TSUKUBA J. MATH.
Vol. 8 No. 1 (1984). 31–54

ISOMETRIC IMMERSIONS OF LORENTZ SPACE WITH PARALLEL SECOND FUNDAMENTAL FORMS

By

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Introduction

In a series of papers, [F1], [F2], [F3], [F4], D. Ferus classified submanifolds of euclidean space with parallel second fundamental forms. These submanifolds form an important class of examples. In [F4], Ferus shows that they appear in many topics in differential geometry.

There has been much recent work on parallel submanifolds of other ambient spaces—notably the work of H. Naitoh, [Na1]–[Na4] and M. Takeuchi [T]. An interesting problem is to classify the parallel submanifolds of euclidean spaces equipped with indefinite metrics.

This paper studies special classes of parallel submanifolds in $\boldsymbol{R}_1^m$, Lorentz space of signature $(1, m-1)$ and in $\boldsymbol{R}_2^m$, euclidean space of signature $(2, m-2)$. All umbilical submanifolds are classified, as well as isometric immersions $\boldsymbol{R}^n \to \boldsymbol{R}_1^{n+k}$, $\boldsymbol{R}_1^n \to \boldsymbol{R}_1^{n+2}$ and $\boldsymbol{R}_1^n \to \boldsymbol{R}_2^{n+2}$ with parallel second fundamental forms. These theorems indicate some of the modifications which will be necessary in order to obtain a complete classification.

The preliminary section (0) gives some basic results about indefinite Riemannian geometry. These include an indefinite version of the result of Allendoerfer and Erbacher for reducing the codimension of an isometric immersion with parallel second fundamental form ($\nabla^*\alpha=0$), and an improved version of Petrov's canonical forms for symmetric transformations of Lorentz space.

Section 1 classifies isometric immersions $\boldsymbol{R}_1^2 \to \boldsymbol{R}_1^4$, $\boldsymbol{R}_1^2 \to \boldsymbol{R}_2^4$ and $\boldsymbol{R}^n \to \boldsymbol{R}_1^{n+k}$ with $\nabla^*\alpha=0$. These maps include a flat n-dimensional umbilic with lightlike mean curvature vector (1.4) and the complex circle of radius $\kappa \in \boldsymbol{C}$ (1.12).

Section 2 contains the main classification results. These state that isometric immersions $\boldsymbol{R}_1^n \to \boldsymbol{R}_1^{n+2}$ and $\boldsymbol{R}_1^n \to \boldsymbol{R}_2^{n+2}$ with $\nabla^*\alpha=0$ are either quadratic in nature, like the flat umbilical immersion with lightlike mean curvature vector, or the product of the identity map and previously determined low dimensional maps.

Most of the results in this paper are taken from the author's Brown University

Received March 28, 1983.

doctoral dissertation. The author gratefully acknowledges Professor Katsumi Nomizu's constant encouragement and patience.

Notation and Terminology

$\boldsymbol{R}^p_{i,j}$ denotes the p-dimensional affine space with the metric $(\ ,\ )$ whose canonical form is

$$\begin{bmatrix} -I_i & & \\ & I_{p-i-j} & \\ & & O_j \end{bmatrix}$$

where I_k is the $k\times k$ identity matrix and O_j is the $j\times j$ O matrix. The metric is non-degenerate iff $j=0$, in which case we write $\boldsymbol{R}^p_i$.

In general, if M is a p-dimensional manifold whose tangent spaces have a metric of signature $(i, p-i-j, j)$ we write $M^p_{i,j}$.

The indefinite sphere, hyperbolic space and lightcone are defined as follows:

$$S^{p-1}_i(r)=\{x\in\boldsymbol{R}^p_i:\ (x,x)=r^2\},\quad r>0$$
$$H^{p-1}_{i-1}(r)=\{x\in\boldsymbol{R}^p_i:\ (x,x)=-r^2\},\quad r<0$$
$$LC^{p-1}_{i-1,1}=\{x\in\boldsymbol{R}^p_i:\ (x,x)=0 \text{ and } x\neq 0\}$$

The isometric immersions

$$t\to(a\cos(t/a),\ a\sin(t/a))$$
$$t\to(b\sinh(t/b),\ b\cosh(t/b))$$
$$t\to(b\cosh(t/b),\ b\sinh(t/b))$$

are called circle maps. The first maps onto $S^1\subset\boldsymbol{R}^2$ or $H^1_1\subset\boldsymbol{R}^2_2$, the second onto $S^1_1\subset\boldsymbol{R}^2_1$ and the third onto $H^1\subset\boldsymbol{R}^2_1$. The circle maps, along with the map

$$t\longmapsto(t,0)$$

are called one-dimensional maps.

A vector X in an indefinite inner product space is called spacelike if $(X,X)>0$, timelike if $(X,X)<0$ and null or lightlike if $(X,X)=0$, where $(\ ,\ )$ is the inner product. A curve is called spacelike, timelike, or null depending on the character of its tangent vectors.

A basis $X_1,\cdots,X_n$ of an indefinite inner product space with the signature $(p,n-p)$ is called orthonormal if the vectors are pairwise orthogonal, the first p are unit timelike and the last $n-p$ are unit spacelike. If the signature is $(1,n-1)$, so that the inner product space is Lorentz, we can define a pseudo-orthonormal basis $\{l,\hat{l},X_1,\cdots,X_{n-2}\}$. In such a basis $(l,l)=0=(\hat{l},\hat{l})=(l,X_i)=(\hat{l},X_i)$, $(l,\hat{l})=1$ and $(X_i,X_j)=\delta_{ij}$, $1\leqslant i,\ j\leqslant n-2$. A $\wedge$ over a vector will always have inner product 1

with the hatless vector.

We recall the following notation from [K-N], vol. II, ch. 7. If $f\colon (M, g)\to(\tilde{M}, \tilde{g})$ is an isometric immersion of one indefinite Riemannian manifold into another, and if the Riemannian connections are denoted by ∇ and $\tilde{\nabla}$, then for vector fields X and Y on M we have the following decomposition.

$$\text{I.}\quad \tilde{\nabla}_X f_* Y = f_* \nabla_X Y + \alpha(X, Y)\,.$$

Here $f_*\nabla_X Y$ is the tangential component and $\alpha(X, Y)$ is the normal component. α is called the second fundamental form of f. For a normal field ξ to M we have

$$\text{II.}\quad \tilde{\nabla}_X \xi = -f_* A_\xi X + \nabla^\perp_X \xi\,.$$

A_ξ is called the shape operator associated to ξ and $\nabla^\perp$ is called the normal connection. A and α are related by the equation

$$g(A_\xi X, Y) = \tilde{g}(\alpha(X, Y), \xi)\,.$$

α is said to be parallel if

$$(\nabla^*_Z \alpha)(X, Y) := \nabla^\perp_Z \alpha(X, Y) - \alpha(\nabla_Z X, Y) - \alpha(X, \nabla_Z Y) = 0\,.$$

0. Preliminaries

Defining the mean curvature vector η as in the Riemannian case, we have the following.

0.1 Proposition. If $f\colon M\to\tilde{M}$ is an isometric immersion of one indefinite Riemannian manifold into another with parallel second fundamental form, then

(i) the mean curvature vector is parallel: $\nabla^\perp_\eta = 0$.

(ii) the first normal spaces $N^1(x)$, $x\in M$ are $\nabla^\perp$-parallel.

$$N^1(x) := \{N^0(x)\}^\perp \quad \text{where } N^0(x) = \{\xi\in N(x)\colon A_\xi \equiv 0\}\,.$$

The next theorem is about isometric immersions with parallel first normal spaces and so can be applied to immersions with parallel second fundamental forms.

0.2 Theorem (Indefinite version of a theorem of Allendoerfer and Erbacher). Let $f\colon M^n_i \to \boldsymbol{R}^m_j$ be an isometric immersion of an indefinite Riemannian manifold with signature $(i, n-i)$ into $\boldsymbol{R}^m_j$. If the first normal spaces are parallel, then there exists a complete $n+k$ dimensional totally geodesic submanifold M^* of $\boldsymbol{R}^m_j$ (where $n=\dim M$ and $k=\dim N^1$) such that $f(M)\subset M^*$.

Note: $M^* = \boldsymbol{R}^{n+k}_{s,t}$ for some s, t and t need not be zero. The proof of the theorem shows that t measures the degenerate part of N^1.

PROOF. The assumption means that for any curve τ from x to y the parallel displacement of normal vectors along τ with respect to the normal connection maps $N^1(x)$ onto $N^1(y)$. Thus the dimension of N^1 is a constant, say k. If ξ is a normal vector field such that $\xi \in N^1(x)$ for each x in M, then $\nabla^\perp_X \xi \in N^1(x)$ for all X in $T_x(M)$. Since $\nabla^\perp$ is a metric connection, the subspaces $N^0(x)$ are also parallel relative to the normal connection. In fact $\xi \in N^1$, $\theta \in N^0$ implies $0=X(\xi,\theta)=(\nabla^\perp_X \xi,\theta)+(\xi,\nabla^\perp_X \theta)$. Since the first term is zero, so is the second.

Let x_0 be a point of M and consider the $n+k$ dimensional subspace E of $\boldsymbol{R}^m_j$ through $f(x_0)$ which is perpendicular to $N^0(x_0)$. $E=T_{x_0}(M)\oplus N^1(x_0)$. The degenerate part of E is $N^0(x_0)\cap N^1(x_0)$. It will be shown that $f(M)\subset E$. Let x_t be any curve in M starting at x_0. For any ξ_0 in $N^0(x_0)$ let ξ_t be the result of $\nabla^\perp$-parallel displacement of ξ_0 along x_t, so that $\xi_t \in N^0(x_t)$. For the euclidean connection we have

$$D_t\xi_t = -f_* A_{\xi_t}(\dot{x}_t) + \nabla^\perp_t \xi_t = 0$$

which means that ξ_t is parallel in $\boldsymbol{R}^m_j$ and so is a constant vector. Now we have

$$\begin{aligned}\frac{d}{dt}(f(x_t)-f(x_0),\xi_0) &= (f_*(\dot{x}_t),\xi_0)\\ &= (f_*(\dot{x}_t),\xi_t)\\ &= 0\end{aligned}$$

so that $f(x_t)$ lies in E. Since x_t is an arbitrary curve in M, $f(M)\subset E$.

Next is an indefinite version of Moore's lemma [Mo 1] which gives a condition that allows an isometric immersion of a product manifold to be decomposed into a product of immersions.

0.3 THEOREM. Let M be an indefinite Riemannian product $M_1\times M_2$. If $f: M\to \boldsymbol{R}^n_j$ is an isometric immersion with $\alpha(X,Y)=0$ whenever $X\in T_x(M_1)$ and $Y\in T_x(M_2)$ then $f=f_1\times f_2: M_1\times M_2 \to \boldsymbol{R}^{n_1}_{j_1}\times \boldsymbol{R}^{n_2}_{j_2}$ and each f_i is an isometric immersion.

The decomposition theorems here involve the shape operators of Riemannian manifolds. A shape operator of a Riemannian manifold is always diagonalizable, but this is not the case for a shape operator of a Lorentzian manifold. In order to analyze the latter, a special case of Petrov's Principal axis theorem for a tensor is needed ([P] pp. 50-55).

0.4 THEOREM. Let V be a real n-dimensional vector space equipped with a Lorentzian metric and let A be linear transformation of V which is symmetric with respect to this metric. Then A can be put into one of the following four forms with respect to bases whose inner products are given by G.

$$A=\begin{pmatrix}\lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n\end{pmatrix} \qquad G=\begin{pmatrix}-1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1\end{pmatrix}$$

$$A=\begin{pmatrix}\lambda & 1 & & & \\ 0 & \lambda & & & \\ & & \lambda_1 & & \\ & & & \ddots & \\ & & & & \lambda_{n-2}\end{pmatrix} \qquad G=\begin{pmatrix}0 & 1 & & & \\ 1 & 0 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1\end{pmatrix}$$

$$A=\begin{pmatrix}\lambda & 0 & 1 & & \\ 0 & \lambda & 0 & & \\ 0 & 1 & \lambda & & \\ & & & \lambda_1 & \\ & & & & \lambda_{n-3}\end{pmatrix} \qquad G=\begin{pmatrix}0 & 1 & & & \\ 1 & 0 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1\end{pmatrix}$$

$$A=\begin{pmatrix}\alpha & \beta & & & \\ -\beta & \alpha & & & \\ & & \lambda_1 & & \\ & & & \ddots & \\ & & & & \lambda_{n-2}\end{pmatrix} \qquad G=\begin{pmatrix}+1 & & & & \\ & -1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1\end{pmatrix} \quad \beta\neq 0.$$

These will be referred to as the case where there: are simple real eigenvalues, is a non-simple eigenvalue of multiplicity 2, is a non-simple eigenvalue of multiplicity 3 or is a complex eigenvalue. The first and fourth cases will be said to have simple eigenvalues.

This is a slightly sharper version of the theorem in [P]. Theorem 2, p. 229 [Mal] allows us to use an orthonormal basis in the case of a complex eigenvalue instead of the more complicated basis which Petrov employs.

The following proposition can be proved using 0.4. Another proof can be found in [Mo 2].

0.5 Proposition. Let $M(c)$ and $\tilde{M}(\tilde{c})$ denote space forms of constant curvature c and $\tilde{c}$. Suppose

(i) $M_0^n(c)$ is isometrically immersed in $\tilde{M}_1^{n+1}(\tilde{c})$
or (ii) $M_1^n(c)$ is isometrically immersed in $\tilde{M}_1^{n+1}(\tilde{c})$
or (iii) $M_1^n(c)$ is isometrically immersed in $\tilde{M}_2^{n+1}(\tilde{c})$.

If $c\neq\tilde{c}$ and $n>2$ then the immersion is umbilical and in case (i) $\tilde{c}>c$, in case (ii) $\tilde{c}<c$ and in case (iii) $\tilde{c}>c$.

It is necessary to know what form a Lorentzian symmetric matrix takes if it

commutes with one of the four standard Lorentzian symmetric matrices.

0.6 PROPOSITION. Let A and B be square matrices which are symmetric with respect to a Lorentzian inner product. If, with respect to a pseudo-orthonormal basis,

$$A=\begin{pmatrix} \lambda & 1 & & & & \\ 0 & \lambda & & & & \\ & & \lambda I_{k_0} & & & \\ & & & \lambda_1 I_{k_1} & & \\ & & & & \ddots & \\ & & & & & \lambda_s I_{k_s} \end{pmatrix}$$

and B commutes with A then

$$B=\left(\begin{array}{cccc|ccc} \mu & b & c_1 \cdots\cdots & c_{k_0} & & & \\ 0 & \mu & 0 & & & & \\ \vdots & c_1 & & & & & \\ \vdots & \vdots & & d^0_{ij} & & & \\ 0 & c_{k_0} & & & & & \\ \hline & & & & \boxed{d^1_{ij}} & & \\ & & & & & \ddots & \\ & & & & & & \boxed{d^s_{ij}} \end{array}\right)$$

where d^r_{ij} is $k_r \times k_r$.

0.7 PROPOSITION. Let A and B be square matrices which are symmetric with respect to a Lorentzian inner product. If, with respect to a pseudo-orthonormal basis,

$$A=\begin{pmatrix} \lambda & 0 & 1 & & & & \\ 0 & \lambda & 0 & & & & \\ 0 & 1 & \lambda & & & & \\ & & & \lambda I_{k_0} & & & \\ & & & & \lambda_1 I_{k_1} & & \\ & & & & & \ddots & \\ & & & & & & \lambda_s I_{k_s} \end{pmatrix}$$

and B commutes with A, then

$$B=\left(\begin{array}{cccc|c} \mu & b & c_1 & c_2\cdots c_{k_0+1} & 0\cdots\cdots\cdots 0 \\ 0 & \mu & 0 & 0\cdots 0 & 0\cdots\cdots\cdots 0 \\ 0 & c_1 & \mu & & \\ 0 & c_2 & & & \\ \vdots & \vdots & & d^0_{ij} & \\ 0 & c_{k_0+1} & & & \\ \hline \vdots & 0 & & & d^1_{ij} \\ \vdots & \vdots & & & \\ \vdots & \vdots & & & \\ 0 & 0 & & & \end{array}\right.$$

$$\left.\begin{array}{c} \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ d^s_{ij} \end{array}\right)$$

where d^r_{ij} is $k_r\times k_r$.

0.8 PROPOSITION. Let A and B be square matrices which are symmetric with respect to a Lorentzian inner product. If, with respect to an orthonomal basis

$$A=\begin{pmatrix} \alpha & \beta & & & & \\ -\beta & \alpha & & & & \\ & & \alpha I_{k_0} & & & \\ & & & \lambda_1 I_{k_1} & & \\ & & & & \ddots & \\ & & & & & \lambda_s I_{k_s} \end{pmatrix} \quad \beta\neq 0$$

and B commutes with A then

$$B=\begin{pmatrix} \gamma & \delta & & & \\ -\delta & \gamma & & & \\ & & d^0_{ij} & & \\ & & & \ddots & \\ & & & & d^s_{ij} \end{pmatrix}$$

where d^r_{ij} is $k_r\times k_r$.

Propositions 0.6, 0.7 and 0.8 can be obtained by letting B have the form of a general symmetric matrix and setting AB and BA equal.

0.9 Partially umbilical immersions. If $f\colon M^n_j\to \boldsymbol{R}^m_k$ is an isometric immersion and there is a globally defined normal vector field θ such that

(i) θ is everywhere non-zero

(ii) $\nabla^\perp\theta=0$

(iii) $A_\theta=\lambda Id \quad \lambda\neq 0$

then $f(M^n_j)$ is contained inside

$$S^{m-1}_k\left(\frac{\sqrt{(\theta,\theta)}}{\lambda}\right) \qquad \text{if} \quad (\theta,\theta)>0\,,$$

$$H^{m-1}_{k-1}\left(\frac{-\sqrt{-(\theta,\theta)}}{\lambda}\right) \quad \text{if} \quad (\theta,\theta)<0 \quad \text{and}$$
$$LC^{m-1}_{k-1,1} \qquad\qquad \text{if} \quad (\theta,\theta)=0\,.$$

0.10 COROLLARY. If θ in 0.9 is the mean curvature vector, then $f(M^n_j)$ is immersed minimally in S^{m-1}_k or H^{m-1}_{k-1}. (Such an immersion is called pseudo-umbilical by Chen and Yano [Y-C].)

PROOFS. Note first that the vector $f(x)+\theta/\lambda$ is constant for all x in M. Denoting this vector by $\vec{c}$ we have

$$(f(x)-\vec{c}, f(x)-\vec{c})=(\theta/\lambda, \theta/\lambda)=\frac{(\theta,\theta)}{\lambda^2}\,.$$

Because θ is parallel this is a constant, and so $f(x)$ is contained in the sphere, hyperbolic space or lightcone with center $\vec{c}$.

Moreover, if θ is the mean curvature vector of $f(M^n_j)$ in $\boldsymbol{R}^m_k$, then θ is parallel to the position vector $f(x)-\vec{c}$ and so $f(M^n_j)$ is minimal in either S^{m-1}_k or H^{m-1}_{k-1}. Here $\lambda\neq 0$ implies $(\theta,\theta)\neq 0$. Q. E. D.

1. Isometric Immersions with Parallel Second Fundamental Forms: $\boldsymbol{R}^n\to\boldsymbol{R}^{n+k}_1$, $\boldsymbol{R}^2_1\to\boldsymbol{R}^4_1$, $\boldsymbol{R}^2_1\to\boldsymbol{R}^4_2$.

The classification of these isometric immersions will follow from some general facts about submanifolds with $\nabla^*\alpha=0$ and global, parallel normal fields.

The first step is to classify one-dimensional submanifolds with parallel second fundamental forms. Thus, the first proposition deals with non-null curves which have parallel mean curvature vector.

1.1 PROPOSITION. Let $x(s)$ be a non-null curve in an indefinite euclidean space $\boldsymbol{R}^m_j$ with parallel mean curvature vector η.

(i) If $\eta=0$ then $x(s)$ is a line segment.

(ii) If $(\eta,\eta)\neq 0$ then $x(s)$ is part of a circle map.

(iii) If $(\eta,\eta)=0$ and $\eta\neq 0$ then $x(s)$ is a curve in $\boldsymbol{R}^2_{0,1}$ or $\boldsymbol{R}^2_{1,1}$ of the form $s\mapsto(s, as^2+bs+c)$. The first map is denoted by U^1, the second by U^1_1.

In case (iii) the metric vanishes on the second coordinate. $\boldsymbol{R}^2_{j,1}$ is imbedded in $\boldsymbol{R}^3_{j+1}$ by sending $(x^1,x^2)\mapsto(x^2,x^1,x^2)$.

PROOF. Let $x(s)$ be a curve in $\boldsymbol{R}^m_j$ with s its arc-length parameter and $X(s)=dx(s)/ds=:\dot{x}(s)$ its unit tangent vector. If D denotes the flat connection in $\boldsymbol{R}^m_j$ we see that $D_sX(s)$ is normal to $x(s)$ and so $D_sX(s)=\alpha(X(s),X(s))$. The mean curvature

vector $\eta(s)=(X(s), X(s))\alpha(X(s)), X(s))$. Because $\eta(s)$ is parallel $(\eta(s), \eta(s))=k$, a constant. If $\eta=0$ then $D_sX(s)=0$ and $x(s)$ is a line segment.

If $k\neq 0$ set $Y(s)=\eta(s)/\sqrt{|k|}$, so that $Y(s)$ is a unit vector normal to $x(s)$. $X(s)$ and $Y(s)$ form an orthonormal frame along $x(s)$ such that

$$D_sX(s)=\sqrt{|k|}(X(s), X(s))Y(s)$$
$$D_sY(s)=\frac{-k}{\sqrt{|k|}}X(s)$$

(ii) follows by 0.2 and inspection.

Now suppose $(\eta, \eta)=0$ and $\eta\neq 0$. By 0.2 $x(s)\subset \boldsymbol{R}^2_{0,1}$ or $\boldsymbol{R}^2_{1,1}$, depending on the length of $X(s)$. Since $x(s)$ is an immersed submanifold

$$x(s)=(s, \phi(s)) \text{ or } (\phi(s), s),$$

where the metric vanishes on the second coordinate. Thus

$$\dot{x}(s)=(1, \phi') \text{ or } (\phi', 1)$$

and $\alpha(X(s), X(s))=(0, \phi'')$ or $(0, 0)$. The second possibility is totally geodesic. Since

$$\nabla^{\perp}_s\alpha(X(s), X(s))=0$$

$\phi'''=0$ and $\phi(s)=as^2+bs+c$. Q. E. D.

One way to obtain submanifolds of an indefinite inner product space with parallel second fundamental forms is to consider umbilical immersions. We will find, besides the sphere and hyperbolic space, a third type of umbilical immersion, and from these a complete classification of umbilical immersions.

1.2 LEMMA. Suppose $f: M^n_i\to \boldsymbol{R}^{n+k+s}_{j,s}$ is an isometric immersion. If $p: \boldsymbol{R}^{n+k+s}_{j,s}\to \boldsymbol{R}^{n+k}_j$ is projection onto the first $n+k$ coordinates then $p\circ f: M^n_i\to \boldsymbol{R}^{n+k}_j$ is an isometric immersion.

PROOF. Choose x_0 in M and let U be a coordinate neighborhood of x_0 with coordinates $(x^1, \cdots, x^n)$. Let $f(x^1, \cdots, x^n)=(f^1, \cdots, f^{n+k}, h^1, \cdots, h^s)$, where the metric vanishes in the last s coordinates. Clearly

$$g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^m}\right)=\left(f_*\frac{\partial}{\partial x^i}, f_*\frac{\partial}{\partial x^m}\right)=\left((p\circ f)_*\frac{\partial}{\partial x^i}, (p\circ f)_*\frac{\partial}{\partial x^m}\right). \quad \text{Q. E. D.}$$

1.3 LEMMA. Let $f: M^n_t\to \boldsymbol{R}^{n+m+s}_{r,s}$ be an isometric immersion with parallel second fundamental form. If $f(x^1, \cdots, x^n)=(f^1, \cdots, f^{n+m}, h^1, \cdots, h^s)$ locally, then

$$\frac{\partial^3 h^p}{\partial x^i\partial x^j\partial x^k}=\sum_{u=1}^n \Gamma^u_{ji}\frac{\partial^2 h^p}{\partial x^u\partial x^k}+\sum_{v=1}^n \Gamma^v_{jk}\frac{\partial^2 h^p}{\partial x^v\partial x^i}.$$

PROOF. If f has parallel second fundamental form then

$$\nabla^{\perp}_{Z}\alpha(X, Y)=\alpha(\nabla_{Z}X, Y)+\alpha(X, \nabla_{Z}Y).$$

If p is projection onto the last s coordinates then

$$\nabla^{\perp}_{Z}p\circ\alpha(X, Y)=p\circ\alpha(\nabla_{Z}X, Y)+p\circ\alpha(X, \nabla_{Z}Y).$$

Replacing X, Y, Z by $\partial/\partial x^{j}$, $\partial/\partial x^{k}$, $\partial/\partial x^{i}$ yields the lemma, since, as is well known

$$\frac{\partial^{2}f}{\partial x^{j}\partial x^{k}}=D_{\partial/\partial x^{j}}f_{*}\left(\frac{\partial}{\partial x^{k}}\right)=f_{*}\left(\nabla_{\partial/\partial x^{j}}\frac{\partial}{\partial x^{k}}\right)+\alpha\left(\frac{\partial}{\partial x^{j}}, \frac{\partial}{\partial x^{k}}\right). \qquad \text{Q. E. D.}$$

1.4 THEOREM. Let $f\colon M^{n}_{k}\to \boldsymbol{R}^{m}_{j}$ be a complete umbilical immersion. Assume the mean curvature vector $\eta\neq 0$, so that f is not totally geodesic.

(i) If $(\eta, \eta)>0$ then $f(M)\subset S^{n}_{k}\subset \boldsymbol{R}^{n+1}_{k}$

(ii) If $(\eta, \eta)<0$ then $f(M)\subset H^{n}_{k}\subset \boldsymbol{R}^{n+1}_{k+1}$

(iii) If $(\eta, \eta)=0$ then $M=\boldsymbol{R}^{n}_{k}$ and $f(M)\subset \boldsymbol{R}^{n+1}_{k,1}$ as an umbilic.

In this last case $f(x^{1}, \cdots x^{n})=(x^{1}, \cdots, x^{n}, a(-\sum_{i=1}^{k}x^{i^{2}}+\sum_{i=k+1}^{n}x^{i^{2}})+b_{1}x^{1}+\cdots+b_{n}x^{n}+c)$ and $\boldsymbol{R}^{n+1}_{k,1}$ is imbedded in $\boldsymbol{R}^{n+2}_{k+1}$ by $(y^{1}, \cdots, y^{n+1})\mapsto(y^{n+1}, y^{1}, \cdots, y^{n}, y^{n+1})$. Both affine spaces have the standard metric. Denote this submanifold by U^{n}_{k}.

PROOF. If $(\eta, \eta)\neq 0$ then by 0.2 there is an $\boldsymbol{R}^{n+1}_{k}$ or $\boldsymbol{R}^{n+1}_{k+1}$ into which M^{n}_{k} is isometrically immersed, depending on the sign of (η, η). This mapping is umbilical and by 0.10 the image is contained in a sphere or hyperbolic space.

If $(\eta, \eta)=0$ the image of M^{n}_{k} is in $\boldsymbol{R}^{n+1}_{k,1}$. By 1.2 $f\colon M^{n}_{k}\to \boldsymbol{R}^{n}_{k}$ is an isometric immersion so that $M^{n}_{k}=\boldsymbol{R}^{n}_{k}$.

1.3 allows us to determine all h such that

$$f(x^{1}, \cdots, x^{n})=(x^{1}, \cdots, x^{n}, h(x^{1}, \cdots, x^{n})).$$

As in 1.3 we have

$$\alpha\left(\frac{\partial}{\partial x^{i}}, \frac{\partial}{\partial x^{j}}\right)=(0, \cdots, 0, h_{ij}).$$

In addition, because f is umbilical,

$$\alpha\left(\frac{\partial}{\partial x^{i}}, \frac{\partial}{\partial x^{j}}\right)=\left(\frac{\partial}{\partial x^{i}}, \frac{\partial}{\partial x^{j}}\right)\eta=(0, \cdots, 0, h_{ij}).$$

It follows easily that h is of the desired form. Q. E. D.

The classification of umbilical immersions allows us, following Walden [Wa], to determine Riemannian immersions with parallel second fundamental forms and trivial normal connections in an indefinite euclidean space. Later this will allow

us to classify isometric immersions $\boldsymbol{R}^n \to \boldsymbol{R}_1^{n+k}$ with parallel second fundamental forms.

1.5 THEOREM. If f: $M^n \to \boldsymbol{R}_j^{n+k}$ is an isometric immersion of a complete Riemannian manifold with parallel second fundamental form and trivial normal connection, then f is a product of umbilical immersions.

$$f(M^n)=\boldsymbol{R}^{l_0}\times U^{l_1}\times\cdots\times U^{l_x}\times S^{m_1}\times\cdots\times S^{m_y}\times H^{n_1}\times\cdots\times H^{n_z}$$

Here $U^{l_p}\subset \boldsymbol{R}_1^{l_p+2}$ as an umbilic $p=1,\cdots,x$

$S^{m_q}\subset \boldsymbol{R}^{m_q+1}$ as an umbilic $q=1,\cdots,y$

$H^{n_r}\subset \boldsymbol{R}_1^{n_r+1}$ as an umbilic $r=1,\cdots,z$

and $\boldsymbol{R}^{l_0}\perp \boldsymbol{R}_1^{l_p+2}\perp \boldsymbol{R}^{m_q+1}\perp \boldsymbol{R}_1^{n_r+1}$. Note $k=2x+y+z$ and $j=x+z$.

PROOF. [Wa].

We now prove some lemmas.

1.6 LEMMA. Let f: $\boldsymbol{R}_j^n \to \boldsymbol{R}_k^m$ be an isometric immersion with parallel second fundamental form. If $(x^1,\cdots,x^n)$ is a global coordinate system with $\nabla_{\partial_i}\partial_j=0$, $i,j=1,\cdots,n$, then $\alpha(\partial_i,\partial_j)$ is a global parallel normal field.

PROOF. This is an easy consequence of the definition of $\nabla^*\alpha=0$.

1.7 LEMMA. Let f: $M_j^n \to \boldsymbol{R}_k^m$ be an isometric immersion. If a global normal field θ is parallel with respect to the normal connection, then A_θ commutes with A_ξ for all normal vectors ξ.

PROOF. Since $R^\perp(X,Y)\theta=\nabla^\perp_X\nabla^\perp_Y\theta-\nabla^\perp_Y\nabla^\perp_X\theta-\nabla^\perp_{[X,Y]}\theta=0$, this result follows from Codazzi's equation and the non-degeneracy of the metric on M.

1.8 PROPOSITION. If f: $\boldsymbol{R}^n \to \boldsymbol{R}_1^{n+k}$ is an isometric immersion with parallel second fundamental form, then it has trivial normal connection, i.e., $R^\perp\equiv 0$.

PROOF. It must be shown that, for any two normal vectors ξ and θ, A_ξ and A_θ commute. By 1.6 and 1.7, if γ is in the first normal space $N^1(x)$ then A_γ commutes with every other shape operator. On the other hand, if $\rho\in N^0(x)$ then $A_\rho\equiv 0$ and A_ρ commutes with every shape operator.

If the metric restricted to the first normal space is non-degenerate this is sufficient because the normal space equals $N^0(x)\oplus N^1(x)$.

If the metric restricted to $N^1(x)$ is degenerate then $N^0(x)+N^1(x)$ is not the entire normal space. In this case there is a unique lightlike direction δ in $N^1(x)\cap N^0(x)$. Choosing a lightlike $\hat{\delta}$ with $(\delta,\hat{\delta})=1$ we see that $N^0(x)$, $N^1(x)$ and $\hat{\delta}$ span

the normal space. Since $A_\xi^\perp$ commutes with itself it is clear that all shape operators commute. Q. E. D.

1.8 allows us, in conjunction with 1.5, to classify isometric immersions of $\boldsymbol{R}^n$ into $\boldsymbol{R}_1^{n+k}$ with parallel second fundamental forms.

1.9 THEOREM. If $f: \boldsymbol{R}^n \to \boldsymbol{R}_1^{n+k}$ is an isometric immersion with parallel second fundamental form, then f has one of the following forms:

(i) f is a product of circle maps and a totally geodesic map.

$$f = s_1 \times \cdots \times s_j \times Id: \boldsymbol{R}^1 \times \cdots \times \boldsymbol{R}^1 \times \boldsymbol{R}^{n-j} \to S^1 \times \cdots \times S^1 \times \boldsymbol{R}^{n-j} \subset \boldsymbol{R}_1^{n+k}$$

or

$$f = h \times s_1 \times \cdots \times s_j \times Id: \boldsymbol{R}^1 \times \boldsymbol{R}^1 \times \cdots \times \boldsymbol{R}^1 \times \boldsymbol{R}^{n-j-1} \to H^1 \times S^1 \times \cdots \times S^1 \times \boldsymbol{R}^{n-j-1} \subset \boldsymbol{R}_1^{n+k}.$$

(ii) f is a product of a flat umbilical map, circle maps, and a totally geodesic map.

$$f = u \times s_1 \times \cdots \times s_j \times Id: \boldsymbol{R}^m \times \boldsymbol{R}^1 \times \cdots \times \boldsymbol{R}^1 \times \boldsymbol{R}^{n-m-j} \to U^m \times S^1 \times \cdots \times S^1 \times \boldsymbol{R}^{n-m-j} \subset \boldsymbol{R}_1^{n+k}$$

u is one of the flat umbilical immersions of Theorem 1.4.

1.10 EXAMPLE. Let $f: \boldsymbol{R}^2 \to \boldsymbol{R}_{0,2}^4$ be the isometric immersion $(x, y) \mapsto (x, y, xy, x^2 - y^2/2)$ with metrics $(+, +)$ and $(+, +, 0, 0)$. The associated mapping into $\boldsymbol{R}_2^6$, where $\boldsymbol{R}_{0,2}^4 \to \boldsymbol{R}_2^6$ by $(y^1, y^2, y^3, y^4) \mapsto (y^4, y^3, y^1, y^2, y^3, y^4)$ neither has trivial normal connection nor splits into a product of umbilics. This shows that in 1.8 and 1.9 the receiving space must be at least Lorentz.

1.11 LEMMA. If θ is a global parallel normal vector field associated to an isometric immersion of $\boldsymbol{R}_1^n$ with parallel second fundamental form then A_θ has constant entries with respect to a flat coordinate system on $\boldsymbol{R}_1^n$.

PROOF. Let $\{x^1, \cdots, x^n\}$ be a global coordinate system such that $\nabla_{\partial_i}\partial_j = 0$, $1 \leqslant i, j \leqslant n$. Since the immersion has $\nabla^*\alpha = 0$

$$0 = A_\theta(\nabla_{\partial_i}\partial_j) = \nabla_{\partial_i}(A_\theta\partial_j) = \nabla_{\partial_i}(a_{1j}\partial_1 + \cdots + a_{nj}\partial_n) = (\partial_i a_{1j})\partial_1 + \cdots + (\partial_i a_{nj})\partial_n.$$

Therefore, $\partial_i a_{kj} = 0$ for all i, j, k. Q. E. D.

In order to classify isometric immersions with $\nabla^*\alpha = 0$ from $\boldsymbol{R}_1^n$ to $\boldsymbol{R}_1^{n+2}$ or $\boldsymbol{R}_2^{n+2}$ it is necessary to determine those from $\boldsymbol{R}_1^2$ to $\boldsymbol{R}_1^4$ or $\boldsymbol{R}_2^4$. For this we need some examples.

1.12 EXAMPLE. Complex circle of radius κ. Fix a non-zero number $c + id = \kappa$ in $\boldsymbol{C}$ and let $a + ib = \kappa^{-1}$. $\boldsymbol{C}^2$ can be identified with $\boldsymbol{R}_2^4$ by sending $(x + iy, u + iv)$ to

(x, u, y, v). The metric on $\boldsymbol{R}_2^4$ is $((x, u, y, v), (x, u, y, v)) = x^2+u^2-y^2-v^2$. The mapping $(x, y) \sim x+iy=z \overset{f'}{\longmapsto} \kappa(\cos z, \sin z) \in \boldsymbol{C}^2$ is an isometric immersion of $\boldsymbol{R}_1^2$ into $\boldsymbol{R}_2^4$ with parallel second fundamental form. Let J be the complex structure on $\boldsymbol{R}_2^4$ which comes from multiplication by i on $\boldsymbol{C}^2$. Then there is a natural orthonormal basis $\{J\xi, \xi\}$ for the normal space such that

$$A_\xi = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \quad \text{and} \quad A_{J\xi} = \begin{bmatrix} -b & -a \\ a & -b \end{bmatrix}.$$

The mean curvature vector $\eta = a\xi + bJ\xi$ and $A_{c\xi+dJ\xi} = -I_2$. By 0.9, $f'(\boldsymbol{R}_1^2)$ is contained in $S_2^3(\sqrt{c^2-d^2})$ or $H_1^3(-\sqrt{-c^2+d^2})$ or $LC_{1,1}^3$.

1.13 EXAMPLE. $f' : \boldsymbol{R}_1^2 \to \boldsymbol{R}_2^4$ where $\boldsymbol{R}_2^4$ has the standard metric.

$$\begin{aligned} \sqrt{2} f'(x, y) = &((1+c)\sin y, (1+c)\cos y, (1-c)\sin y, (1-c)\cos y) \\ &+(x+cy)(-\cos y, \sin y, \cos y, -\sin y), \quad c \in \boldsymbol{R}. \end{aligned}$$

This mapping has, with respect to a pseudo-orthonormal basis for the normal space $\{\eta, \hat{\eta}\}$, where η is the mean curvature vector, shape operators

$$A_\eta = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad A_{\hat{\eta}} = \begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix}.$$

The basis for $T(\boldsymbol{R}_1^2)$ is $\{\partial/\partial x, \partial/\partial y\}$. Since $A_{\hat{\eta}-c\eta} = I_2$, $f'(\boldsymbol{R}_1^2)$ is contained in $S_2^3(\sqrt{-2c})$ or $H_1^3(-\sqrt{2c})$ or $LC_{1,1}^3$. The shape operators associated to the immersions into the sphere or hyperbolic space have non-simple real eigenvalues and non-zero trace. Thus by the fundamental theorem of surfaces they are the unique immersions with these shape operators.

1.14 EXAMPLE. B-scroll over the null cubic C in $\boldsymbol{R}_1^3$ ([G 1]). In $\boldsymbol{R}_1^3$ take a null curve $x(v)$ with a null frame, that is, a set of vector fields $A(v)$, $B(v)$, $C(v)$ such that $\dot{x}(v) = A(v)$, $(A, A) = (B, B) = (A, C) = (B, C) = 0$ and $(A, B) = (C, C) = 1$. If these satisfy the following system of equations

$$\begin{aligned} \dot{A}(v) &= k_1(v)A(v) + k_2(v)C(v) \\ \dot{B}(v) &= k_1(v)B(v) \\ \dot{C}(v) &= -k_2(v)B(v) \end{aligned}$$

then $f'(u, v) = x(v) + uB(v)$ is a Lorentz surface in $\boldsymbol{R}_1^3$ called a B-scroll over $x(v)$. If $k_1 \equiv 0$ and $k_2 \equiv 1$ then $\dddot{A} = 0$ and the curve is called the null cubic C. ([B], p. 240). In this case we have the B-scroll over the null cubic C.

1.14′ EXAMPLE. B'-scroll over the null cubic C' in $\boldsymbol{R}_2^3$. This is the same as 1.14, with obvious modifications.

1.15 THEOREM. If $f: \boldsymbol{R}_1^2 \to \boldsymbol{R}_1^4$ is an isometric immersion with parallel second fundamental form then, up to a rigid motion, f has one of the following forms.

(i) f is totally geodesic

(ii) f is a product of one-dimensional maps

(iii) $f: \boldsymbol{R}_1^2 \to \boldsymbol{R}_1^3 \subset \boldsymbol{R}_1^4$ is a B-scroll over the null cubic C.

Note: In case (iii) the mean curvature vector is zero, even though the immersion is not totally geodesic.

1.16 LEMMA. If $f: \boldsymbol{R}_j^2 \to \boldsymbol{R}_j^4$ or $\boldsymbol{R}_{j+1}^4$ $j=0,1$ is an isometric immersion with parallel second fundamental form and the mean curvature vector $\eta=0$ then the first normal space has dimension less than two.

PROOF of 1.16: Assume that the first normal space is two-dimensional and let ξ_1, ξ_2 be an orthonormal basis of $N(x)$. Using 0.4 put A_{ξ_1} into the appropriate canonical form. Thus

$$A_{\xi_1}=\begin{bmatrix} a & 0 \\ 0 & -a \end{bmatrix} \text{ or } \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \text{ or } \begin{bmatrix} 0 & \beta \\ -\beta & 0 \end{bmatrix}.$$

Since A_{ξ_2} commutes with A_{ξ_1} it is of the form

$$\begin{bmatrix} b & 0 \\ 0 & -b \end{bmatrix} \text{ or } \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} \text{ or } \begin{bmatrix} 0 & \gamma \\ -\gamma & 0 \end{bmatrix}.$$

A calculation of α on the basis of $T_x(M)$ shows that the first normal spaces are at most one-dimensional. Q. E. D.

PROOF of 1.15: There are two cases to consider $\eta=0$ and $\eta\neq 0$.

CASE I. $\eta=0$. By 1.16 the first normal space is zero or one-dimensional. If it is zero-dimensional, then f is totally geodesic. If it is one-dimensional, let ξ span the first normal space. Then by 0.2 $f(\boldsymbol{R}_1^2)\subset \boldsymbol{R}_1^3$ and A_ξ has rank one and trace zero, so

$$A_\xi=\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

[G 1] shows that this is the B-scroll over the null cubic.

CASE II. $\eta\neq 0$. Put A_η into one of the three possible canonical forms. We will see that A_η can only be diagonal. Let η' be the unit vector in the direction of η and choose another unit normal vector ξ such that $\xi \perp \eta$.

If $A_\eta=\begin{bmatrix}\lambda & 1\\ 0 & \lambda\end{bmatrix}$ then $A_\xi=\begin{bmatrix}\mu & a\\ 0 & \mu\end{bmatrix}$ since A_η and A_ξ commute. But $\mu=1/2$ trace A_ξ $=(\xi,\eta)=0$. If $\{l,\hat{l}\}$ is the pseudo-orthonormal basis with respect to which A_η and A_ξ are represented, then the Gauss equation yields

$$\begin{aligned}0&=(A_\eta l,l)(A_\eta\hat{l},\hat{l})-(A_\eta l,\hat{l})^2+(A_\xi l,l)(A_\xi\hat{l},\hat{l})-(A_\xi l,\hat{l})^2\\&=-(\text{const. }\lambda)^2.\end{aligned}$$

This implies that $\lambda=0$, contradicting $\eta\neq 0$.

Similarly, if $A_{\eta'}=\begin{bmatrix}a & b\\ -b & a\end{bmatrix}$ then $A_\xi=\begin{bmatrix}0 & c\\ -c & 0\end{bmatrix}$. The Gauss equation gives $a=$ $b=c=0$ contradicting $\eta\neq 0$.

If A_η is diagonalizable with distinct eigenvalues, then all other A_ξ are diagonalizable and 0.3 shows that $f\colon \boldsymbol{R}^2_1\to\boldsymbol{R}^4_1$ can be decomposed as

$$\begin{aligned}f_1\times f_2\colon \boldsymbol{R}^1_1\times\boldsymbol{R}^1&\to\boldsymbol{R}^3_1\times\boldsymbol{R}^1_1 \quad\text{or}\\ &\quad\ \boldsymbol{R}^2_1\times\boldsymbol{R}^2 \quad\text{or}\\ &\quad\ \boldsymbol{R}^1_1\times\boldsymbol{R}^3.\end{aligned}$$

Each f_i has parallel mean curvature vector, so by 1.1

$$\begin{aligned}&f_1(\boldsymbol{R}^1_1)=S^1_1;\ f_2(\boldsymbol{R}^1)=\boldsymbol{R}^1 \quad\text{or}\\ &f_1(\boldsymbol{R}^1_1)=S^1_1;\ f_2(\boldsymbol{R}^1)=S^1 \quad\text{or}\\ &f_1(\boldsymbol{R}^1_1)=\boldsymbol{R}^1_1;\ f_2(\boldsymbol{R}^1)=S^1.\end{aligned}$$

If $A_\eta=\lambda I_2$ then there is a minimal immersion of $\boldsymbol{R}^2_1$ into $S^3_1(c)$. The shape operator A of this minimal immersion can, a priori, be one of the three types. Once again, the Gauss equation shows that A must be diagonalizable. In this final case the map splits as above. Q. E. D.

1.17 THEOREM. If $f\colon \boldsymbol{R}^2_1\to\boldsymbol{R}^4_2$ is an isometric immersion with parallel second fundamental form, then, up to a rigid motion, f has one of the following forms.

(i) f is totally geodesic

(ii) f is a product of one-dimensional maps

(iii) f is a complex circle, as in 1.12

(iv) f is a B-scroll over the null cubic, as in 1.14, 1.14′

(v) f is as in 1.13

(vi) $f\colon \boldsymbol{R}^2_1\to\boldsymbol{R}^3_{1,1}\subset\boldsymbol{R}^4_2$ where $(x,y)\mapsto(q(x,y),x,y,q(x,y))$ and $q(x,y)=ax^2+bxy+cy^2+dx+ey+g$. η is lightlike.

PROOF. The proof is divided into three cases: case I $(\eta,\eta)\neq 0$, case II $(\eta,\eta)=0$ and $\eta\neq 0$, and case III $\eta=0$.

CASE I. $(\eta,\eta)\neq 0$. Suppose first that A_η is diagonal. If the eigenvalues are distinct then, as before, the map splits into a product of one-dimensional maps, giving (ii). If the eigenvalues are equal, say to λ, then $\boldsymbol{R}^2_1$ immerses minimally into S^3_2 or H^3_1. By examining the Gauss equations, we see that the shape operator A of this minimal immersion can only be of the form

$$A=A^\perp_\eta=\begin{bmatrix}0 & \beta\\ -\beta & 0\end{bmatrix}$$

with respect to an orthonormal basis $\{e,f\}$ with $(f,f)=-1$. If S^3_2 has constant curvature $\tilde{c}>0$ then $0=\tilde{c}-[(-\beta^2)/-1]=\tilde{c}-\beta^2$. If H^3_1 has constant curvature $\tilde{c}<0$ then $0=\tilde{c}+(-\beta^2/-1)=\tilde{c}+\beta^2$. Both occur. In fact, in 1.12 let $c=1/\sqrt{\lambda}$, $d=0$ if $\lambda>0$ and $d=1/\sqrt{-\lambda}$, $c=0$ if $\lambda<0$. We have two maps: the given f and the f' in 1.12. We see that $A_{\eta_f}=A_{\eta_{f'}}$, $A^\perp_{\eta_f}=A^\perp_{\eta_{f'}}$, $(\eta_f,\eta_f)=(\eta_{f'},\eta_{f'})$, $(\eta^\perp_f,\eta^\perp_f)=(\eta^\perp_{f'},\eta^\perp_{f'})$ and $0=\nabla^\perp_{\eta f}=\nabla^\perp_{\eta^\perp_f}=\nabla^\perp_{\eta f'}=\nabla^\perp_{\eta^\perp_{f'}}$ so that we have a map $\tilde{f}: N_f\to N_{f'}$, from the normal bundle of f to the normal bundle of f', covering an isometry, which preserves bundle metrics, bundle connections, and shape operators. By the uniqueness theorem for immersions, there is a rigid motion from one image onto the other.

A_η cannot have a non-simple real eigenvalue. If it did, the Gauss equation would imply that $(\eta,\eta)=0$.

$$\text{If}\quad A_\eta=\begin{bmatrix}\alpha & \beta\\ -\beta & \alpha\end{bmatrix}\quad\text{then}\quad A^\perp_\eta=\begin{bmatrix}0 & \gamma\\ -\gamma & 0\end{bmatrix}.$$

This isometric immersion is also rigidly equivalent to a complex circle.

CASE II. $(\eta,\eta)=0$ and $\eta\neq 0$. Assume that

$$A_\eta=(1)\begin{bmatrix}a & 0\\ 0 & -a\end{bmatrix}\text{ or }(2)\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\text{ or }(3)\begin{bmatrix}0 & \beta\\ -\beta & 0\end{bmatrix}$$

where $a\neq 0$ and $\beta\neq 0$. Trace $A_\eta=0$ because $(\eta,\eta)=0$. Choosing $\hat{\eta}$ in the normal space so that $\{\eta,\hat{\eta}\}$ forms a pseudo-orthonormal basis, and using the fact that trace $A_{\hat{\eta}}=2$, we see that

$$A_{\hat{\eta}}=(1)\begin{bmatrix}b & 0\\ 0 & 2-b\end{bmatrix}\text{ or }(2)\begin{bmatrix}1 & c\\ 0 & 1\end{bmatrix}\text{ or }(3)\begin{bmatrix}1 & \gamma\\ -\gamma & 1\end{bmatrix}.$$

If (1) holds the map splits and the possibilities have been classified. If (2) holds, the isometric immersion is equivalent to 1.13. If case (3) holds $\gamma=0$ and the map is equivalent to a complex circle.

We must also consider what happens if $A_\eta=0$. This implies that the first normal space is one-dimensional and lightlike, so $f(\boldsymbol{R}^2_1)\subset\boldsymbol{R}^3_{1,1}$. 1.3 shows that

$$f(x, y)=(q(x, y), x, y, q(x, y))$$

and $$q(x, y)=ax^2+bxy+cx^2+dx+ey+g\,.$$

CASE III. $\eta=0$. By 1.16 the first normal space is one-dimensional if the map is not totally geodesic. Let θ span the first normal space. If $(\theta, \theta)\neq 0$ then $f(\boldsymbol{R}_1^2)$ is contained in $\boldsymbol{R}_1^3$ or $\boldsymbol{R}_2^3$ and A_θ must have rank one and trace zero. Thus $A_\theta=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}$. In this case the immersion is equivalent to 1.14 or 1.14′. If $(\theta, \theta)=0$ then $A_\theta=0$ and the map falls into (vi). Q. E. D.

For completeness we state the classification of isometric immersions from $\boldsymbol{R}^2$ to $\boldsymbol{R}^4$ with parallel second fundamental form.

1.18 THEOREM. If f: $\boldsymbol{R}^2\to\boldsymbol{R}^4$ is an isometric immersion with parallel second fundamental form, then f is either totally geodesic or a product of one-dimensional maps.

2. The n-Dimensional Classification.

Here we classify isometric immersions from $\boldsymbol{R}_1^n$ into $\boldsymbol{R}_1^{n+2}$ or $\boldsymbol{R}_2^{n+2}$ with parallel second fundamental forms. All such maps break into two classes. Immersions in the first class are quadratic in nature, like the flat umbilical immersion with lightlike mean curvature vector in 1.4. Immersions in the second class are products of the identity map and one of the two-dimensional immersions in 1.15, 1.17 or 1.18 or the B-D scroll immersion of $\boldsymbol{R}_1^3$ into $\boldsymbol{R}_2^5$ given below.

2.1 EXAMPLE. B-D scroll over the null quintic Q. Consider a null curve $x(v)$ in $\boldsymbol{R}_2^5$ with a frame $A(v)$, $B(v)$, $C(v)$, $D(v)$ and $E(v)$ along $x(v)$ such that $\dot{x}(v)=A(v)$, A, B are lightlike, C unit timelike, D, E unit spacelike, $C, D, E\perp A, B$, $(A, B)=1$ and C, D, E are mutually orthogonal. In addition, suppose $\dot{A}=bE$, $\dot{B}=0$, $\dot{C}=-D$, $\dot{D}=-C+E$, and $\dot{E}=-bB-D$. Note that the fifth derivative of A is zero, so that $x(v)$ is a quintic Q. Let $f(u, v, w)=x(v)+uB(v)+wD(v)$. This is an isometric immersion of $\boldsymbol{R}_1^3$ into $\boldsymbol{R}_2^5$ with $\{\partial/\partial u, \partial/\partial v, \partial/\partial w\}$ forming a pseudo-orthonormal basis of $\boldsymbol{R}_1^3$. If $\xi=-wB(v)+C(v)$ and $\xi^\perp=-wB(v)+E(v)$ then $\{\xi, \xi^\perp\}$ form an orthonormal basis of the normal space, $\nabla^\perp\xi=0=\nabla^\perp\xi^\perp$

$$A_\xi=\begin{bmatrix}0 & 0 & 1\\ 0 & 0 & 0\\ 0 & 1 & 0\end{bmatrix} \quad\text{and}\quad A_{\xi^\perp}=\begin{bmatrix}0 & b & 1\\ 0 & 0 & 0\\ 0 & 1 & 0\end{bmatrix}$$

with respect to $\{\partial/\partial u, \partial/\partial v, \partial/\partial w\}$.

2.2 LEMMA. Let f: $M_1^n\to\boldsymbol{R}_j^{n+k}$ be an isometric immersion with $\nabla^*\alpha=0$ and a

definite first normal space. If $x_0 \in M$ and ξ is a parallel normal field in a neighborhood of x_0 such that $A_{\xi_{x_0}}$ has a complex eigenvalue, then f decomposes, in a neighborhood of x_0, as $f_1 \times f_2 : M_1^2 \times M^{n-2} \to \boldsymbol{R}_{j_1}^{n_1} \times \boldsymbol{R}_{j_2}^{n_2}$.

PROOF. Let $\alpha(x) \pm i\beta(x)$, $\lambda_1(x), \cdots, \lambda_m(x)$ be the distinct eigenvalues of A_ξ at x. First we show that $\lambda_k(x)$ is constant and that

$$T_x^{\lambda_k} = \{X \in T_x(M) : A_{\xi_x} X = \lambda_k X\}$$

is a parallel distribution on M_1^n.

Let $X \in T^\lambda$, $Z \in T(M)$. Then

$$\begin{aligned} 0 &= \nabla_Z(A_\xi X) - A_\xi(\nabla_Z X) \\ &= \nabla_Z(\lambda X) - A_\xi(\nabla_Z X) \\ &= (Z\lambda)X + \lambda(\nabla_Z X) - A_\xi(\nabla_Z X) \\ &= (Z\lambda)X + \lambda(\nabla_Z X)^\perp - A_\xi((\nabla_Z X)^\perp) \end{aligned}$$

where $(\ )^\perp$ denotes the orthogonal complement of T^λ. Thus $(Z\lambda)=0$ and $A_\xi(\nabla_Z X) = \lambda(\nabla_Z X)$. It follows that each T^{λ_k} is parallel and each λ_k is constant.

Now let $C_x = (T_x^{\lambda_1})^\perp \cap \cdots \cap (T_x^{\lambda_m})^\perp$. C is also a parallel distribution. In fact, let $Z \in TM$, $Y \in C$ and $X \in T^{\lambda_1} \oplus \cdots \oplus T^{\lambda_m}$. Then $0 = Z(X, Y) = (\nabla_Z X, Y) + (X, \nabla_Z Y) = (X, \nabla_Z Y)$ so $\nabla_Z Y \in C$. Therefore $M_1^n = M_1^2 \times M^{n-2}$ locally. Because A_ξ and A_θ commute for all normal vectors θ, each A_θ has the same form by 0.8 and so $\alpha(X, Y) = 0$ whenever X is tangent to M_1^2 and Y is tangent to M^{n-2}. The theorem follows by 0.3. Q. E. D.

Let V and W be vector spaces. For a bilinear map $\alpha : V \times V \to W$ define the relative nullity space of α, $N(\alpha)$, by

$$N(\alpha) = \{X \in V : \alpha(X, Y) = 0 \text{ for all } Y \text{ in } V\}.$$

α is said to be flat with respect to an inner product $(\ ,\ )$ on W if

$$(\alpha(X, Z), \alpha(Y, W)) = (\alpha(X, W), \alpha(Y, Z))$$

for all X, Y, Z, W in V. We then have

2.3 PROPOSITION. ([Mo 2])

(i) If $\alpha : V \times V \to W$ is flat with respect to a positive definite inner product on W then $\dim N(\alpha) \geq \dim V - \dim W$.

(ii) If $\alpha : V \times V \to W$ is flat with respect to a Lorentzian inner product on W and $\dim W = 2$ then either $\dim N(\alpha) \geq \dim V - \dim W$ or $\alpha(X, Y)$ is null for all X, Y in V.

Note: This will be referred to as Moore's nullity result.

2.4 THEOREM. If $f: \boldsymbol{R}_1^n \to \boldsymbol{R}_1^{n+2}$ is an isometric immersion with parallel second fundamental form then

$$f=f_1\times Id: \boldsymbol{R}_1^2\times\boldsymbol{R}^{n-2}\to\boldsymbol{R}_1^4\times\boldsymbol{R}^{n-2} \quad \text{or}$$
$$f=Id\times f_1: \boldsymbol{R}_1^{n-2}\times\boldsymbol{R}^2\to\boldsymbol{R}_1^{n-2}\times\boldsymbol{R}^4$$

where f_1 is one of the two-dimensional maps of 1.15 or 1.18.

PROOF. We consider the two cases $\eta=0$ and $\eta\neq 0$.

CASE I. $\eta=0$. We will see below that in this case the first normal space is 0- or 1-dimensional. If it is 1-dimensional, then $f(\boldsymbol{R}_1^n)\subset\boldsymbol{R}_1^{n+1}$ non-trivially. If θ spans the first normal space then A_θ can be put into the only rank one, trace zero canonical form. It is clear then that f can be decomposed as $f_1\times Id: \boldsymbol{R}_1^2\times\boldsymbol{R}^{n-2}\to\boldsymbol{R}_1^4\times\boldsymbol{R}^{n-2}$ where f_1 is the B-scroll over the null cubic.

Now it is shown that the first normal space cannot be two-dimensional. Suppose that it is two-dimensional and let $\{\xi,\xi^\perp\}$ be an orthonormal basis of the first normal space.

We will assume A_ξ can be put into one of the canonical forms and then show that the first normal space is not two-dimensional.

If A_ξ were diagonal with respect to an orthonormal basis, then by minimality and 2.3 it has at most two non-zero eigenvalues $\pm\lambda$ and the eigenspaces are one-dimensional. The same reasoning shows that this is also true for $A_{\xi^\perp}$, and in fact the eigenspaces are the same as those of A_ξ. If this is true, then the first normal space is one-dimensional.

If A_ξ had a non-simple eigenvalue of multiplicity 2 then combining 2.3, minimality and the fact that A_ξ and $A_{\xi^\perp}$ commute, we have

$$A_\xi=\begin{pmatrix}0&1&&\\0&0&&\\&&&\\&&&O_{n-2}\\&&&\end{pmatrix} \quad\text{and}\quad A_{\xi^\perp}=\begin{pmatrix}0&b&0\cdots c_j 0\cdots 0\\0&0&0\cdots\cdots\cdots 0\\0&0&&\\\vdots&\vdots&&\\\vdots&0&&\\\vdots&c_j&&O_{n-2}\\\vdots&0&&\\\vdots&\vdots&&\\0&0&&\end{pmatrix}$$

with respect to a pseudo-orthonormal basis $\{l,\hat{l},e_1,\cdots,e_{n-2}\}$. This gives $\alpha(\hat{l},\hat{l})=\xi+b\xi^\perp$, $\alpha(e_j,e_j)=0$ and $\alpha(e_j,\hat{l})=c_j\xi^\perp$. Flatness of α gives $0=(c_j\xi^\perp,c_j\xi^\perp)=c_j^2$ so that the first normal space is one-dimensional.

If A_ξ has a complex eigenvalue then $A_{\xi^\perp}$ does also and so

$$A_\xi=\begin{pmatrix} 0 & \beta & & \\ -\beta & 0 & & \\ & & & \\ & & O_{n-2} & \end{pmatrix} \text{ and } A_{\xi^\perp}=\begin{pmatrix} 0 & \gamma & & \\ -\gamma & 0 & & \\ & & & \\ & & O_{n-2} & \end{pmatrix}.$$

The first normal space has dimension one.

If A_ξ had a non-simple eigenvalue of multiplicity 3 we see that

$$A_\xi=\begin{pmatrix} 0 & 0 & 1 & & \\ 0 & 0 & 0 & & \\ 0 & 1 & 0 & & \\ & & & & \\ & & & O_{n-3} & \end{pmatrix} \text{ and } A_{\xi^\perp}=\begin{pmatrix} 0 & b & c & & \\ 0 & 0 & 0 & & \\ 0 & c & 0 & & \\ & & & & \\ & & & O_{n-3} & \end{pmatrix}$$

with respect to a pseudo-orthonormal basis $\{l, \hat{l}, e, e_1, \cdots, e_{n-3}\}$. Here $\alpha(\hat{l}, e)=\xi+c\xi^\perp$ and $\alpha(e, e)=0$ so that $1+c^2=0$, which is impossible.

CASE II. $\eta\neq 0$. Here we show that A_η is diagonalizable. Choose η' to be a unit vector in the direction of η and $\eta^\perp$ to be a unit vector perpendicular to η.

Suppose first that $A_{\eta'}$ is of the form

$$A_{\eta'}=\begin{pmatrix} \lambda & t & & & & \\ 0 & \lambda & & & & \\ & & \lambda_1 & & & \\ & & & \ddots & & \\ & & & & \lambda_{n-2} & \end{pmatrix}, \qquad t\neq 0$$

with respect to a pseudo-orthonormal basis $\{l, \hat{l}, e_1, \cdots, e_{n-2}\}$. By Moore's result, we see that either $\lambda\neq 0$ and all $\lambda_j=0$ or $\lambda=0$ and exactly one $\lambda_{j_0}\neq 0$.

$$A_{\eta^\perp}=\left(\begin{array}{cc|c} \mu & b & c_1\cdots c_{n-2} \\ 0 & \mu & 0\cdots 0 \\ \hline & c_1 & \\ 0 & \vdots & d_{ij} \\ & c_{n-2} & \end{array}\right).$$

We have $\alpha(l, l)=0$ and $\alpha(l, \hat{l})=\lambda\eta'+\mu\eta^\perp$. The Gauss equation applied to the plane spanned by l and $\hat{l}$ shows $\mu=0=\lambda$. Thus $\lambda_{j_0}\neq 0$ for some j_0. Moore's result then gives each $c_j=0$ and possibly one $d_{j_0j_0}\neq 0$. However, trace $A_{\eta^\perp}=0$ forces $d_{j_0j_0}=0$. Therefore $\alpha(e_{j_0}, e_{j_0})=\lambda_{j_0}\eta'$, $\alpha(e_{j_0}, \hat{l})=0$ and $\alpha(\hat{l}, \hat{l})=t\eta'+b\eta^\perp$. From this we get $\lambda_{j_0}=0$ and so $A_{\eta'}$ cannot have a non-simple eigenvalue of multiplicity two.

Suppose $A_\eta=\begin{pmatrix} \lambda & 0 & c & & & \\ 0 & \lambda & 0 & & & \\ 0 & c & \lambda & & & \\ & & & \lambda_1 & & \\ & & & & \ddots & \\ & & & & & \lambda_{n-3} \end{pmatrix}$. Then 2.3 implies $\lambda=0=\lambda_j$, contradicting $(\eta,\eta)\neq 0$.

If A_η had a complex eigenvalue, then by 2.2 the map splits, but the two-dimensional argument shows this splitting can't occur.

Finally, if A_η is diagonalizable, it has at most two non-zero entries, and the map splits into a product of the identity map and one of the possible two-dimensional maps. Q. E. D.

If the normal space is Lorentzian, then there are more possibilities.

2.5 THEOREM. If $f\colon \boldsymbol{R}_1^n \to \boldsymbol{R}_2^{n+2}$ is an isometric immersion with parallel second fundamental form, then f is one of the following.

(i) $f=f_1\times Id\colon \boldsymbol{R}_1^2\times\boldsymbol{R}^{n-2}\to \boldsymbol{R}_2^4\times\boldsymbol{R}^{n-2}$

(ii) $f=f_1\times Id\colon \boldsymbol{R}_1^{n-2}\times\boldsymbol{R}^2\to \boldsymbol{R}_1^{n-2}\times\boldsymbol{R}_1^4$ where f_1 is one of the two-dimensional maps.

(iii) $f=f_1\times Id\colon \boldsymbol{R}_1^3\times\boldsymbol{R}^{n-3}\to \boldsymbol{R}_2^5\times\boldsymbol{R}^{n-3}$ where f_1 is the *B-D* scroll over the null quintic.

(iv) f is a quadratic map:

$$(x^1,\cdots,x^n)\mapsto (q(\vec{x}),x^1,\cdots,x^n,q(\vec{x})) \quad \text{with}$$
$$q(x^1,\cdots,x^n)=\sum a_{ij}x^ix^j+\sum b_ix^i+c\,.$$

PROOF. We consider three cases. Case I: $(\eta,\eta)\neq 0$, Case II: $(\eta,\eta)=0$, $\eta\neq 0$, Case III: $\eta=0$.

CASE I. We show that A_η cannot have any non-simple real roots, and that if it has simple roots it splits as in (i) and (ii).

Let η' be a unit vector in the direction of η and let $\eta^\perp$ be a vector orthogonal to η satisfying $(\eta',\eta')+(\eta^\perp,\eta^\perp)=0$. Define $\sigma=(\eta',\eta')$ and $\tau=(\eta^\perp,\eta^\perp)$. If

$$A_{\eta'}=\begin{pmatrix} \lambda & t & & & \\ 0 & \lambda & & & \\ & & \lambda_1 & & \\ & & & \ddots & \\ & & & & \lambda_{n-2} \end{pmatrix},\qquad A_{\eta^\perp}=\left(\begin{array}{cc|c} \mu & b & c_1\cdots c_{n-2} \\ 0 & \mu & \\ \hline & c_1 & \\ & \vdots & d_{ij} \\ & c_{n-2} & \end{array}\right)$$

with respect to a pseudo-orthonormal basis $\{l,\hat{l},e_1,\cdots,e_{n-2}\}$ with $t\neq 0$, then Moore's

result says $A_{\eta'}$ has rank $\leqslant 2$ and non-zero trace. Then either $\lambda=0=c_1=\cdots=c_{n-2}$ and one $\lambda_{j_0}\neq 0$ or $\lambda\neq 0$ and $d_{ij}=0=\lambda_i=c_i$. If $\lambda\neq 0$ then f would split into the product of the identity map and a non-existent two-dimensional map, so assume $\lambda=0$. Since $\alpha(l,l)=0$ and $\alpha(l,\hat{l})=\tau\mu\eta^{\perp}$, $\tau\mu^2=0$ implying that $\mu=0$. By 2.3 and the fact that trace $A_{n^{\perp}}=0$ all $d_{ij}=0$. Then $\alpha(e_{j_0},\hat{l})=0$, $\alpha(e_{j_0},e_{j_0})=\lambda_{j_0}\sigma\eta'$ and $\alpha(\hat{l},\hat{l})=\sigma t\eta'+b\tau\eta^{\perp}$ so $0=(\lambda_{j_0}\sigma\eta',\sigma t\eta'+b\tau\eta^{\perp})=\lambda_{j_0}\sigma t$ which would imply $\lambda_{j_0}=0$, which cannot happen.

$A_{\eta'}$ cannot have a non-simple real eigenvalue of multiplicity 3, non-zero trace and rank $\leqslant 2$.

If $A_{\eta'}$ is diagonalizable then, as above, it has one or two non-zero entries and the map splits into a product of the identity map and a two-dimensional map. The same is true when $A_{\eta'}$ has a complex eigenvalue.

CASE II. $\eta\neq 0$ and $(\eta,\eta)=0$.

If $A_\eta\neq 0$ and has simple eigenvalues the map splits into the product of the identity map and an appropriate two-dimensional map.

When

$$A_\eta=\begin{pmatrix}\lambda & 1 & & & & \\ 0 & \lambda & & & & \\ & & \lambda_1 & & & \\ & & & \ddots & & \\ & & & & & \lambda_{n-2}\end{pmatrix},\qquad A_{\hat{\eta}}=\left(\begin{array}{cc|c}\mu & b & c_1\cdots c_{n-2}\\ 0 & \mu & \\ \hline & c_1 & \\ & \vdots & d_{ij}\\ & c_{n-2} & \end{array}\right)$$

where $\hat{\eta}$ is a lightlike normal vector satisfying $(\eta,\hat{\eta})=1$ we see that $\lambda=0=\lambda_j$. If $\mu\neq 0$ then $c_j=0=d_{ij}$ and the map reduces to a product of the identity map and a known two-dimensional immersion. On the other hand, μ cannot be zero. If $\mu=0$ then for some j_0, $d_{j_0j_0}\neq 0$ and possibly $c_{j_0}\neq 0$, while all the other entries are 0. This would give $\alpha(\hat{l},\hat{l})=\hat{\eta}+b\eta$, $\alpha(e_{j_0},\hat{l})=c_{j_0}\eta$ and $\alpha(e_{j_0},e_{j_0})=d_{j_0j_0}\eta$. Because α is flat we have $0=(\hat{\eta}+b\eta,d_{j_0j_0}\eta)=d_{j_0j_0}$, contradicting trace $A_{\hat{\eta}}\neq 0$.

When A_η has a non-simple eigenvalue of multiplicity three then $A_{\hat{\eta}}$ is of the form

$$\left(\begin{array}{ccc|c}\mu & b & c_1 & c_2\cdots c_{n-2}\\ 0 & \mu & 0 & \\ 0 & c_1 & \mu & \\ \hline & c_2 & & \\ & \vdots & & d_{ij}\\ & c_{n-2} & & \end{array}\right).$$

Using the arguments above, we have $\mu=0=d_{ij}$, contradicting once again that trace $A_{\hat{\eta}}\neq 0$.

If $A_\eta=0$ it is easy to see using 1.3 that f is quadratic in nature and we get (iv).

CASE III. $\eta=0$.

If the first normal space is one-dimensional and definite then the immersion is the product of the identity and the B-scroll. If the first normal space is one-dimensional and lightlike, the immersion is quadratic.

If the normal space is two-dimensional, arguments similar to those above show that if $\{\xi, \xi^\perp\}$ is an orthonormal basis of the first normal space then the only canonical form possible for A_ξ and $A_{\xi^\perp}$ is

$$A_\xi=\left(\begin{array}{ccc|c} 0 & 0 & 1 & \\ 0 & 0 & 0 & \\ 0 & 1 & 0 & \\ \hline & & & \\ & & & O_{n-3} \\ & & & \end{array}\right), \quad A_{\xi^\perp}=\left(\begin{array}{ccc|c} 0 & b & 1 & \\ 0 & 0 & 0 & \\ 0 & 1 & 0 & \\ \hline & & & \\ & & & O_{n-3} \\ & & & \end{array}\right)$$

and f splits as the product of the identity map and the B-D scroll over Q. Q. E. D.

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