodanov [11] held at Sofia University, at the end of the 70's, a seminar on dualities and spectral spaces, suggesting some similar results: we recall those by Dimov [2], Stoyanov [14] and the first author [4].

Dimov proved that Stone duality is the unique duality between the category of Hausdorff compact totally disconnected spaces and the category of Boolean rings.

Let (A, σ) be a compact ring and denote by $\mathcal{L}\text{-}A_\sigma$ $(A_\sigma\text{-}\mathcal{L})$ the category of locally compact right (left) topological modules over (A, σ). L. Stoyanov proved that, if A is commutative, then the unique duality between $\mathcal{L}\text{-}A_\sigma$ and $A_\sigma\text{-}\mathcal{L}$ is the Pontryagin duality, by using the Theorem of Kaplansky and Zelinsky on the decomposition of a *commutative* compact ring as a product of local rings.

Stoyanov's theorem has been extended by the first author [4] to the non commutative case, by using his results on equivalences between closed categories of modules [5].

Unfortunately the activity of Ivan Prodanov, who inspired this line of research, was interrupted by his untimely death in April 1985.

0.2. If we use the result of Stoyanov and Gregorio, it is easy to show that if (A, σ) and (R, τ) are compact rings, then, if a duality $H=(H_1, H_2)$

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$$\mathcal{L}\text{-}A_\sigma \underset{H_2}{\overset{H_1}{\rightleftarrows}} R_\tau\text{-}\mathcal{L}$$

between $\mathcal{L}\text{-}A_\sigma$ and $R_\tau\text{-}\mathcal{L}$ exists, it is unique.

0.3. The present paper is subdivided into two parts. In the first part (Sections 1 to 4) we give a completely new proof of the uniqueness stated above by showing, in the meantime, how the duality H acts on modules.

Namely, if we set

$$K_A=H_2(R, \tau) \quad \text{and} \quad {}_RK=H_1(A, \sigma),$$

the *discrete* bimodule ${}_RK_A$ is faithfully balanced, in the sense that $R\cong\mathrm{End}(K_A)$ and $A\cong\mathrm{End}({}_RK)$ canonically (see [9]). Moreover K_A is an injective cogenerate of the category Mod-A_σ of all discrete right A-modules which are topological modules over (A, σ) and, similarly, ${}_RK$ is an injective cogenerator of the category R_τ-Mod.

The crucial result is the fact that the structure of the bimodule ${}_RK_A$ depends only on the rings (A, σ) and (R, τ) and not on the duality $H=(H_1, H_2)$. Moreover we can prove that, for all $M\in\mathcal{L}\text{-}A_\sigma$, $H_1(M)$ is naturally (and topopologically) isomorphic to the left R-module $\mathrm{Chom}_A(M, K_A)$ of continuous morphisms of M into K_A, endowed with the compact-open topology. A similar result holds, of course, for all modules $N\in R_\tau\text{-}\mathcal{L}$.

If a duality H exists, then the modules K_A and ${}_RK$ have *finite grade*, that is, the isotypic components of their socles are finitely generated.

In the second part of the paper (Section 5), given a compact ring (A, σ), we determine all compact rings (R, τ) such that there exists a duality between $\mathcal{L}\text{-}A_\sigma$ and $R_\tau\text{-}\mathcal{L}$ in the followingw ay. Let K_A be an injective cogenerator of Mod$-A_\sigma$ with finite grade and set $R=\mathrm{End}(K_A)$, with its K-topology τ. Then (R, τ) is compact, ${}_RK$ is an injective cogenerator, with finite grade, of R_τ-Mod and the bimodule ${}_RK_A$ is faithfully balanced. Let $M\in\mathcal{L}\text{-}A_\sigma$ and let $H_1(M)$ be the left R-module $\mathrm{Chom}_A(M, K_A)$, with the compact-open topology: then $H_1(M)\in R_\tau\text{-}\mathcal{L}$. If we define analogously a functor $H_2: R_\tau\text{-}\mathcal{L}\to\mathcal{L}\text{-}A_\sigma$, we get a duality $H=(H_1, H_2)$ between $\mathcal{L}\text{-}A_\sigma$ and $R_\tau\text{-}\mathcal{L}$. Finally, we give necessary and sufficient conditions under which (R, τ) is topologically isomorphic to (A, σ).

0.4. All rings considered in this paper have identity $1\neq 0$ and all modules are unital. The categories and functors we consider are always additive and subcategories are full; since we deal only with categories of (topological) modules, we use the convention of writing all morphisms on the side opposite to the scalars, unless the contrary is explicitly stated. All ring and module topologies are assumed to be Hausdorff. The symbol (M, ε) generally means that the module M is endowed with the topology ε.

1. Preliminary results

1.1. Let (R, τ) be a compact ring. It is known that (R, τ) is a linearly topologized ring having as a base of neighborhoods of zero a family of two-sided ideals which, of course, have finite index.

Let R_τ-$\mathcal{L}$ be the category of locally compact left modules over the topological ring (R, τ), where the morphisms are the R-linear continuous morphisms. It is known that every object M in R_τ-$\mathcal{L}$ is linearly topologized; more precisely, M has a base of neighborhoods of zero consisting of compact and open R-submodules (for an account of this see, e.g. [12]).

1.2. Let $\mathcal{I}_\tau$ be the family of all open two-sided ideals of (R, τ). Observe that (R, τ) is topologically artinian and noetherian on both sides, since R/I is a finite module for any $I \in \mathcal{I}_\tau$, hence artinian and noetherian. Denote by R_τ-Mod the full subcategory of R-Mod defined as follows.

$$R_\tau\text{-Mod} = \{M \in R\text{-Mod} : \forall x \in M,\ \mathrm{Ann}_R(x) \geqq I, \text{ for some } I \in \mathcal{I}_\tau\}.$$

Thus R_τ-Mod is the category of all left R-modules which, with the discrete topology, are topological modules over the topological ring (R, τ). Note that, for any $M \in R_\tau$-Mod, every finitely generated submodule of M is finite. For any $M \in R$-Mod we set

$$\mathrm{t}_\tau(M) = \{x \in M : \mathrm{Ann}_R(x) \geqq I, \text{ for some } I \in \mathcal{I}_\tau\}.$$

The class R_τ-Mod, together with the usual morphisms in R-Mod, is a Grothendieck category, so that R_τ-Mod has enough injectives. If $M \in R_\tau$-Mod, then its injective envelope $\mathrm{E}_\tau(M)$ in R_τ-Mod is

$$\mathrm{E}_\tau(M) = \mathrm{t}_\tau(\mathrm{E}(M)),$$

where $\mathrm{E}(M)$ is the injective envelope of M in R-Mod. Finally, it is obvious that $R_\tau\text{-Mod} \subseteqq R_\tau\text{-}\mathcal{L}$.

We denote by R_τ-CM the full subcategory of R_τ-$\mathcal{L}$ consisting of all compact modules. Then $(R, \tau) \in R_\tau$-CM. The meaning of the symbols $\mathcal{L}$-R_τ, Mod-R_τ and CM-R_τ should be clear.

1.3. Let (A, σ) and (R, τ) be two compact rings and assume we are given a duality $H = (H_1, H_2)$ between $\mathcal{L}$-A_σ and R_τ-$\mathcal{L}$:

$$\mathcal{L}\text{-}A_\sigma \underset{H_2}{\overset{H_1}{\rightleftarrows}} R_\tau\text{-}\mathcal{L}\,.$$

Arguing as in [1], Proposition 1.7, we can show that H induces a duality between CM-A_σ and R_τ-Mod and one between Mod-A_σ and R_τ-CM.

By some results of [9], there exists a faithfully balanced discrete bimodule ${}_RK_A$ such that $H_1(A_\sigma)={}_RK$ and $H_2(R_\tau)=K_A$. Moreover, for any $M\in\mathcal{L}\text{-}A_\sigma$ and any $N\in R_\tau\text{-}\mathcal{L}$ there are *algebraic canonical* isomorphisms

$$H_1(M)\cong \mathrm{Chom}_A(M, K_A)\,; \qquad H_2(N)\cong \mathrm{Chom}_R(N, {}_RK)\,.$$

1.4. For the rest of this section we shall study the situation settled in 1.3.

1.5 PROPOSITION. *Let ${}_RK=H_1(A_\sigma)$. Then ${}_RK$ is an injective cogenerator of* R_τ-Mod. *Similarly $K_A=H_2(R_\tau)$ is an injective cogenerator of* Mod-A_σ.

PROOF. We prove this fact by showing that (A, σ) is a projective generator of CM-A_σ; the proof relies on the following facts:

(1) for any $M\in$CM-A_σ, $\mathrm{Chom}_A(A_\sigma, M)=\mathrm{Hom}_A(A, M)$;

(2) epimorphisms in CM-A_σ are surjective.

Since A_A is a projective generator of Mod-A, it follows from (1) that (A, σ) is a generator of CM-A_σ, while it follows from (2) that (A, σ) is projective in CM-A_σ. Now, by applying the duality between CM-A_σ and R_τ-Mod, we get that ${}_RK$ is an injective cogenerator of R_τ-Mod. □

Denote by R_τ-LT (resp. LT-A_σ) the category of left (resp. right) linearly topologized modules over the topological ring (R, τ) (resp. (A, σ)).

Evidently (see 1.1):

$$\mathcal{L}\text{-}A_\sigma\subseteqq \mathrm{LT}\text{-}A_\sigma \quad\text{and}\quad R_\tau\text{-}\mathcal{L}\subseteqq R_\tau\text{-LT}\,.$$

1.6 PROPOSITION. *${}_RK$ is an injective cogenerator of* R_τ-LT.

PROOF. Let $M\in R_\tau$-LT, X be a submodule of M with the relative topology and $f: X\to K$ be a continuous morphism. We want to show that f can be extended to a continuous morphism from M into K. Since K is discrete, $\mathrm{Ker}\, f\geqq X\cap V$, where V is an open submodule of M; setting $f'(x+v)=f(x)$, for $x\in X$ and $v\in V$, gives a continuous morphism from $X+V$ into K, so that there is no loss in generality if we assume that X is open in M. Consider the diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "\pi"'] & M \arrow[d, "\pi'"] \\ X/\mathrm{Ker}\, f \arrow[r] \arrow[d, "\bar{f}"'] & M/\mathrm{Ker}\, f \arrow[dl, "g"] \\ K & \end{tikzcd}$$

where π and π' are the canonical projections and $\bar{f}\circ\pi=f$. Then $\bar{f}$ extends to a morphism $g: M/\mathrm{Ker}\, f \to K$, since $X/\mathrm{Ker}\, f$ and $M/\mathrm{Ker}\, f$ belong to R_τ-Mod and ${}_RK$ is injective in R_τ-Mod; thus $g\circ\pi'$ is an extension of f.

Let now $x\in M$, $x\neq 0$; there exists an open submodule V of M such that $x\neq V$. Let $\pi: M\to M/V$ be the canonical projection and endow M/V with the discrete topology, so that π is continuous and $M/V\in R_\tau$-Mod. There exists $\xi: M/V\to K$ such that $\xi(\pi(x))\neq 0$, since ${}_RK$ is a cogenerator of R_τ-Mod; hence ${}_RK$ is a cogenerator of R_τ-LT. □

1.7 COROLLARY. *The topology of* $M\in R_\tau$-CM *coincides with the weak topology of* $\mathrm{Chom}_R(M, K)$.

PROOF. By Proposition 1.6, the weak topology of $\mathrm{Chom}_R(M, K)$ is Hausdorff so that it coincides with the topology of M, which is compact. □

1.8 COROLLARY. *The topology τ on R coincides with the K-topology, that is the topology having as a base of neighborhoods of zero the annihilators in R of the finite subsets of K.*

1.9 REMARK. The preceding corollaries hold also in CM-A_σ.

2. The structure of K_A and ${}_RK$

2.1. In this Section we work under the hypotheses settled in 1.3. Denote by (D_1, D_2) the duality between CM-A_σ and R_τ-Mod induced by (H_1, H_2):

$$\text{CM-}A_\sigma \underset{D_2}{\overset{D_1}{\rightleftarrows}} R_\tau\text{-Mod}\,.$$

Note that (A, σ) and (R, τ), as inverse limits of finite—in particular artinian—rings, are *strictly linearly compact* (s. l. c.) in the sense of Leptin [6].

Let $(W_\lambda)_{\lambda\in\Lambda}$ be a system of representatives of all simple non isomorphic modules in Mod-A_σ. It is apparent that, for any $\lambda\in\Lambda$, $W_\lambda\in$CM-A_σ, since it is finite. Moreover, any finitely generated submodule of K_A is finite, so that K_A has essential socle. We have

$$\mathrm{Soc}\,(K_A)=\bigoplus_{\lambda\in\Lambda} W_\lambda^{(m_\lambda)} \quad \text{and} \quad K_A=\bigoplus_{\lambda\in\Lambda} \mathrm{E}_\sigma(W_\lambda)^{(m_\lambda)}$$

where the m_λ's are cardinal numbers uniquely determined by K_A. The second equality holds since (A, σ) is topologically artinian, so that every module in Mod-A_σ has essential socle, and topologically noetherian, so that a direct sum of injectives in Mod-A_σ is injective. Let $D_\lambda=\mathrm{End}_A(W_\lambda)$; since W_λ is finite, D_λ is a finite field.

For all $\lambda\in\Lambda$, we set $V_\lambda=D_1(W_\lambda)$. Since (D_1, D_2) is a duality, $(V_\lambda)_{\lambda\in\Lambda}$ is a system of representatives of all non isomorphic simple modules in R_τ-Mod. It is clear that $\mathrm{Soc}\,({}_RK)$ is essential in ${}_RK$ and that $\mathrm{End}_R(V_\lambda)\cong D_\lambda$ canonically.

Let $\mathrm{J}(A)$ be the Jacobson radical of A. Since (A, σ) is linearly compact, we have

$$A/\mathrm{J}(A)\cong\prod_{\lambda\in\Lambda}\mathrm{End}_{D_\lambda}(W_\lambda)$$

by a well-known result of Leptin ([6]).

Let n_λ be the dimension of W_λ as a left vector space over D_λ; since W_λ is finite, then also n_λ is finite and therefore we can write

$$\mathrm{End}_{D_\lambda}(W_\lambda)\cong I_\lambda^{n_\lambda}$$

where I_λ is a minimal right ideal of $\mathrm{End}_{D_\lambda}(W_\lambda)$. Recall that $I_\lambda\cong W_\lambda$ in Mod-A and hence in CM-A_σ, since W_λ is finite.

2.2 LEMMA. *Let* $\mathrm{J}(A)$ *be the Jacobson radical of* A. *Then*

a) $\mathrm{Ann}_A\,\mathrm{Soc}\,({}_RK)=\mathrm{J}(A)$;

b) $\mathrm{Ann}_A\,\mathrm{Ann}_K\,\mathrm{J}(A)=\mathrm{J}(A)$.

PROOF. a) Apply to the exact sequence

$$0\longrightarrow\mathrm{Soc}\,({}_RK)\longrightarrow{}_RK\longrightarrow{}_RK/\mathrm{Soc}\,({}_RK)\longrightarrow 0$$

the functor $\mathrm{Hom}_R(-, {}_RK)$, to get the exact sequence

$$0\longrightarrow\mathrm{Ann}_A\,\mathrm{Soc}\,({}_RK)\longrightarrow A\longrightarrow\mathrm{End}_R(\mathrm{Soc}\,({}_RK))\longrightarrow 0\,.$$

Since ${}_RK$ is quasi-injective and $A=\mathrm{End}({}_RK)$, we have that $\mathrm{J}(A)$ coincides with the ideal of A consisting of the endomorphisms of ${}_RK$ with essential kernel (see [3]). But $\mathrm{Soc}({}_RK)$ is the intersection of all essential submodules of ${}_RK$ and so

$$\mathrm{J}(A)\subseteq\mathrm{Ann}_A\,\mathrm{Soc}\,({}_RK)\,.$$

On the other hand, $\mathrm{Soc}({}_RK)$ is essential in ${}_RK$, hence

$$\mathrm{Ann}_A \operatorname{Soc}({}_RK) \subseteq \mathrm{J}(A)\,.$$

b) Put $J=\mathrm{J}(A)$ and assume there exists $a\in A$ such that $a\in \mathrm{Ann}_A \mathrm{Ann}_K(J)\setminus J$. Since (A, σ) is linearly compact, J is closed in A, so that there exists a continuous morphism $f: A \to K_A$ such that $f(J)=0$ and $f(a)\neq 0$. Thus we can find $x\in K$ such that $xJ=0$, but $xa\neq 0$. This is a contradiction, since $x\in \mathrm{Ann}_K(J)$ and $a\in \mathrm{Ann}_A \mathrm{Ann}_K(J)$. □

2.3 THEOREM. *Let $(W_\lambda)_{\lambda\in\Lambda}$ and $(V_\lambda)_{\lambda\in\Lambda}$ be systems of representatives of all non isomorphic simple modules in* Mod-A_σ *and* R_τ-Mod *respectively. Let n_λ and m_λ be the dimensions of W_λ and V_λ respectively as vector spaces over $D_\lambda=$* $\mathrm{End}_A(W_\lambda)=\mathrm{End}_R(V_\lambda)$. *Then*

a) *for all $\lambda\in\Lambda$, n_λ is finite and ${}_RK=\bigoplus_{\lambda\in\Lambda}\mathrm{E}_\tau(V_\lambda)^{n_\lambda}$;*

b) *for all $\lambda\in\Lambda$, m_λ is finite and $K_A=\bigoplus_{\lambda\in\Lambda}\mathrm{E}_\sigma(W_\lambda)^{m_\lambda}$.*

Hence the structures of K_A and ${}_RK$ depend only on the pair of compact rings (A, σ) and (R, τ) and not on the duality (H_1, H_2) under consideration.

PROOF. We shall prove only a), for b) follows by symmetry.

As we have seen before, n_λ is finite, for every $\lambda\in\Lambda$. Set $J=\mathrm{J}(A)$ and consider the exact sequence

(1) $$0 \longrightarrow J \longrightarrow A \longrightarrow A/J \longrightarrow 0$$

and set

$$(A/J)^* = \mathrm{Chom}_A(A/J, K_A) = \mathrm{Chom}_A\Big(\prod_{\lambda\in\Lambda} \mathrm{End}_{D_\lambda}(W_\lambda), K_A\Big)\,.$$

Observe that, since $I_\lambda \cong W_\lambda$ in CM-A_σ and $V_\lambda = H_1(W_\lambda)$, there are canonical algebraic isomorphisms

$$(A/J)^* \cong \bigoplus_{\lambda\in\Lambda} H_1(I_\lambda^{n_\lambda}) \cong \bigoplus_{\lambda\in\Lambda} V_\lambda^{n_\lambda}\,.$$

By Proposition 1.6, applying $\mathrm{Chom}_A(-, K_A)$ to the sequence (1) gives the exact sequence

$$0 \longrightarrow (A/J)^* = \bigoplus_{\lambda\in\Lambda} V_\lambda^{n_\lambda} \longrightarrow {}_RK \longrightarrow \mathrm{Chom}_A(J, K_A)\,.$$

We want to show that $(A/J)^*$, which we can identify with $\mathrm{Ann}_K(J)$, is the socle of ${}_RK$, for from this fact the conclusion will follow, since ${}_RK$ is injective with essential socle. Obviously $(A/J)^*=\mathrm{Ann}_K(J)$ is semisimple, so that $\mathrm{Ann}_K(J) \subseteq \operatorname{Soc}({}_RK)$.

Assume, by contradiction, that $\mathrm{Ann}_K(J)\neq \operatorname{Soc}({}_RK)$: then there exists an endomorphism $a\in A=\mathrm{End}({}_RK)$ such that

$$\mathrm{Ann}_K(J)a=0 \quad\text{and}\quad \operatorname{Soc}({}_RK)a\neq 0\,.$$

From Lemma 2.2(b), it follows $a \in \mathrm{Ann}_A \mathrm{Ann}_K(J)=J$, while, from Lemma 2.2(a), it follows $a \notin \mathrm{Ann}_A \mathrm{Soc}({}_RK)=J$, a contradiction. □

3. Uniqueness of the duality induced between CM-A_σ and R_τ-Mod

3.1. Assume there exists a duality (H_1, H_2) between $\mathcal{L}$-A_σ and R_τ-$\mathcal{L}$. Then (H_1, H_2) induces a duality (D_1, D_2) between CM-A_σ and R_τ-Mod

$$\text{CM-}A_\sigma \underset{D_2}{\overset{D_1}{\rightleftarrows}} R_\tau\text{-Mod}\,.$$

3.2 PROPOSITION. *There is a functorial isomorphism*

$$D_1 \cong \mathrm{Chom}_A(-, K_A)\,.$$

PROOF. As we know, the functor D_1 is, from algebraic point of view, naturally equivalent to the functor $\mathrm{Chom}_A(-, K_A)$. Hence, it is sufficient to show that, for any $M \in$ CM-A_σ, $\mathrm{Chom}_A(M, K_A) \in R_\tau$-Mod. Let $f \in \mathrm{Chom}_A(M, K_A)$; then

$$\mathrm{Ann}_R(f)=\{r \in R : rf=0\}=\{r \in R : rf(M)=0\}=\mathrm{Ann}_R(f(M))\,.$$

As M is compact, $f(M) \leqq {}_RK$ is finite, so that $\mathrm{Ann}_R(f)$ is an open left ideal in (R, τ). □

3.3 LEMMA. *Let F be a finite module belonging to R_τ-Mod. Then $\mathrm{Hom}_R(F, {}_RK)$ is finite and, when endowed with the discrete topology, it is an object of* CM-A_σ.

PROOF. Since F is finite, it has a composition series, say of length p. Assume first that $p=1$, so that F is simple; then F is isomorphic to V_λ for some $\lambda \in \Lambda$. By Theorem 2.3, $\mathrm{Hom}_R(F, {}_RK) \cong \mathrm{Hom}_R(V_\lambda, V_\lambda^{(n_\lambda)})$ is finite.

Assume now that the thesis holds for modules of length $p-1$. Then, if S is a simple submodule of F, the module F/S has length $p-1$. Applying the functor $\mathrm{Hom}_R(-, {}_RK)$ to the exact sequence $0 \to S \to F \to F/S \to 0$ ends the proof. □

Note that, in this proof, we use only the structure of ${}_RK$.

3.4. For every $N \in R_\tau$-Mod we will denote by $\mathrm{Hom}_R^p(N, {}_RK)$ the module $\mathrm{Hom}_R(N, {}_RK)$ endowed with the topology of pointwise convergence.

3.5 LEMMA. *For every $N \in R_\tau$-Mod, $\mathrm{Hom}_R^p(N, {}_RK)$ belongs to* CM-A_σ.

PROOF. The discrete module N is the direct limit of a family $(F_\lambda)_{\lambda \in \Lambda}$ of finite submodules and we have:

$$\mathrm{Hom}_R^c(N,\ {}_RK)=\mathrm{Hom}_R^c(\varinjlim F_\lambda,\ {}_RK) \overset{\alpha}{\cong} \varprojlim \mathrm{Hom}_R^c(F_\lambda,\ {}_RK)$$

where $\mathrm{Hom}_R^c(F_\lambda,\ {}_RK)\in \mathrm{CM}\text{-}A_\sigma$ by Lemma 3.3. The isomorphism α is topological, provided we endow the inverse limit with the inverse limit topology (i.e. the topology it has as a submodule of the direct product). □

Let $N\in R_\tau$-LT and let F be a subset of N. We set

$$\mathcal{W}(F)=\{\xi\in \mathrm{Chom}_R(N,\ {}_RK): (F)\xi=0\}$$

and we use a similar notation for subsets of $M\in\mathrm{LT}\text{-}A_\sigma$. If $F=\{y\}$, we write $\mathcal{W}(F)=\mathcal{W}(y)$.

3.6 PROPOSITION. *Let $M\in\mathrm{CM}\text{-}A_\sigma$ and $N\in R_\tau$-Mod. There exists an abelian group isomorphism*

$$\varphi: \mathrm{Chom}_A(M,\ \mathrm{Hom}_R^c(N,\ {}_RK)) \longrightarrow \mathrm{Hom}_R(N,\ \mathrm{Chom}_A(M,\ K_A))$$

which is natural in the variables M and N.

PROOF. In this proof, to keep the notation not too heavy, we shall skip the convention of writing the morphisms on the opposite side to the scalars and we shall write morphisms on the left; since we are not considering endomorphism rings, there is no harm in doing this. Let $f: M\to \mathrm{Hom}_R^c(N,\ {}_RK)$ be an A-linear continuous morphism. Define the morphism $\hat{f}: N\to \mathrm{Chom}_A(M,\ K_A)$ by setting, for $x\in M$ and $y\in N$,

$$\hat{f}(y)(x)=f(x)(y)\,.$$

Then $\hat{f}$ is R-linear, since

$$[\hat{f}(ry)](x)=[f(x)](ry)=r[f(x)](y)=[r\hat{f}(y)](x)\,.$$

Let us verify that $\hat{f}$ is continuous. Indeed, for $y\in N$, we have:

$$\mathrm{Ker}\,(\hat{f}(y))=\{x\in M: \hat{f}(y)(x)=f(x)(y)=0\}=f^{-1}(\mathcal{W}(y))$$

which is open in M. Thus we can set $\varphi(f)=\hat{f}$.

The morphism φ can be inverted by the abelian groups morphism

$$\psi: \mathrm{Hom}_R(N,\ \mathrm{Chom}_A(M,\ K_A)) \longrightarrow \mathrm{Chom}_A(M,\ \mathrm{Hom}_R^c(N,\ {}_RK))$$

defined, for any morphism $g: N\to \mathrm{Chom}_A(M,\ K_A)$, by $\psi(g)=\check{g}$, where $\check{g}: M\to \mathrm{Hom}_R^c(N,\ {}_RK)$ is defined by

$$\check{g}(x)(y)=g(y)(x)\,,$$

for $x\in M$ and $y\in N$. We prove that $\check{g}$ is continuous, leaving the verification of the A-linearity to the reader. If $y \in N$, we have

$$\check{g}^{-1}(\mathcal{W}(y)) = \{x \in M : \check{g}(x)(y) = g(y)(x) = 0\} = \mathrm{Ker}\,(g(y))$$

which is open in M. □

3.7. A consequence of the preceding proposition is that the functor $\mathrm{Hom}_R^p(-, {}_RK) : R_\tau\text{-Mod} \to \mathrm{CM}\text{-}A_\sigma$ is a right adjoint to the functor $D_1 : \mathrm{CM}\text{-}A_\sigma \to R_\tau\text{-Mod}$. Thus, by Proposition 3.2, $\mathrm{Hom}_R^p(-, {}_RK)$ is naturally equivalent to D_2.

Therefore we have the following

3.8 THEOREM. *Let $H_1 : \mathcal{L}\text{-}A_\sigma \to R_\tau\text{-}\mathcal{L}$, $H_2 : R_\tau\text{-}\mathcal{L} \to \mathcal{L}\text{-}A_\sigma$ be a duality and let (D_1, D_2) be the induced duality*

$$\mathrm{CM}\text{-}A_\sigma \underset{D_2}{\overset{D_1}{\rightleftarrows}} R_\tau\text{-Mod}.$$

Then, if ${}_RK_A$ is the bimodule such that $H(A_\sigma) = {}_AK$ and $H_2(R_\tau) = K_A$, there are the functorial isomorphisms

$$D_1 \cong \mathrm{Chom}_A(-, K_A), \qquad D_2 \cong \mathrm{Hom}_R^p(-, {}_RK).$$

3.9 REMARK. An analogous statement holds for the induced duality between $\mathrm{Mod}\text{-}A_\sigma$ and $R_\tau\text{-CM}$.

4. The general case

4.1. We assume throughout this Section that we are given a duality (H_1, H_2) between $\mathcal{L}\text{-}A_\sigma$ and $R_\tau\text{-}\mathcal{L}$:

$$\mathcal{L}\text{-}A_\sigma \underset{H_2}{\overset{H_1}{\rightleftarrows}} R_\tau\text{-}\mathcal{L}.$$

Let ${}_RK_A$ be the canonical bimodule of the duality (H_1, H_2).

4.2. Let $M \in \mathcal{L}\text{-}A_\sigma$ and let $\mathcal{C}(M)$ denote the set of all compact topological submodules of M. If $C, C' \in \mathcal{C}(M)$ and $C \leq C'$, we denote by $f_{C'}^C : C \to C'$ the inclusion of C into C'. For all $C \in \mathcal{C}(M)$, $i_C : C \to M$ is the inclusion of C into M.

4.3 LEMMA. *Let $M \in \mathcal{L}\text{-}A_\sigma$. Then*

$$M = \varinjlim_{C \in \mathcal{C}(M)} (C\,; f_{C'}^C) \tag{1}$$

in the category $\mathcal{L}\text{-}A_\sigma$. A similar result holds in the category $R_\tau\text{-}\mathcal{L}$.

PROOF. To begin with, we observe that, for all $x\in M$, the submodule xA is compact, since the map $A\to xA$ which sends 1 into x is continuous. It is obvious that $\mathcal{C}(M)$ contains a compact open submodule C_0.

To prove (1), let $L\in\mathcal{L}\text{-}A_\sigma$ and assume there exists a family of continuous morphisms $(f_C: C\to L)_{C\in\mathcal{C}(M)}$ such that the diagrams

(2)
$$\begin{tikzcd} C \arrow[rr, "f_{C'}^{C}"] \arrow[dr, "i_C"] \arrow[ddr, "f_C"'] & & C' \arrow[dl, "i_{C'}"'] \arrow[ddl, "f_{C'}"] \\ & M \arrow[d, dashed, "\varphi"] & \\ & L & \end{tikzcd}$$

are commutative, for all $C, C'\in\mathcal{C}(M)$ with $C\leqq C'$ i.e., $f_{C'}|C=f_C$. Let us prove that there exists a unique continuous morphism $\varphi: M\to L$ such that, for any $C\in\mathcal{C}(M)$, the diagram

(3)
$$\begin{tikzcd} C \arrow[r, hook, "i_C"] \arrow[dr, "f_C"'] & M \arrow[d, "\varphi"] \\ & L \end{tikzcd}$$

is commutative.

The morphism φ is given by

$$\varphi(x)=f_{xA}(x) \qquad (x\in M);$$

we show that φ is well-defined and that the diagrams (3) are commutative by recalling that the diagrams (2) are commutative, so that, if $x\in C$, we have $xA\leqq C$ and

$$f_C(x)=(f_C\,|\,xA)(x)=f_{xA}(x)=\varphi(x)=\varphi(i_C(x)).$$

To end the proof, we show that φ is continuous. Let V be an open submodule of L. Then $\varphi^{-1}(\mathrm{V})\geqq f_{C_0}^{-1}(\mathrm{V})$ and this is open in C_0. But, since C_0 is open in M, $f_{C_0}^{-1}(\mathrm{V})$ is also open in M. □

4.4 LEMMA. *In the preceding situation, let $M\in\mathcal{L}\text{-}A_\sigma$. Then:*

(1)
$$H_1(M)=\varprojlim_{C\in\mathcal{C}(M)}(H_1(C)\,;H_1(f_{C'}^{C}))$$

in the category $R_\tau\text{-}\mathcal{L}$ and the equality (1) *holds in the category R_τ-LT as well.*

A similar result holds also for the modules in R_τ-$\mathcal{L}$.

PROOF. By the preceding Lemma we have, in R_τ-$\mathcal{L}$,

$$H_1(M)=H_1\Big(\varinjlim_{C\in\mathcal{C}(M)}(C\,;f_{C'}^{C})\Big)=\varprojlim_{C\in\mathcal{C}(M)}(H_1(C)\,;H_1(f_{C'}^{C}))\,.$$

Since $H_1(M)$ is complete and linearly topologized we have, in the category R_τ-LT,

$$H_1(M)=\varprojlim_{V\in\mathcal{F}(H_1(M))}H_1(M)/\mathrm{V}\,,$$

where we denote by $\mathcal{F}(H_1(M))$ the filter of all open submodules of $H_1(M)$. However, for every $C\in\mathcal{C}(M)$, there exists one and only one open submodule V of $H_1(M)$ such that $H_1(M)/\mathrm{V}$ is canonically isomorphic to the discrete module $H_1(C)$. Hence the two inverse limits coincide. □

4.5 REMARK. In the preceding inverse limit, the canonical morphisms from $H_1(M)$ to $H_1(C)$ (for $C\in\mathcal{C}(M)$) are the morphisms $H_1(i_C)$.

4.6 THEOREM. *Under the hypotheses of* 4.1, *if $M\in\mathcal{L}$-A_σ then the module $H_1(M)$ is canonically isomorphic to* $\mathrm{Chom}_A(M, K_A)$ *endowed with the topology of the uniform convergence on the compact subsets of M; this topology coincides with the topology of uniform convergence on the compact submodules of M, which has as a basis of neighborhoods of zero the submodules $\mathcal{W}(C)=\{\xi\in\mathrm{Chom}_A(M, K_A):\xi(C)=0\}$, for $C\in\mathcal{C}(M)$.*

A completely analogous result holds for R_τ-$\mathcal{L}$. Consequently any duality between $\mathcal{L}$-A_σ and R_τ-$\mathcal{L}$ is unique.

PROOF. Let $M\in\mathcal{L}$-A_σ. As we know, $H_1(M)$ and $\mathrm{Chom}_A(M, K_A)$ are canonically isomorphic as abstract modules. Moreover:

$$H_1(M)=\varprojlim_{C\in\mathcal{C}(M)}(H_1(C)\,;H_1(f_{C'}^{C}))=\varprojlim_{C\in\mathcal{C}(M)}(\mathrm{Chom}_A(C, K_A)\,;(f_{C'}^{C})^*)\,,$$

where the * denotes the transposed morphism. If we identify $H_1(M)$ with $\mathrm{Chom}_A(M, K_A)$, a basis of neighborhoods of zero in $H_1(M)$ is given by the kernels of the projections $i_C^*:\mathrm{Chom}_A(M, K_A)\to\mathrm{Chom}_A(C, K_A)$ and

$$\begin{aligned}\mathrm{Ker}\,(i_C^*)&=\{\xi\in\mathrm{Chom}_A(M, K_A):i_C^*(\xi)=0\}\\&=\{\xi\in\mathrm{Chom}_A(M, K_A):\xi(C)=0\}=\mathcal{W}(C)\,.\end{aligned}$$

Denote by w this topology and by w' the topology of uniform convergence on the compact subsets of M. Of course $w\subseteqq w'$. To prove the converse inclusion it is sufficient to prove that every compact subset F of M is contained in a compact submodule. So, let C be an open compact submodule of M; there exist $x_1, \cdots, x_n \in F$ such that

$$F \subseteq \bigcup_{i=1}^{n} (x_i + C) \subseteq \bigcup_{i=1}^{n} (x_i A + C) \subseteq \sum_{i=1}^{n} (x_i A + C) = C + \sum_{i=1}^{n} x_i A$$

which is a compact submodule of M. □

4.7. We want to show now, as an example, that Pontryagin duality over a compact ring (R, τ) can be represented as stated in Theorem 4.6.

Let $\boldsymbol{T} = \boldsymbol{R}/\boldsymbol{Z}$ be the circle group and recall that, if $M \in \mathcal{L}\text{-}R_\tau$, the Pontryagin dual $\Gamma_1(M)$ of M_R is the left R-module $\mathrm{Chom}_{\boldsymbol{Z}}(M, \boldsymbol{T})$, endowed with the topology of uniform convergence on the compact subsets of M; in an analogous way we define the Pontryagin dual $\Gamma_2(N)$ of a left locally compact module $N \in R_\tau\text{-}\mathcal{L}$. It is not difficult to show that these functors—the action on morphisms being the obvious one—yield a duality between the categories $\mathcal{L}\text{-}R_\tau$ and $R_\tau\text{-}\mathcal{L}$.

As in the proof of Theorem 4.6, we can see that the topology on the Pontryagin dual of M conincides with the topology of uniform convergence on the compact submodules M; thus a basis of neighborhoods of zero in $\Gamma_1(M)$ is given by the submodules of the form

$$\mathcal{W}_{\boldsymbol{T}}(C) = \{\chi \in \mathrm{Chom}_{\boldsymbol{Z}}(M, \boldsymbol{T}) : \chi(C) = 0\},$$

where C runs over all compact submodules of M. Indeed, a basis should be the family of subsets of the form

$$\mathcal{W}_{\boldsymbol{T}}(C\,;U) = \{\chi \in \mathrm{Chom}_{\boldsymbol{Z}}(M, \boldsymbol{T}) : \chi(C) \subseteq U\},$$

where C is a compact submodule of M and U is a neightborhood of zero in $\boldsymbol{T}$, but, without loss of generality, we can assume U is a *small* neighborhood, i.e., one containing no subgroups of $\boldsymbol{T}$. In this case $\mathcal{W}_{\boldsymbol{T}}(C\,;U) = \mathcal{W}_{\boldsymbol{T}}(C)$.

Let now K be the Pontryagin dual of R_τ: it is obvious that it does not matter whether we consider R_τ as a left or a right module over itself, so that K carries a natural structure of R-R-bimodule. Consider $M \in \mathcal{L}\text{-}R_\tau$; then the map

$$\varphi_M : \mathrm{Chom}_{\boldsymbol{Z}}(M, \boldsymbol{T}) \longrightarrow \mathrm{Chom}_R(M, K_R)$$

$$\chi \longmapsto \hat{\chi}$$

where, for $x \in M$ and $r \in R$,

$$\hat{\chi}(x)(r) = \chi(xr)$$

is an isomorphism.

Endow $\mathrm{Chom}_R(M, K_R)$ with the topology of uniform convergence on the compact submodules of M, so that a fundamental neighborhood of zero is of the form

$$\mathcal{W}(C)=\{\xi\in\mathrm{Chom}_R(M, K_R):\xi(C)=0\},$$

where C is a compact submodule of M. Now it is only a matter of calculations to verify that φ_M is a topological isomorphism:

$$\begin{aligned}\varphi_M^{-1}(\mathcal{W}(C)) &= \{\chi\in\mathrm{Chom}_{\boldsymbol{Z}}(M, \boldsymbol{T}):\hat{\chi}(C)=0\}\\ &= \{\chi\in\mathrm{Chom}_{\boldsymbol{Z}}(M, \boldsymbol{T}):\chi(xr)=0,\ \forall x\in C,\ \forall r\in R\}\\ &= \{\chi\in\mathrm{Chom}_{\boldsymbol{Z}}(M, \boldsymbol{T}):\chi(C)=0\}\\ &= \mathcal{W}_{\boldsymbol{T}}(C).\end{aligned}$$

5. Existence of dualities

In this section, given a compact ring (A, σ), we construct all compact rings (R, τ) for which there exists a duality between $\mathcal{L}\text{-}A_\sigma$ and $R_\tau\text{-}\mathcal{L}$.

5.1 LEMMA. *Let (A, σ) be a right linearly topologized ring, K_A an injective cogenerator of* Mod-A_σ, $R=\mathrm{End}(K_A)$. *Then the simple submodules of ${}_RK$ are exactly those isomorphic to* $\mathrm{Hom}_A(\mathrm{V}, K_A)$, *when* V *runs over the simple submodules of K_A. Moreover* $\mathrm{Soc}({}_RK)=\mathrm{Soc}(K_A)$ *and* $\mathrm{Soc}({}_RK)$ *is essential in ${}_RK$.*

PROOF. Let $\mathcal{P}$ be the set of open right maximal ideals of (A, σ). We prove that, for any $P\in\mathcal{P}$, $\mathrm{Ann}_K(P)$ is a simple submodule of ${}_RK$ and that every simple submodule of ${}_RK$ can be obtained in this way. Let us fix $P\in\mathcal{P}$ and let $x, y\in\mathrm{Ann}_K(P)$ be non zero. Since P is open in (A, σ) and K_A is a cogenerator of LT-A_σ, it is $\mathrm{Ann}_A(x)=P=\mathrm{Ann}_A(y)$. Thus there exists a morphism $f: xA\to yA$ such that $f(x)=y$ and this morphism f extends to an endomorphism of K_A. Hence there is $r\in R$ such that $y=rx$ and this proves that $\mathrm{Ann}_K(P)$ is a simple submodule of ${}_RK$.

Conversely, let ${}_RS$ be a simple submodule of ${}_RK$ and $x\in S$ be non zero. There exists $P\in\mathcal{P}$ such that $\mathrm{Ann}_A(x)\subseteq P$. Let $y\in\mathrm{Ann}_K(P)$, $y\neq 0$: since $\mathrm{Ann}_A(y)=P$, there exists a surjective morphism $f: xA\to yA$ such that $f(x)=y$. Then $y=rx$, for some $r\in R$, so that $\mathrm{Ann}_K(P)\leqq{}_RS$; since ${}_RS$ is simple, we have $S=\mathrm{Ann}_K(P)$.

Let now ${}_RS$ be a simple submodule of ${}_RK$ and let $P\in\mathcal{P}$ be such that $S=\mathrm{Ann}_K(P)$. Since K_A is a cogenerator of Mod-A_σ, the simple module A/P is isomorphic to a simple submodule V of K_A. The assignment $f\mapsto f(e)$, where $e=1+P\in A/P$ clearly defines an isomorphism of left R-modules $\mathrm{Hom}_A(A/P, K_A)\to\mathrm{Ann}_K(P)$ and composing this with the isomorphism $\mathrm{Hom}_A(\mathrm{V}, K_A)\to\mathrm{Hom}_A(A/P, K_A)$

yields the desired isomorphism.

Now $\mathrm{Soc}(K_A)=\sum\{\mathrm{Ann}_K(P)\mid P\in\mathscr{P}\}$, since K_A is a cogenerator of Mod-A_σ, and so the equality $\mathrm{Soc}({}_RK)=\mathrm{Soc}(K_A)$ is proved.

To end the proof it is sufficient to show that $\mathrm{Soc}(K_A)$, as an R-submodule of ${}_RK$, is essential. Let $x\in{}_RK$, $x\neq 0$; there exists $P\in\mathscr{P}$ such that $\mathrm{Ann}_A(x)\subseteq P$ and K_A contains a submodule V isomorphic to A/P. Thus there exists a morphism $f: xA\to \mathrm{V}$ such that $f(x)\neq 0$ and, by extending this via the injectivity of K_A, we get $r\in R=\mathrm{End}(K_A)$ such that $rx\in\mathrm{Soc}({}_RK)=\mathrm{Soc}(K_A)$ and $rx\neq 0$. □

5.2 DEFINITION. Let (A, σ) be a right linearly topologized ring and let $(W_\lambda)_{\lambda\in\Lambda}$ be a system of representatives of all simple non isomorphic modules in Mod-A_σ. If $M\in$ Mod-A_σ, then

$$\mathrm{Soc}(M)=\bigoplus_{\lambda\in\Lambda} W_\lambda^{(m_\lambda)}$$

where, for all $\lambda\in\Lambda$, m_λ is a suitable cardinal number. The family $(m_\lambda)_{\lambda\in\Lambda}$ is called the *grade* of M. We say that M has *finite grade* if every m_λ is finite.

5.3 LEMMA. *Let (A, σ) be a right linearly topologized ring, K_A an injective cogenerator of* Mod-A_σ, $R=\mathrm{End}(K_A)$ *and endow R with the K-topology τ. If (R, τ) is compact, then K_A has finite grade.*

PROOF. Let N be the set of non negative integers and assume, by contradiction, that there exists a simple submodule S of K_A such that K_A contains an infinite direct sum $S^{(N)}$ of copies of S. Denote by S_n the n-th component of $S^{(N)}$, let $x_0\in S_0$, $x_0\neq 0$, and set $I=\mathrm{Ann}_R(x_0)$. Then I is an open ideal of (R, τ), so that R/I is finite. Consider, for $n>0$ the element $x_n\in S^{(N)}$ having the n-th component equal to x_0 and the other components equal to zero. Let $\varphi_n: S_0\to S_n$ be the A-morphism such that $\varphi_n(x_0)=x_n$ (i.e., the identity); then φ_n extends to an endomorphism $f_n: K_A\to K_A$. The morphisms $f_n\in R$ are all distinct and non zero and clearly $f_n\notin I$. Moreover, if $\pi: R\to R/I$ is the canonical projection, it is obvious that $\pi(f_n)\neq\pi(f_m)$, if $n\neq m$. Since R/I is finite, this is absurd. □

5.4. Let ${}_RK\in R$-Mod be a faithful module and endow R with the K-topology τ, which is Hausdorff.

(5.4.1) *The following conditions are equivalent:*

(i) *${}_RK$ is an injective object in* R_τ-Mod,

(ii) *${}_RK$ is an injective object in* R_τ-LT,

(iii) ${}_RK$ *is quasi-injective.*

(cf. [8], Proposition 6.6.)

Recall that ${}_RK$ called *strongly quasi-injective* if it is quasi-injective and, for every submodule B of ${}_RK$ and every $x \in K \setminus B$, there exists an endomorphism α of ${}_RK$ such that $B\alpha=0$ and $x\alpha \neq 0$.

(5.4.2) *The following conditions are equivalent:*

(i) ${}_RK$ *is an injective cogenerator in* R_τ-Mod,

(ii) ${}_RK$ *is an injective cogenerator in* R_τ-Mod,

(iii) ${}_RK$ *is strongly quasi-injective.*

(5.4.3) *Let* (A, σ) *be a compact ring,* K_A *a cogenerator of* Mod-A_σ *and* $R=\mathrm{End}(K_A)$. *Then the bimodule* ${}_RK_A$ *is faithfully balanced and* ${}_RK$ *is quasi-injective.*

(cf. [7], Main Theorem.)

(5.4.4) *Let* ${}_RK_A$ *be a faithfully balanced bimodule and assume* K_A *strongly quasi-injective. Then the following conditions are equivalent:*

(i) ${}_RK$ *is strongly quasi-injective;*

(ii) A *is linearly compact in the* K*-topology and* $\mathrm{Soc}(K_A)$ *is essential in* K_A.

(cf. [7], Theorem 10.)

5.5 THEOREM. *Let* (A, σ) *be a compact ring,* K_A *an injective cogenerator of* Mod-A_σ *of finite grade,* $R=\mathrm{End}(K_A)$ *and let* R *have the* K*-topology* τ. *Then:*

a) *the bimodule* ${}_RK_A$ *is faithfully balanced;*

b) $\mathrm{Soc}(K_A)=\mathrm{Soc}({}_RK)$ *and both are essential in* K_A *and* ${}_RK$ *respectively;*

c) ${}_RK$ *is an injective cogenerator of* R_τ-Mod;

d) (R, τ) *is compact and* ${}_RK$ *has finite grade.*

PROOF. a) By 5.4.1, the bimodule ${}_RK_A$ is faithfully balanced.

b) $\mathrm{Soc}(K_A)$ is essential in K_A since (A, σ) is compact (see 2.1); by Lemma 5.2, $\mathrm{Soc}(K_A)=\mathrm{Soc}({}_RK)$ and $\mathrm{Soc}({}_RK)$ is essential in ${}_RK$.

c) Observe that the K-topology on A coincides with σ by Corollaries 1.7 and 1.8. Therefore it follows from 5.4.4 and the preceding a) and b) that ${}_RK$ is strongly quasi-injective. Thus ${}_RK$ is an injective cogenerator of R_τ-Mod by 5.4.2.

d) Since (R, τ) is complete, it suffices to show that every finitely generated module in R_τ-Mod is finite.

We prove first that a finitely generated module M in R_τ-Mod has finite length. Since M embeds in a finite power of ${}_RK$, which is a cogenerator of R_τ-Mod, and any submodule of ${}_RK^n$ is contained in the sum of its projections, we need only to show that every finitely generated submodule of ${}_RK$ has finite length; but this follows from [10], Proposition 2.3.

Thus we are reduced to seeing that every simple module S in R_τ-Mod is finite (cf. Lemma 3.3). By Lemma 5.1, we have $S\cong\mathrm{Hom}_A(W, K_A)$, for some simple submodule W of K_A; since K_A has finite grade, it follows that

$$S\cong\mathrm{Hom}_A(W, K_A)=\mathrm{Hom}_A(W, \mathrm{Soc}(K_A))=\mathrm{Hom}_A(W, W^n)$$

for some integer n, and this module is finite since W_A is.

Finally ${}_RK$ has finite grade by Lemma 5.3. □

5.6. From this point on we will denote by (A, σ) a compact ring, K_A an injective cogenerator with finite grade of Mod-A_σ, $R=\mathrm{End}(K_A)$ endowed with its K-topology τ. According to the preceding theorem, (R, τ) is a compact ring, ${}_RK$ is an injective cogenerator with finite grade of R_τ-Mod and the bimodule ${}_RK_A$ is faithfully balanced.

If $M\in\mathcal{L}$-A_σ, we denote by $H_1(M)$ the left R-module $\mathrm{Chom}_A(M, K_A)$ endowed with the topology of uniform convergence on the compact submodules of M. If $\mathcal{C}(M)$ is the family of all compact submodules of M, then a typical neighborhood of zero in $H_1(M)$ is

$$\mathcal{W}(F)=\{\xi\in\mathrm{Chom}_A(M, K_A)\,|\,\xi(F)=0\}$$

for $F\in\mathcal{C}(M)$, since K_A is discrete.

Analogous notations and definitions hold also for every $N\in R_\tau$-$\mathcal{L}$, though we denote by $H_2(N)$ the right A-module with the topology of uniform convergence on the compact submodules of N.

Arguing as in Theorem 4.6, we see that this topology coincides with the usual compact-open topology.

5.7 LEMMA. *For every $M\in\mathcal{L}$-A_σ, $H_1(M)$ belongs to R_τ-$\mathcal{L}$. Analogously $H_2(N)\in\mathcal{L}$-A_σ, for every $N\in R_\tau$-$\mathcal{L}$.*

PROOF. Let us prove, first of all, that $H_1(M)$ is complete in its canonical uniformity. Let $(\xi_\lambda)_{\lambda\in\Lambda}$ be a Cauchy net in $H_1(M)$: then, for any $F\in\mathcal{C}(M)$, there exists $\lambda\in\Lambda$ such that, for all λ', $\lambda''\geqq\lambda_F$ in Λ,

$$\xi_{\lambda'}-\xi_{\lambda''}\in\mathcal{W}(F)$$

and this implies that the net is eventually constant on every compact submodule of M. Therefore the net converges uniformly on all compact submodules of M to a morphism $\xi: M \to K_A$, which is continuous, since it is continuous on a compact open submodule of M.

To prove $H_1(M)$ is locally compact, it is sufficient to show that it has a precompact open neighborhood of 0.

Let F be a compact open submodule of M: then a basis of neighborhoods of zero in $\mathcal{W}(F)$ consists of the modules $\mathcal{W}(G)$, where $G \supseteq F$ is a compact submodule of M. Thus we have to show that, under these conditions, $\mathcal{W}(F)/\mathcal{W}(G)$ is finite. Let $\varphi: \mathcal{W}(F) \to H_1(G/F)$ be the morphism defined, for $\xi \in \mathcal{W}(F)$ and $x \in G$ by

$$\varphi(\xi)(x+F)=\xi(x) .$$

This is a well-defined morphism, since $\xi(F)=0$ and, of course, $\mathrm{Ker}(\varphi)=\mathcal{W}(G)$, so that φ induces an injection $\mathcal{W}(F)/\mathcal{W}(G) \hookrightarrow H_1(G/F)$. Arguing as in Lemma 3.3, we get that $H_1(G/F)$ is finite and hence also $\mathcal{W}(F)/\mathcal{W}(G)$ is. Thus $\mathcal{W}(F)$ is precompact and, being open in $H_1(M)$, it is complete, hence compact.

To finish the proof, it is sufficient to show that $H_1(M)$ is a left topological module over (R, τ). Since $H_1(M)$ is a linearly topologized module, this amounts to showing that, for any $\xi \in H_1(M)$ and any $F \in \mathcal{C}(M)$,

$$(\mathcal{W}(F):\xi)=\{r \in R \mid r\xi \in \mathcal{W}(F)\}$$

is an open left ideal in (R, τ). Indeed,

$$(\mathcal{W}(F):\xi)=\{r \in R \mid r\xi(F)=0\}=\mathrm{Ann}_R(\xi(F))$$

and, since $\xi(F)$ is a compact submodule of K_A, it is finite, so that $\mathrm{Ann}_A(\xi(F))$ is open in the K-topology τ of R. □

5.8 LEMMA. *Let $f: L \to M$ be a continuous morphism in $\mathcal{L}$-A_σ. Then the transposed morphism $H_1(f): H_1(M) \to H_1(L)$ is a continuous morphism in R_τ-$\mathcal{L}$.*

The proof is standard.

5.9 THEOREM. *By defining H_1 and H_2 as in 5.6, we get a pair of contravariant functors*

$$\mathcal{L}\text{-}A_\sigma \underset{H_2}{\overset{H_1}{\rightleftarrows}} R_\tau\text{-}\mathcal{L} .$$

5.10 REMARK. If F and G are compact submodules of $M \in \mathcal{L}$-A_σ and F is open, then $F+G$ is an open compact submodule of M.

5.11 DEFINITION. Let $M\in\mathcal{L}\text{-}A_\sigma$ and $\mathcal{F}$ be a basis of neighborhoods of zero in M. We say that $\mathcal{F}$ is a *good basis* for M if

(1) $\mathcal{F}$ consists of open compact submodules of M;

(2) if V_1, V_2 are in $\mathcal{F}$, then there exists $V\in\mathcal{F}$ such that $V_1+V_2\subseteq V$;

(3) for any $x\in M$ there exists $V\in\mathcal{F}$ such that $x\in V$.

If $C_0(M)$ denotes the family of all open compact submodules of M, then $C_0(M)$ is a good basis for M. Indeed, conditions (1) and (2) are trivially verified; for condition (3), we have that xA is a compact submodule of M and, given $V\in C_0(M)$, $xA+V\in C_0(M)$.

5.12 LEMMA. *Let $\mathcal{F}$ be a good basis for $M\in\mathcal{L}\text{-}A_\sigma$. If X is a compact subset of M, there exists $V\in\mathcal{F}$ such that $X\subseteq V$ and so the topology on $H_1(M)$ coincides with the topology of uniform convergence on the elements of $\mathcal{F}$. Moreover the family*

$$\tilde{\mathcal{F}}=\{\mathcal{W}(V)\mid V\in\mathcal{F}\}$$

is a good basis for $H_1(M)$.

PROOF. Fix $V\in\mathcal{F}$: there exists a finite subset $\{x_1, x_2, \cdots, x_n\}$ of X such that

$$X\subseteq\bigcup_{i=1}^{n}(x_i+V)\subseteq\sum_{i=1}^{n}(V_i+V)$$

where $x_i\in V_i\in\mathcal{F}$. By the definition of a good basis, there is $F\in\mathcal{F}$ such that $X\subseteq F$.

Let us prove that $\tilde{\mathcal{F}}$ is a good basis for $H_1(M)$. As we have seen in 5.6, $\mathcal{W}(F)$ is open and compact in $H_1(M)$, for all $F\in\mathcal{F}$. Moreover, if $F_1, F_2\in\mathcal{F}$,

$$\mathcal{W}(F_1)+\mathcal{W}(F_2)\subseteq\mathcal{W}(F)$$

where $F\in\mathcal{F}$ and $F\subseteq F_1\cap F_2$.

Finally, let $\xi\in H_1(M)$ and take $F\in\mathcal{F}$ such that $F\subseteq\operatorname{Ker}(\xi)$ (recall that $\operatorname{Ker}(\xi)$ is open in M); then $\xi\in\mathcal{W}(F)$. □

5.13 DEFINITION. Let $M\in\mathcal{L}\text{-}A_\sigma$. The *canonical morphism* $\omega_M: M\to H_2H_1(M)$ is defined by setting, for $x\in M$ and $\xi\in H_1(M)$,

$$\omega_M(x)(\xi)=\xi(x),$$

where, for the sake of clarity, we write all morphisms on the left. This is well-defined, since $\omega_M(x)$ is a continuous morphism of $H_1(M)$ into $_RK$. Indeed, let $x\in M$: then

$$\operatorname{Ker}(\omega_M(x)) = \{\xi \in H_1(M) \mid \xi(x)=0\} = \mathcal{W}(xA)$$

which is, by definition an open submodule in $H_1(M)$.

It is clear that ω_M is injective, since $\operatorname{Chom}_A(M, K_A)$ separates the points of M.

We define in a similar way the canonical morphism ω_N, for all $N \in R_\tau\text{-}\mathcal{L}$.

To maintain the convention of writing morphisms to the opposite side to the scalars, we set $\omega_M(x)=\hat{x}$.

5.14 PROPOSITION. *Let $M \in \mathcal{L}\text{-}A_\sigma$; then the canonical morphism*

$$\omega_M : M \longrightarrow H_2H_1(M)$$

is injective, continuous and open onto its image i.e., it is a topological and algebraic embedding of M into $H_2H_1(M)$.

A similar result holds for any $N \in R_\tau\text{-}\mathcal{L}$.

PROOF. We have already remarked that ω_M is injective. The proof will be complete if we show that

$$\omega_M(F) = \mathcal{W}(\mathcal{W}(F)) \cap \operatorname{Im}(\omega_M), \tag{1}$$

for every $F \in \mathcal{C}_0(M)$, since we know that $\mathcal{C}_0(M)$ is a good basis for M.

Let us prove the "$\subseteq$" inclusion, which is equivalent to the continuity of ω_M. If $x \in F$, then $\omega_M(x) \in \mathcal{W}(\mathcal{W}(F))$, since, for all $\xi \in \mathcal{W}(F)$, $(\xi)\hat{x}=\xi(x)=0$.

For the converse inclusion, we take $x \in M$ such that $\hat{x}=\omega_M(x) \in \mathcal{W}(\mathcal{W}(F))$. Assume that $x \notin F$. Since F is an open compact submodule of M and K_A is an injective cogenerator of $\mathcal{L}\text{-}A_\sigma$, there exists a continuous morphism $\eta : M \to K_A$ such that $\eta(F)=0$ and $\eta(x) \neq 0$. Thus $(\eta)\hat{x} \neq 0$ with $\eta \in \mathcal{W}(F)$ and this contradicts the fact that $\hat{x} \in \mathcal{W}(\mathcal{W}(F))$. □

5.15. Let $M \in \text{CM-}A_\sigma$: then it is clear that $H_1(M) \in R_\tau\text{-Mod}$. Conversely, if $N \in R_\tau\text{-Mod}$, then $H_2(N)$, which is $\operatorname{Hom}_R(N, {}_RK)$ with the topology of pointwise convergence, is in $\text{CM-}A_\sigma$, by reasoning as in Lemma 3.3 and Proposition 3.5. Of course, analogous results hold for modules in $R_\tau\text{-CM}$ and $\text{Mod-}A_\sigma$.

5.16 PROPOSITION. *For any $M \in \text{CM-}A_\sigma$, the canonical morphism ω_M is a topological isomorphism. A similar result holds for any $N \in R_\tau\text{-CM}$.*

PROOF. We have already shown that ω_M is a topological embedding; since $\operatorname{Im}(\omega_M)$ is compact, we need only to prove that it is dense in $H_2H_1(M)$.

Let $L=\langle \xi_1, \cdots, \xi_n \rangle$ be a finitely generated submodule of the discrete module

$H_1(M)$ and let $f: L \to {}_RK$ be a morphism. We want to show that there exists $x \in M$ such that the restriction of $\hat{x} = \omega_M(x)$ to L coincides with f. Consider the submodule X of K_A^n defined by

$$X = \{((\xi_1)\hat{x}, \cdots, (\xi_n)\hat{x}) \mid x \in M\}$$

and set $y = ((\xi_1)f, \cdots, (\xi_n)f)$. Assume by contradiction that $y \notin X$. Since K_A is an injective cogenerator of Mod-A_σ, there exists a morphism $\varphi \in \mathrm{Hom}_A(K^n, K)$ such that $\varphi(y) \neq 0$ and $\varphi(X) = 0$. Identify φ with an n-tuple $(r_1, \cdots, r_n)$, with $r_i \in R = \mathrm{End}(K_A)$. By the R-linearity of $\hat{x}$, for all $x \in M$, we have

$$\sum_{i=1}^{n} r_i(\xi_i)\hat{x} = \Big(\sum_{i=1}^{n} r_i\xi_i\Big)\hat{x} = \Big(\sum_{i=1}^{n} r_i\xi_i\Big)(x) = 0$$

so that the morphism $\sum_{i=1}^{n} r_i\xi_i = 0$ and this is absurd, since otherwise

$$\Big(\sum_{i=1}^{n} r_i\xi_i\Big)f = \sum_{i=1}^{n} (r_i\xi_i)f = \sum_{i=1}^{n} r_i(\xi_i)f = \varphi(y) \neq 0\,.$$

Let $g \in H_2H_1(M)$; a typical neighborhood of g in $H_2H_1(M)$ is of the form $g + \mathcal{W}(F)$, where $F = \{\xi_1, \cdots, \xi_n\}$ is a finite subset of $H_1(M)$. By the preceding result, there exists $x \in M$ such that $(\xi_i)g = (\xi_i)\hat{x} = \xi_i(x)$, for $i = 1, \cdots, n$. Therefore $\hat{x} \in (g + \mathcal{W}(F)) \cap \mathrm{Im}(\omega_M)$ and this proves the density.

5.17 PROPOSITION. *For any* $N \in R_\tau$-Mod *the canonical morphism* ω_N *is an isomorphism. A similar result holds in* Mod-A_σ.

PROOF. We need only to show that ω_N is surjective. Let $Y = \mathrm{Im}(\omega_N)$ and assume there exists $y \in H_1H_2(N) \setminus \mathrm{Im}(\omega_N)$. Since ${}_RK$ is an injective cogenerator of R_τ-Mod, there is a continuous morphism $\beta: H_1H_2(N) \to {}_RK$ such that $(Y)\beta = 0$ and $(y)\beta \neq 0$. Thus $\beta \in H_2H_1H_2(N)$ and $\beta \neq 0$. Set $M = H_2(N) \in \mathrm{CM}\text{-}A_\sigma$; by Proposition 5.16, there exists $\alpha \in M = H_2(N)$ such that $\omega_M(\alpha) = \hat{\alpha} = \beta$. Then, for all $x \in N$, we have $\hat{x} \in Y$ and so

$$0 = (\hat{x})\beta = (\hat{x})\hat{\alpha} = \hat{x}(\alpha) = (x)\alpha$$

which implies that $\alpha = 0$; contradiction. □

We can now state the main theorem of this section, which summarizes and generalizes the previous results.

5.19 THEOREM. *Let* (A, σ) *be a compact ring,* K_A *an injective cogenerator of* Mod-A_σ *with finite grade,* $R = \mathrm{End}(K_A)$ *and* τ *be the* K*-topology on* R. *Then* (R, τ) *is compact,* ${}_RK$ *is an injective cogenerator of* R_τ-Mod *with finite grade and the bimodule* ${}_RK_A$ *is faithfully balanced. Moreover, for all* $M \in \mathcal{L}\text{-}A_\sigma$ *and* $N \in$

R_τ-$\mathcal{L}$, the canonical morphism ω_N and ω_N are topological isomorphisms. In particular (H_1, H_2) is a duality between $\mathcal{L}$-A_σ and R_τ-$\mathcal{L}$.

PROOF. Let $M \in \mathcal{L}$-A_σ and let F be a compact open submodule of M. Then we have the exact sequence

$$0 \longrightarrow F \xrightarrow{i} M \xrightarrow{\pi} D \longrightarrow 0$$

so that we obtain the commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & F \arrow[r, "i"] \arrow[d, "\omega_F"'] & M \arrow[r, "\pi"] \arrow[d, "\omega_M"'] & D \arrow[r] \arrow[d, "\omega_D"'] & 0 \\
0 \arrow[r] & H_2H_1(F) \arrow[r, "H_2H_1(i)"] & H_2H_1(M) \arrow[r, "H_2H_1(\pi)"] & H_2H_1(D) \arrow[r] & 0
\end{tikzcd}$$

and, since ω_F and ω_D are isomorphisms, also ω_M is. Since ω_M is a topological embedding, it is a topological isomorphism. □

5.19. Let (A, σ) be a compact ring. Among all injective cogenerators with finite grade of Mod-A_σ, there is one which realizes the Pontryagin duality between $\mathcal{L}$-A_σ and A_σ-$\mathcal{L}$. We want to determine it.

Let $(W_\lambda)_{\lambda \in \Lambda}$ be a system of representatives of the non isomorphic simple modules in Mod-A_σ, $D_\lambda = \mathrm{End}_A(W_\lambda)$ and n_λ be the dimension of W_λ as a left vector space over D_λ. As we already know, n_λ is finite, for every $\lambda \in \Lambda$. Set

$$\dim(A, \sigma) = (n_\lambda)_{\lambda \in \Lambda}.$$

5.20 THEOREM. *Let (A, σ) be a compact ring, K_A an injective cogenerator of* Mod-A_σ *with finite grade. Then K_A realizes the Pontryagin duality between $\mathcal{L}$-A_σ and A_σ-$\mathcal{L}$ if and only if the grade of K_A coincides with* $\dim(A, \sigma)$.

PROOF. The condition is sufficient by the structure theorem 2.3.

To show the necessity, let $\dim(A, \sigma) = (n_\lambda)_{\lambda \in \Lambda}$, so that

$$K_A = \bigoplus_{\lambda \in \Lambda} E_\sigma(W_\lambda)^{n_\lambda}.$$

Set $R = \mathrm{End}(K_A)$ and consider, on R, the K-topology τ.

Let $\Gamma\colon A_\sigma$-$\mathcal{L} \to \mathcal{L}$-$A_\sigma$ be the Pontryagin duality and set $T_A = \Gamma(A_\sigma)$. Then T_A is an injective cogenerator of Mod-A_σ with finite grade and (A, σ) is topologically isomorphic to $\mathrm{End}(T_A)$ with the T-topology. By Theorem 2.3, the grade of T_A coincides with $\dim(A, \sigma)$, so that $T_A \cong K_A$. Hence (R, τ) is topologically isomorphic to (A, σ). □

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