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HARMONIC FOLIATIONS ON A COMPLEX PROJECTIVE SPACE

By

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1. Introduction.

In 1970, D. Ferus [6] gave an estimation on the codimension of a totally geodesic foliation on a sphere and a complex projective space, and successively P. Dombrowski [1] improved his results. Moreover, R. Escobales classified Riemannian foliations satisfying a certain condition on a sphere and a complex projective space in a series of his papers [2], [3], [4], [5].

On the other hand, F. Kamber and Ph. Tondeur [7], [8] studied the index of harmonic foliations with bundle-like metric on a sphere from a view point of harmonic mappings.

Recently, H. Nakagawa and R. Takagi [11] showed that any harmonic foliations on a compact Riemannian manifold of non-negative constant sectional curvature is totally geodesic if the normal plane field is minimal.

In this paper we will prove

THEOREM. *Let* $\mathbf{P}_m(\mathbf{C})$ *be a complex projective space of complex dimension* m *with the metric of constant holomorphic sectional curvature. If* $\mathcal{F}$ *is a harmonic foliation on* $\mathbf{P}_m(\mathbf{C})$ *snch that the normal plane field is minimal, then* $\mathcal{F}$ *is totally geodesic.*

I am grateful to professor Ryoichi Takagi for his kind guidance and constant encouragement.

2. Preliminaries.

We first establish some basic notations and formulas in the theory of foliated Riemannian manifolds. For details, see [9], [10], [11], [13].

Let (M, g) be an n-dimentional Riemannian manifold and $\mathcal{F}$ a foliation with codimension q on M. Considering $\mathcal{F}$ as an $(n-q)$-dimensional integrable distribution on M, we denote the orthogonal distribution of $\mathcal{F}$ by $\mathcal{F}^{\perp}$, which is called the normal plane field.

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Therefore if we denote the space of vector fields on M by $\mathfrak{X}(M)$, each $X\in\mathfrak{X}(M)$ can be decomposed as $X=X'+X''$, where $X'_x\in\mathcal{F}_x$ and $X''_x\in\mathcal{F}_x^{\perp}$ for each $x\in M$. Then two tensor fields A and h of type (1.2) on M are defined by

$$\begin{aligned} A(X, Y) &= -(\nabla_{Y''}X'')', \\ h(X, Y) &= (\nabla_{Y'}X')'', \qquad X, Y\in\mathfrak{X}(M). \end{aligned} \tag{1.1}$$

The ristriction of h to each leaf of $\mathcal{F}$ is so-called the second fundamental form of the leaf.

Now, according to [11], we express them with respect to locally defined orthonormal frame field.

As for the range of indices the following convention will be used throughout this paper unless otherwise stated:

$$\begin{aligned} A, B, C, \cdots &=1, 2, 3, \cdots, n \\ i, j, k, \cdots &=1, 2, 3, \cdots, p \\ \alpha, \beta, \gamma, \cdots &=p+1, \cdots, n, \end{aligned}$$

where $p=n-q$ is the dimension of $\mathcal{F}$.

Let $\{e_1, e_2, \cdots, e_n\}$ be a locally defined orthonormal frame field of M such that $e_1, e_2, \cdots, e_p$ are always tangent to $\mathcal{F}$. Denote its dual by $\{\omega_1, \omega_2, \cdots, \omega_n\}$.

The Riemannian connection form $\{\omega_{AB}\}$ with respect to $\{\omega_A\}$ are defined by the followings:

$$\begin{aligned} \omega_{AB}+\omega_{BA} &= 0, \\ d\omega_A+\sum\omega_{AB}\wedge\omega_B &= 0. \end{aligned} \tag{1.2}$$

A relation between ω_{AB} and ∇ is given by

$$\nabla_{e_A}e_B = \sum\omega_{CB}(e_A)e_C. \tag{1.3}$$

Then the components h^A_{BC} (resp. A^B_{CD}) of h (resp. A) with respect to $\{e_A\}$ and $\{\omega_A\}$ are given by

$$h^\alpha_{ij} = \omega_{\alpha i}(e_j) \quad (\text{resp. } A^i_{\alpha\beta} = \omega_{\alpha i}(e_\beta)), \tag{1.4}$$

and any other components vanish.

Since the distribution $\omega_\alpha=0$ is integrable,

$$h^\alpha_{ij} = h^\alpha_{ji}. \tag{1.5}$$

The foliation $\mathcal{F}$ is said to be *harmonic* or *minimal* (resp. *totally geodesic*) provided that $\sum h^\alpha_{ii}=0$ (resp. $h^\alpha_{ij}=0$), and owing to [9], [13], the normal plane field $\mathcal{F}^{\perp}$ is said to be *minimal* provided that $\sum A^i_{\alpha\alpha}=0$.

A necessary and sufficient condition for the distribution $\omega_i=0$ to be integrable is $A^i_{\alpha\beta}=A^i_{\beta\alpha}$. On the contrary, the Riemannian metric g is *bundle-like* if and only if

$$A^i_{\alpha\beta}=-A^i_{\beta\alpha}. \tag{1.6}$$

The curvature form $\Omega=(\Omega_{AB})$ of M is defined by

$$\Omega_{AB}=d\omega_{AB}+\sum\omega_{AC}\wedge\omega_{CB}, \tag{1.7}$$

and we define its components R_{ABCD} by

$$\Omega_{AB}=-(1/2)\sum R_{ABCD}\omega_C\wedge\omega_D, \qquad R_{ABCD}+R_{ABDC}=0. \tag{1.8}$$

Then the equalities $R_{ABCD}=-R_{BACD}=R_{CDAB}$ hold.

Now for an (r, s)-tensor field $T=(T^{A_1A_2\cdots A_r}_{B_1B_2\cdots B_s})$ on M, we define the coveriant derivative $\nabla T=(T^{A_1A_2\cdots A_r}_{B_1B_2\cdots B_sD})$ by

$$\begin{aligned}\sum T^{A_1A_2\cdots A_r}_{B_1B_2\cdots B_sC}\omega_C &= dT^{A_1A_2\cdots A_r}_{B_1B_2\cdots B_s}\\ &\quad-\sum_{a=1}^{r}T^{A_1\cdots A_{a-1}CA_{a+1}\cdots A_r}_{B_1\cdots\cdots\cdots\cdots B_s}\omega_{CA_a}\\ &\quad-\sum_{b=1}^{s}T^{A_1\cdots\cdots\cdots\cdots A_r}_{B_1\cdots B_{b-1}CB_{b+1}\cdots B_s}\omega_{CB_b}.\end{aligned} \tag{1.9}$$

Then we have followings ([11]):

$$h^\alpha_{ijk}-h^\alpha_{ikj}=R_{\alpha ijk}, \tag{1.10}$$

$$h^\alpha_{ij\beta}-A^i_{\alpha\beta j}-\sum h^\alpha_{ik}h^\beta_{kj}-\sum A^i_{\alpha\gamma}A^j_{\gamma\beta}=R_{\alpha ij\beta}, \tag{1.11}$$

$$A^i_{\alpha\beta\gamma}-A^i_{\alpha\gamma\beta}+\sum h^\alpha_{ij}(A^j_{\beta\gamma}-A^j_{\gamma\beta})=-R_{\alpha i\gamma\beta}. \tag{1.12}$$

From now on, we consider the case where M is the complex projective space $\mathbf{P}_m(\mathbf{C})$ of complex dimension m $(=n/2)$ with the metric of constant holomorphic sectional curvature $4c$.

Let J denote the complex structure of $\mathbf{P}_m(\mathbf{C})$ and put $J(e_A)=\sum J_{BA}(e_B)$. Then (J_{AB}) satisfies

$$\begin{aligned}&J_{AB}+J_{BA}=0,\\ &\sum J_{AC}J_{CB}=-\delta_{AB},\end{aligned} \tag{1.13}$$

$$dJ_{AB}=\sum(J_{AC}\omega_{CB}-J_{BC}\omega_{CA}). \tag{1.14}$$

The last equation means that $\nabla J=0$. Moreover the curvature form $\Omega=(\Omega_{AB})$ and its components R_{ABCD} defined by (1.7) and (1.8) respectively are given by

$$\Omega_{AB}=c\omega_A\wedge\omega_B+c\sum(J_{AC}J_{BD}+J_{AB}J_{CD})\omega_C\wedge\omega_D, \tag{1.15}$$

$$R_{ABCD}=c(\delta_{AD}\delta_{BC}-\delta_{AC}\delta_{BD})+c(J_{AD}J_{BC}-J_{AC}J_{BD}-2J_{AB}J_{CD}). \tag{1.16}$$

Therefore we obtain

$$R_{ABCDE} = 0. \tag{1.17}$$

3. Proof of the main theorem.

In this section we give the proof of our main theorem. In the case where $p=1$, any harmanic foliation is necessarily totally geodesic. Therefore we may assume $p \geqq 2$.

Consider the global vector field $v=\sum v_A e_A$ on $\mathbf{P}_m(\mathbf{C})$ defined by

$$v_k = \sum h_{ij}^{\alpha} h_{ijk}^{\alpha}, \qquad v_{\alpha} = 0.$$

We first calculate the divergence δv of v.

In general H. Nakagawa and R. Takagi showed the following lemma ([11]):

LEMMA 2.1. *Let $(M, g, \mathcal{F})$ be a faliated Riemannian manifold and v a vector field on M defined above. Then*

(1) *the divergence δv of v is given by*

$$\begin{aligned}\delta v = &\sum v_i A_{\alpha\alpha}^{i} + \sum h_{ijk}^{\alpha} h_{ijk}^{\alpha} + \sum h_{ij}^{\alpha} R_{\alpha ijkk} \\ &+ \sum h_{ij}^{\alpha} R_{\alpha kikj} + \sum h_{ij}^{\alpha} h_{kk}^{\beta} h_{ij\beta}^{\alpha} + \sum h_{ij}^{\alpha} h_{kkij}^{\alpha} \\ &+ \sum (h_{ik}^{\beta} R_{\alpha\beta jk} + h_{lk}^{\alpha} R_{iljk} + h_{il}^{\alpha} h_{kljk}) h_{ij}^{\alpha} \\ &+ \sum h_{ij}^{\alpha} h_{lk}^{\alpha} h_{ij}^{\beta} h_{lk}^{\beta} + 2\sum h_{ij}^{\alpha} h_{ik}^{\beta} h_{jl}^{\alpha} h_{lk}^{\beta},\end{aligned}$$

and

(2) *if the foliation $\mathcal{F}$ is harmonic,*

$$\sum h_{iijk}^{\alpha} = -2\sum h_{ij}^{\beta} h_{il}^{\alpha} h_{lk}^{\beta}.$$

Therefore if the foliation $\mathcal{F}$ is harmonic and the normal plane field $\mathcal{F}^{\perp}$ minimal, we obtain

$$\begin{aligned}\delta v = &\sum h_{ijk}^{\alpha} h_{ijk}^{\alpha} + \sum h_{ij}^{\alpha} h_{lk}^{\alpha} h_{ij}^{\beta} h_{lk}^{\beta} \\ &+ 2\sum Tr(H^{\alpha} H^{\alpha} H^{\beta} H^{\beta} - H^{\alpha} H^{\beta} H^{\alpha} H^{\beta}) \\ &+ \sum (h_{ik}^{\beta} R_{\alpha\beta jk} + h_{lk}^{\alpha} R_{iljk} + h_{il}^{\alpha} R_{kljk}) h_{ij}^{\alpha},\end{aligned} \tag{2.1}$$

where H^{α} denotes the $p \times p$ matrix (h_{ij}^{α}).

The essential part of the proof is to show that δv is non-negative on $\mathbf{P}_m(\mathbf{C})$. For it, putting

$$X = \sum (h_{ik}^{\beta} R_{\alpha\beta jk} + h_{lk}^{\alpha} R_{iljk} + h_{il}^{\alpha} R_{kjlk}) h_{ij}^{\alpha},$$

we have only to show $X \geqq 0$, since

$$Tr(H^\alpha H^\alpha H^\beta H^\beta - H^\alpha H^\beta H^\alpha H^\beta) \geqq 0 \quad \text{holds ([11])}.$$

For simplicity we put

$$\xi_{ijk} = \sum h_{ij}^\alpha J_{\alpha k}, \qquad \eta_{i\beta}^\alpha = \sum h_{ij}^\alpha J_{\beta j}, \qquad \mu_{ij}^\alpha = \sum h_{ik}^\alpha J_{kj}.$$

Then from (1.13), (1.16), we have

$$X = \sum_{\alpha,i,j} cp(h_{ij}^\alpha)^2 + cY \tag{2.2}$$

$$+3c\sum_\alpha \left\{ 2\sum_i (\mu_{ii}^\alpha)^2 + \sum_{i<k} (\mu_{ik}^\alpha + \mu_{ki}^\alpha)^2 \right\},$$

where we put

$$Y = \sum h_{ij}^\alpha h_{ik}^\beta (J_{\alpha k} J_{\beta j} - J_{\alpha j} J_{\beta k} - 2J_{\alpha\beta} J_{jk}).$$

Next lemma gives the key inequality.

LEMMA 2.2. *For the Y above, the following inequality holds*:

$$Y \geqq -\{((p-1)^2+1)/(p-1)\} \sum_{i,j,\alpha} (h_{ij}^\alpha)^2$$

$$+(p-2)\sum_i \sum_{j\neq k} (\xi_{ijk})^2 + \sum_i \sum_{j<k} (\xi_{ijk}+\xi_{ikj})^2$$

$$+\sum_i \sum_{j<k} (\xi_{ijj}-\xi_{ikk})^2 + (p-1)^{-1} \sum_{i,\alpha,\beta} (\eta_{i\alpha}^\beta)^2.$$

PROOF of lemma 2.2. For any real number $t \neq 0$, an inequality $(t\sum h_{ij}^\alpha J_{\alpha\beta} - t^{-1}\sum h_{ik}^\beta J_{jk})^2 \geqq 0$ holds, which implies

$$-2\sum h_{ij}^\alpha h_{ik}^\beta J_{\alpha\beta} J_{jk} \geqq -t^2 \sum h_{ij}^\alpha J_{\alpha\beta} h_{ij}^\gamma J_{\gamma\beta} - t^{-2} \sum h_{ik}^\beta J_{jk} h_{il}^\beta J_{jl}.$$

By (1.10), the right hand side of this equation is equal to

$$= -t^2 \sum h_{ij}^\alpha h_{ij}^\gamma (-\sum J_{\alpha k} J_{\gamma k} + \delta_{\alpha\gamma}) - t^{-2} \sum h_{ik}^\beta h_{il}^\beta (-J_{\alpha k} J_{\alpha l} + \delta_{kl})$$

$$= -(t^2+t^{-2}) \sum_{i,j,\alpha} (h_{ij}^\alpha)^2 + t^2 \sum_{i,j,k} (\xi_{ijk})^2 + t^{-2} \sum_{i,\alpha,\beta} (\eta_{i\alpha}^\beta)^2.$$

Therefore, putting $t = \sqrt{p-1}$, we obtain

$$Y + \{((p-1)^2+1)/(p-1)\} \sum_{i,j,\alpha} (h_{ij}^\alpha)^2$$

$$\geqq \sum_{i,j,k} \xi_{ijk}\xi_{ikj} - \sum_{i,j,k} \xi_{ijj}\xi_{ikk} + (p-1) \sum_{i,j,k} (\xi_{ijk})^2 + (p-1)^{-1} \sum_{i,\alpha,\beta} (\eta_{i\alpha}^\beta)^2$$

$$= \sum_{i,j} (\xi_{ijj})^2 + 2\sum_i \sum_{j<k} \xi_{ijk}\xi_{ikj} - \sum_{i,j} (\xi_{ijj})^2 - 2\sum_i \sum_{j<k} \xi_{ijj}\xi_{ikk}$$

$$+(p-1)\sum_{i,j} (\xi_{ijj})^2 + (p-1)\sum_i \sum_{j\neq k} (\xi_{ijk})^2 + (p-1)^{-1} \sum_{i,\alpha,\beta} (\eta_{i\alpha}^\beta)^2$$

$$= (p-2)\sum_i \sum_{j\neq k} (\xi_{ijk})^2 + \sum_i \sum_{j<k} (\xi_{ijk}+\xi_{ikj})^2 + \sum_i \sum_{j<k} (\xi_{ijj}-\xi_{ikk})^2 + (p-1)^{-1} \sum_{i,\alpha,\beta} (\eta_{i\alpha}^\beta)^2,$$

which is the required inequality. (q. e. d.)

We are now in a position to complete the proof of the theorem. Owing to lemma 2.2, (2.1) and (2.2), we obtain

$$
\begin{aligned}
\delta v \geqq & \sum_{i,j,k,\alpha}(h^{\alpha}_{ijk})^2+\sum_{i,j,k,l,\alpha}(\sum h^{\alpha}_{ij}h^{\alpha}_{lk})^2+2\sum_{\alpha,\beta}Tr(H^{\alpha}H^{\alpha}H^{\beta}H^{\beta}-H^{\alpha}H^{\beta}H^{\alpha}H^{\beta}) \\
& +c\{(p-2)/(p-1)\}\sum_{i,j,\alpha}(h^{\alpha}_{ij})^2+c(p-2)\sum_{i}\sum_{j\neq k}(\xi_{ijk})^2+c\sum_{i}\sum_{j<k}(\xi_{ijk}+\xi_{ikj})^2 \\
& +c\sum_{i}\sum_{j<k}(\xi_{ijj}-\xi_{ikk})^2+\{c/(p-1)\}\sum_{i,\alpha,\beta}(\eta^{\beta}_{i\alpha})^2 \\
& +3c\sum_{\alpha}\Big\{2\sum_{i}(\mu^{\alpha}_{ii})^2+\sum_{i<k,\alpha}(\mu^{\alpha}_{ik}+\mu^{\alpha}_{ki})^2\Big\}\geqq 0,
\end{aligned}
$$

since $p\geqq 2$ by assumption.

Since $\mathbf{P}_m(\mathbf{C})$ is orientable and compact, we have

$$\int_{\mathbf{P}_m(\mathbf{C})}\delta v *1 = 0,$$

where $*1$ denotes the volume element of $\mathbf{P}_m(\mathbf{C})$. This together with the above inequality shows

$$\sum h^{\alpha}_{ij}h^{\alpha}_{kl} = 0, \qquad \text{and so} \quad h^{\alpha}_{ij} = 0.$$

The theorem is now completely proved. (q. e. d.)

Next corollary is now obvious:

COROLLARY. *Let* $\mathbf{P}_m(\mathbf{C})$ *be the complex projective space of complex dimension* m *with the metric of constant holomorphic sectional curvature. Let* $\mathcal{F}$ *be a harmonic foliation for which the metric is bundle-like. Then the foliation* $\mathcal{F}$ *is totally geodesic.*

4. Some other results and remarks.

In this section the preceding notations are kept.

We call a foliation on $\mathbf{P}_m(\mathbf{C})$ *Kähler* (resp. *totally real*) if $J_{\alpha i}=0$ (resp. $J_{ij}=0$) at each point.

Let $\mathcal{F}$ be a totally geodesic foliation on $\mathbf{P}_m(\mathbf{C})$. Then from (1.10) and (1.16) we obtain

$$J_{\alpha k}J_{ij}-J_{\alpha j}J_{ik}-2J_{\alpha i}J_{jk} = 0.$$

Therefore

$$0 = \sum(J_{\alpha k}J_{ij}-J_{\alpha j}J_{ik}-2J_{\alpha i}J_{jk})J_{\alpha j}J_{ik}$$

$$= -\sum_{\alpha,i}\Bigl(\sum_{j} J_{\alpha j}J_{ij}\Bigr)^2 - \sum_{i,j,k,\alpha}(J_{\alpha j}J_{ik})^2,$$

which implies

(3.1) $$J_{\alpha j}=0 \quad \text{or} \quad J_{ik}=0 \quad \text{at each point}.$$

PROPOSITION 3.1. *Let $\mathcal{F}$ be a totally geodesic foliation on $\mathbf{P}_m(\mathbf{C})$. Then $\mathcal{F}$ is Kähler or totally real.*

PROOF.

Set $K=\{x\in\mathbf{P}_m(\mathbf{C})\,|\,\mathcal{F}$ is Kähler at $x\}$ and $T=\{x\in\mathbf{P}_m(\mathbf{C})\,|\,\mathcal{F}$ is totally real at $x\}$. Then (3.1) implies the followings:

(a) K and T are open in $\mathbf{P}_m(\mathbf{C})$,

(b) $K\cap T=\emptyset$,

(c) $K\cup T=\mathbf{P}_m(\mathbf{C})$.

These (a), (b), (c) and connectedness of $\mathbf{P}_m(\mathbf{C})$ show the assertion. (q. e. d.)

REMARK 1. There is a well-known example of a foliation on a complex projective space which is induced by the fiber bundle

$$\begin{array}{ccc} \mathbf{P}_1(\mathbf{C}) & \longrightarrow & \mathbf{P}_{2n+1}(\mathbf{C}) \\ & & \downarrow \\ & & \mathbf{P}_n(\mathbf{H}) \end{array}$$

where $\mathbf{P}_n(\mathbf{H})$ denotes the quaternionic projective n-space.

R. Escobales [5] has proved that the above example is the only non-trivial Riemannian foliation on $\mathbf{P}_n(\mathbf{C})$ by $\mathbf{P}_k(\mathbf{C})$ by making use of his results [3], [4] and Ucci's result [15].

REMARK 2. The above example is totally geodesic and Kähler. The auther does not know examples of totally geodesic and totally real foliations on a complex projective space.

Does there exist a totally geodesic foliation on a complex projective space which is totally real?

This question seems to be of interest.

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