

A CHARACTERIZATION OF COMPLEX PROJECTIVE SPACES BY LINEAR SUBSPACE SECTIONS

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1. Introduction

It is conjectured in [2] that a complex projective space will be characterized from the standpoint of the positivity of sectional curvature. This conjecture is partially supported. Namely, a compact Kähler manifold (M, g) with positive curvature is biholomorphically homeomorphic to a complex projective space, if, for examples, one of the following conditions is satisfied;

- i) $\dim_{\mathbb{C}} M = 2$ ([2]),
- ii) the Kähler metric g is Einstein ([1]),
- iii) the group of holomorphic transformations acts on M transitively ([6]) and
- iv) $\dim_{\mathbb{C}} M = 3$ or 4 and $H^*(M; \mathbb{Z}) \cong H^*(\mathbb{P}_n(\mathbb{C}); \mathbb{Z})$, $n = \dim_{\mathbb{C}} M$ ([4]).

These conditions play essential role in each result.

In this connection, we are in a position to consider the following assertion.

ASSERTION If a compact complex manifold M admits a closed complex submanifold, in particular, a closed complex hypersurface which is biholomorphically homeomorphic to a complex projective space, then M itself is biholomorphically homeomorphic to a complex projective space.

If this assertion is verified, the conjecture due to Frankel can be reduced to the following conjecture.

CONJECTURE A compact Kähler manifold with positive curvature will admit a closed complex submanifold endowed with a Kähler metric of positive curvature.

Of course, the submanifold of positive curvature may not be a Kähler submanifold of the ambient manifold.

In general, the assertion is false. For example, a product manifold $\mathbb{P}_n(\mathbb{C}) \times M$, where M is a compact complex manifold, has $\mathbb{P}_n(\mathbb{C})$ as a closed complex submanifold, but the total manifold can never be biholomorphically homeomorphic to a complex projective space. Hence, the submanifold in the assertion must satisfy further as-

sumptions.

A compact Kähler manifold with positive curvature has the positive definite Ricci tensor, hence its first Chern class c_1 is positive. It is, then, an algebraic variety of a complex projective space by the aid of Kodaira's imbedding theorem. Therefore, the compact complex manifold stated in the assertion is furthermore assumed to be a closed submanifold of a complex projective space.

The assertion is held under the conditions that the submanifold is given as a section by a linear subspace in an ambient projective space and that it is biholomorphically homeomorphic to a complex projective space. This fact is precisely stated in Theorem 1.

The main purpose of this paper is to give a proof of Theorem 1. It is shown by the aid of a generalized Lefschetz's theorem ([5]) together with a characterization theorem of a complex projective space in terms of Chern classes ([7]).

2. Theorem and Corollaries

The following theorem characterizes a complex projective space by a linear subspace section in an ambient complex projective space.

THEOREM 1. *Let M be an n -dimensional closed complex submanifold in an N -dimensional complex projective space $\mathbf{P}_N(\mathbf{C})$.*

Assume that there is a linear subspace V in $\mathbf{P}_N(\mathbf{C})$ of codimension r ($\leq n-2$) such that a section $M \cap V$ of M by V is biholomorphically homeomorphic to $\mathbf{P}_{n-r}(\mathbf{C})$. Then, M itself is biholomorphically homeomorphic to $\mathbf{P}_n(\mathbf{C})$.

Note that $r \leq n-2$ is necessary in proving Theorem 1, since the surjectivity of $\iota_*: H_2(M \cap V; \mathbf{Z}) \rightarrow H_2(M; \mathbf{Z})$ is guaranteed under the requirement of r .

The following is an immediate conclusion from Theorem 1.

COROLLARY 2. *Let M be as in Theorem 1. If there is a sequence of linear subspaces $\{V^1, \dots, V^k\}$ of $\mathbf{P}_N(\mathbf{C})$, $r = \sum_{i=1}^n r_i \leq n-2$, $r_i = \text{codim}_{\mathbf{C}} V^i$ such that*

- i) $M^{(i)}$ is a closed complex submanifold of $M^{(i-1)}$, $i=1, \dots, k$,
- ii) $M^{(k)}$ is biholomorphically homeomorphic to $\mathbf{P}_{n-r}(\mathbf{C})$,

where $M^{(i)} = M \cap V^1 \cap \dots \cap V^i$, $i=1, \dots, k$ and $M^{(0)} = M$, then M is biholomorphically homeomorphic to $\mathbf{P}_n(\mathbf{C})$.

Since $M^{(k)}$ is biholomorphically homeomorphic to $\mathbf{P}_{n-r}(\mathbf{C})$, $M^{(k-1)}$ is also biholomorphically homeomorphic to a complex projective space by the result of Theorem 1. Hence an inductive argument verifies Corollary 2.

COROLLARY 3. *Let M be as in Theorem 1. If there is a closed complex hypersurface S in $\mathbf{P}_N(\mathbf{C})$ such that a section $M \cap S$ is biholomorphically homeomorphic to $\mathbf{P}_{n-1}(\mathbf{C})$, then M is also biholomorphically homeomorphic to $\mathbf{P}_n(\mathbf{C})$.*

If, moreover, there is a sequence of closed complex hypersurfaces $\{S^1, \dots, S^k\}$, $k \leq n-2$ such that

- i) $M^{(i)}$ is a hypersurface of $M^{(i-1)}$, $i=1, \dots, k$ and
- ii) $M^{(k)}$ is biholomorphically homeomorphic to $\mathbf{P}_{n-k}(\mathbf{C})$,

where $M^{(i)} = M \cap S^1 \cap \dots \cap S^i$, $i=1, \dots, k$ and $M^{(0)} = M$, then M itself is biholomorphically homeomorphic to $\mathbf{P}_n(\mathbf{C})$.

Corollary 3 is shown by the aid of Veronese mapping. Veronese mapping $v_m: \mathbf{P}_N(\mathbf{C}) \rightarrow \mathbf{P}_{N'}(\mathbf{C})$, $N' = \binom{N+m}{m} - 1$, is defined as follows ([8]). Let $u_{i_0 i_1 \dots i_N}$'s be homogeneous coordinates in $\mathbf{P}_{N'}(\mathbf{C})$ where i_0, i_1, \dots, i_N are nonnegative integers such that $i_0 + i_1 + \dots + i_N = m$. v_m is defined by $u_{i_0 i_1 \dots i_N} \circ v_m = z_0^{i_0} \cdot z_1^{i_1} \cdot \dots \cdot z_N^{i_N}$, where z_0, z_1, \dots, z_N are the homogeneous coordinates in $\mathbf{P}_N(\mathbf{C})$. It follows from the definition that the Veronese mapping is an imbedding.

Since the hypersurface S of $\mathbf{P}_N(\mathbf{C})$ in Corollary 3 is given as zero points of a homogeneous polynomial of degree m , $\sum_{i_0+i_1+\dots+i_N=m} \alpha_{i_0 i_1 \dots i_N} z_0^{i_0} \cdot z_1^{i_1} \cdot \dots \cdot z_N^{i_N}$, S is imbedded onto $v_m(S) = H \cap v_m(\mathbf{P}_N(\mathbf{C}))$, where H is a hyperplane in $\mathbf{P}_{N'}(\mathbf{C})$ defined by $\sum \alpha_{i_0 i_1 \dots i_N} u_{i_0 i_1 \dots i_N} = 0$. Thus, $M \cap S$ is imbedded onto $v_m(M) \cap v_m(S) = v_m(M) \cap H$ which is biholomorphically homeomorphic to $\mathbf{P}_{n-1}(\mathbf{C})$ by the assumption. From Theorem 1, $v_m(M)$, hence, M is biholomorphically homeomorphic to $\mathbf{P}_n(\mathbf{C})$. Hence we have the first part of Corollary 3. The second part of the corollary is easily obtained.

3. Proof of Theorem 1

Let $\iota: M' \rightarrow M$ and $j: M' \rightarrow \mathbf{P}_N(\mathbf{C})$ be the imbeddings, where $M' = M \cap V$. Let τ_M , $\tau_{M'}$ and ν be the tangent bundle of M , the tangent bundle of M' and the normal bundle of M' in M , respectively.

If we denote by $[V]$ the vector bundle over $\mathbf{P}_N(\mathbf{C})$ defined by V , then the normal bundle of V in $\mathbf{P}_N(\mathbf{C})$ is the pullback of $[V]$. Moreover, it is well-known that ν is isomorphic to the pullback of the normal bundle of V in $\mathbf{P}_N(\mathbf{C})$. Therefore we have

$$\iota^* \tau_M = \tau_{M'} \oplus \iota^* j^* [V].$$

Since V is a linear subspace of codimension r , there is a hyperplane H in $\mathbf{P}_N(\mathbf{C})$ such that $[V] = r[H]$, where $[H]$ is the line bundle over $\mathbf{P}_N(\mathbf{C})$ defined by H . Hence we have

$$(1) \quad \iota^* c_1(M) = c_1(M') + r \iota^* j^* c_1([H]),$$

where c_1 's denote the first Chern classes.

Since $[V]$ is positive in the sense of Griffiths ([5]) and $M' = M \cap V$ is a non-singular zero locus of a non-trivial global section of $\mathcal{O}(j^*[V])$, by the aid of a generalized Lefschetz's theorem (see Theorem H in [5]), we obtain the following two exact sequences under the condition $r \leq n-2$;

$$H_2(M'; \mathbf{Z}) \xrightarrow{\iota^*} H_2(M; \mathbf{Z}) \longrightarrow 0$$

and

$$0 \longrightarrow H_1(M'; \mathbf{Z}) \xrightarrow{\iota^*} H_1(M; \mathbf{Z}) \longrightarrow 0.$$

Since M' is homeomorphic to a complex projective space, we have $H_2(M'; \mathbf{Z}) \cong \mathbf{Z}$ and $H_1(M'; \mathbf{Z}) = 0$. Hence we obtain the following exact sequence;

$$0 \longrightarrow H_2(M'; \mathbf{Z}) \xrightarrow{\iota^*} H_2(M; \mathbf{Z}) \longrightarrow 0$$

which, together with $H_1(M'; \mathbf{Z}) = H_1(M; \mathbf{Z}) = 0$, implies that $\iota^*: H^2(M; \mathbf{Z}) \rightarrow H^2(M'; \mathbf{Z})$ is an isomorphism.

If α is a positive generator of $H^2(M; \mathbf{Z}) \cong \mathbf{Z}$, then $\iota^*\alpha$ is also a positive generator of $H^2(M'; \mathbf{Z})$. Thus we have $\iota^*j^*c_1([H]) \cong \iota^*\alpha$, and hence, $\iota^*c_1(M) \cong c_1(M') + r\iota^*\alpha$. Since $c_1(M') = (n-r+1)\iota^*\alpha$, which is derived from the fact that M' is biholomorphically homeomorphic to an $(n-r)$ -dimensional complex projective space, we have $c_1(M) \cong (n-r+1)\alpha + r\alpha = (n+1)\alpha$ by the injectivity of $\iota^*: H^2(M; \mathbf{Z}) \rightarrow H^2(M'; \mathbf{Z})$.

Therefore, Theorem 1 follows from a result of [7].

4. Further Remarks

1) A linear subspace of a complex projective space is also a complex projective space. And its section by another linear subspace gives a linear subspace again. This is a trivial example which supports Theorem 1. We have a non-trivial example for Theorem 1 as follows. Recall the Veronese mapping $v_m: \mathbf{P}_n(\mathbf{C}) \rightarrow \mathbf{P}_N(\mathbf{C})$, $N = \binom{n+m}{m} - 1$. The section of $v_m(\mathbf{P}_n(\mathbf{C}))$ by the hyperplane of a form; $u_{m0\dots 0} = 0$, in $\mathbf{P}_N(\mathbf{C})$ gives a hyperplane $z_0 = 0$ in $\mathbf{P}_n(\mathbf{C})$. On the contrary, the section by the hyperplane of a form; $u_{m0\dots 0} + u_{0m0\dots 0} + \dots + u_{0\dots 0m} = 0$, gives the hypersurface of degree m ; $\sum_{j=0}^n z_j^m = 0$ in $\mathbf{P}_n(\mathbf{C})$.

2) In [3], a pair (V, L) of a compact variety V and a line bundle L is called a polarized variety, if L is ample. If a compact complex manifold M is imbedded in a complex projective space, then a hyperplane section $M \cap H$ of M induces a polarized variety $(M, [M \cap H])$, since $[M \cap H]$ is very ample. Theorem 1 is, if es-

pecially $r=1$, implicated with Theorem 6.1 in [3].

3) With respect to Conjecture stated in Introduction, we have the following consideration.

Let (M, g) be a compact Kähler manifold with positive curvature. Then, M is a closed complex submanifold of a complex projective space. A hyperplane section M' of M gives a hypersurface which is defined by a certain holomorphic function, which we denote by f , locally. A relation between the holomorphic bisectonal curvature $H'\sigma, \tau$ of M' with respect to the induced metric and $H\sigma, \tau$ of (M, g) is given as follows;

$$(2) \quad H'\sigma, \tau = H\sigma, \tau - \frac{|H_f(Z, W)|^2}{\|df\|^2 \|Z\|^2 \|W\|^2}.$$

Here σ and τ are holomorphic planes tangent to M' , $\sigma = X \wedge IX, \tau = Y \wedge IY$ and $Z = X - \sqrt{-1} IX, W = Y - \sqrt{-1} IY$. H_f denotes the complex Hessian of f , i.e., $H_f = (\Gamma_i \Gamma_j f)$ and $\|df\|^2 = \Sigma g^{j\bar{i}} \frac{\partial f}{\partial z^i} \overline{\frac{\partial f}{\partial z^j}}$.

(2) is obtained by the similar argument as that in [9].

From (2), we have the following statement which locally supports Conjecture with respect to the holomorphic bisectonal curvature.

For any point p of M and an arbitrary positive number ϵ , there are a neighborhood U of p and a holomorphic function f defined on U which satisfy the following;

i) $\{q \in U; f(q) = 0\}$ is a hypersurface of M which contains p

and

ii) on the hypersurface endowed with the induced metric,

$|H'\sigma, \tau - H\sigma, \tau| < \epsilon$ for any pair of holomorphic planes σ and τ tangent to the hypersurface.

This statement is observed as follows. Among all charts around p we can choose a certain normal chart $(U', x^i), x^i(p) = 0$, with respect to which the components $g_{i\bar{j}}$'s of the metric g satisfy

$$(3) \quad g_{i\bar{j}}(x^i) = \delta_{ij} + \sum_{s,t} R_{i\bar{j}s\bar{t}}(p) x^s \bar{x}^t + o(r^3),$$

where $r = (\sum_t |x^t|^2)^{1/2}$, and $R_{i\bar{j}s\bar{t}}$'s are the components of the curvature tensor R .

Assume that a holomorphic function f on U' is of the form, $f(x^i) = \sum_k a^k x^k + o(r^1), (a^i) \neq 0$, of course, such an f exists indeed. Then, $\Gamma_i \Gamma_j f(p) = \partial^2 f / \partial x^i \partial x^j(p) - \sum_k \Gamma_{ij}^k(p) \partial f / \partial x^k(p) = 0$, where Γ_{ij}^k 's are the Christoffel's symbols, that is, $\Gamma_{ij}^k = \sum_u^k g^{\bar{u}k} \partial g_{i\bar{u}} / \partial x^j$. Hence, the complex Hessian $H_f = (\Gamma_i \Gamma_j f)$ vanishes at p . Therefore we can choose a sufficiently small neighborhood U around p such that

$$|H'\sigma, \tau - H\sigma, \tau| = \frac{|H_f(Z, W)|^2}{\|df\|^2 \|Z\|^2 \|W\|^2} < \varepsilon$$

for any pair of holomorphic planes σ and τ tangent to $\{q \in U; f(q) = 0\}$.

Since p is arbitrary, M is covered with such a system $\{(U_p, f_p)\}_{p \in M}$ which gives local hypersurfaces. In order for the system to define a global hypersurface in M , it must satisfy the property that there is a subsystem $\{(U_\alpha, f_\alpha)\}_{\alpha \in A}$, which covers M and f_α/f_β gives a non-vanishing holomorphic function on $U_\alpha \cap U_\beta (\neq \emptyset)$, that is, the subsystem induces a non-singular holomorphic divisor.

It should be noticed that if this is verified, Conjecture can be supported with respect to the holomorphic bisectonal curvature, since we only need to set $\varepsilon = 1/2 \cdot \min H\sigma, \tau$ over all pairs of holomorphic planes of M .

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