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MINIMAL MODELS OF MINIMAL THEORIES

By

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1. Introduction

The algebraic closure $\bar{\mathbf{Q}}$ of the rationals $\mathbf{Q}$ in the complex number field $\mathbf{C}$ is small in the following two senses: (i) There is no proper elementary subfield K of $\bar{\mathbf{Q}}$, and (ii) every field which is elementarily equivalent to $\bar{\mathbf{Q}}$ has a copy of $\bar{\mathbf{Q}}$ in it. In general model theory we have to distinguish these two notions. The notion expressing the first property is called *minimal*, and the other for the the second *prime* (see Definition 1). The following is an example of a theory having a minimal non-prime model:

EXAMPLE (Fuhrken [2]). The theory T_0 is defined as follows: For each $\nu\in{}^{<\omega}2$ we define a function $F_\nu: {}^{\omega}2\to{}^{\omega}2$ by $(F_\nu(\eta))(i)=\nu(i)+\eta(i) \bmod 2$ for $\eta\in{}^{\omega}2$, $i<\omega$. And for $\eta\in{}^{<\omega}2$, $P_\eta=\{\tau\in{}^{\omega}2: \eta\prec\tau\}$. Let $M=({}^{\omega}2, \{F_\nu\}_{\nu\in{}^{<\omega}2}, \{P_\eta\}_{\eta\in{}^{<\omega}2})$ and $T_0=Th(M)$. Then each model generated by only one element ($\in M$) is minimal and non-prime.

Our concern is the number of minimal models of a theory with no prime model (In fact if a theory has a prime model then it has at most one minimal model). In [3] Marcus showed that if T is a theory of one unary function symbol and T has a minimal non-prime model then T has $2^{\aleph_0}$ such models. On the other hand, Shelah proved that for every κ, $1\leqq\kappa\leqq\aleph_0$, there is a theory with exactly κ minimal non-prime models (see [4]).

Here we extent Marcus' result: Theories of one unary function symbol may have the Lascar rank greater than 1 ($U(T)>1$), however if such a theory T has a minimal model then any element a of the model has the minimum Lascar rank (i.e. $U(a)\leqq 1$). Moreover a theory of one unary function symbol is *trivial* (*see* Definition 3). In this paper we show that if a trivial theory T has a minimal non-prime model and every element of the model has the minimum Lascar rank then T has $2^{\aleph_0}$ minimal models. Our result does not depend on the language.

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2. Definitions and Preliminary results

Our notations and conventions are standard. We fix a complete theory T formulated in a countable language L. We work in a big model C of T. A, $B, \cdots$ are used to denote small subsets of C. $\bar{a}, \bar{b}, \cdots$ are used to denote finite sequences of elements in C. $\varphi, \psi, \cdots$ are used to denote formulas (with parameters). $p, q, \cdots$ are used to denote types (with parameter). The types of a over A is denoted by $\mathrm{tp}(a/A)$. φ^B denotes the set of realizations of φ in a set B. The Lascar rank of p is denoted by $U(p)$. We simply write $U(a/A)$ instead of $U(\mathrm{tp}(a/A))$. $U(a)$ means $U(a/\emptyset)$.

DEFINITION 1. Let M be a model of the theory T.

(1) M is said to be *minimal* if there is no proper elementary submodel of M.

(2) M is said to be *prime* if M can be elementarily embedded in any model of T.

DEFINITION 2. (1) Let A be a set. Then an $L(A)$-type $\Gamma(x)$ (not necessarily complete) is said to be *principal over* A if it is generated by one $L(A)$-formula $\varphi(x)$ (φ need not be a formula in Γ).

(2) A formular $\varphi(x)\in L$ is said to be *atomless* if there is no formula $\psi(x)$ with the following properties:

(i) $T\vdash\forall x(\psi(x)\rightarrow\varphi(x))$;

(ii) $\psi(x)$ is complete i.e. $\psi(x)$ determines a complete type $p(x)$.

If $S(\emptyset)=\bigcup_{n<\omega}S^n(\emptyset)$ is countable, then there is a prime (and atomic) model. On the other hand, if $S(\emptyset)$ is uncountable then there is an atomless formula.

We prove a version of Lemma 1.3 of [3].

LEMMA. *Let $\Gamma(\bar{x})$ be a non-pirncipal (possibly incomplete) type over a countable set A. Suppose that there is an atomless formula $\psi(y)$ over $\emptyset$ such that any realization d of ψ independent from A. Then there are $2^{\aleph_0}$ countable models ($\supset A$) omitting Γ.*

PROOF. First we show the following claim:

CLAIM 1. *Let $\theta(\bar{x}, y)$ and $\varphi(y)$ be $L(A)$-formulas. If $\theta(\bar{x}, y)\wedge\varphi(y)$ is consistent then there is an $L(A)$-formula $\varphi^*(y)$ with $\varphi^{*C}\subset\varphi^C$ such that $\theta(\bar{x}, d)$ does not generate Γ for any realization d of φ^*.*

PROOF. Since Γ is non-principal over A there is a realization d of φ such that $\theta(\bar{x}, d)$ does not generate Γ. So we can pick $\gamma\in\Gamma$ such that $\theta(\bar{x}, d)\wedge\neg\gamma(\bar{x})$ is consistent. Define $\varphi^*(y)=(\exists\bar{x})(\varphi(y)\wedge\theta(\bar{x}, y)\wedge\neg\gamma(\bar{x}))$. Then φ^* is a consistent $L(A)$-formula. It is clear that Γ is not generated by $\theta(x, d)$ for any $d\in\varphi^{*C}$.

Let $\Gamma(\bar{x})$ have k-variables. Let $\theta_n(\bar{x}, y)$ $(n<\omega)$ be an enumeration of all $L(A)$-formula with $(k+1)$-variables.

CLAIM 2. *We can define inductively $L(A)$-formulas $\psi_\eta(y)$ and L-formula $\alpha_\eta(y)$ $(\eta\in{}^{<\omega}2)$ satisfying the following conditions: for each $\eta\in{}^{<\omega}2$,*

(1) $\psi_{\langle\rangle}(y)=\psi(y)$;

(2) $\models(\forall y)(\psi_{\eta\frown i}(y)\rightarrow\psi_\eta(y))$ $(i=0, 1)$;

(3) *there is an L-formula $\alpha_\eta(y)$ such that $\models(\forall y)(\psi_{\eta\frown 0}(y)\rightarrow\alpha_\eta(y))$ and $\models(\forall y)(\psi_{\eta\frown 1}(y)\rightarrow\neg\alpha_\eta(y))$;*

(4) *If $\psi_\eta(y)\wedge\theta_n(\bar{x}, y)$ is consistent then $\theta_n(\bar{x}, a)$ does not generate Γ for any realization a of ψ_η (the length of η is $n+1$).*

PROOF. Suppose that ψ_η's (the length of η is $\leqq n+1$) have been defined. Fix any η with length $n+1$. First we see that there is an L-formula $\alpha(y)$ such that both $\alpha(y)\wedge\psi_\eta(y)$ and $\neg\alpha(y)\wedge\psi_\eta(y)$ are consistent. If not, ψ_η generates some complete L-type q. Since ψ is atomless q is non-principal. On the other hand, by the assumption, ψ_η does not fork over $\emptyset$. So ψ_η is realized by every model. This means that q is principal, which is a contradiction. Therefore we get such an $\alpha(y)$. Put $\alpha_\eta(y)=\alpha(y)$. Let $\psi_0(y)=\alpha_\eta(y)\wedge\psi_\eta(y)$ and $\psi_1(y)=\neg\alpha_\eta(y)\wedge\psi_\eta(y)$. Suppose that $\psi_0(y)\wedge\theta_{n+1}(x, y)$ is consistent. By claim 1 we obtain an $L(A)$-formula $\psi_0^*(\psi)$ with $\psi_0^{*C}\subset\theta_0{}^C$ such that $\theta_{n+1}(\bar{x}, d)$ does not generate $\Gamma(\bar{x})$ for any realization d of ψ_0^*. Put $\psi_{\eta\frown 0}=\psi_0^*$. Similarly we can get $\psi_{\eta\frown 1}$. Then they satisfy our requirement. This completes our construction.

For $\tau\in{}^\omega 2$, define $\Sigma_\tau(y)=\{\psi_\tau(y)=\{\psi_{\tau\restriction n}(y): n<\omega\}$. It is easy to see that Σ_τ's are $L(A)$-types which satisfy that i) $\tau\neq\lambda$ implies $\mathrm{tp}(d_\tau)\neq\mathrm{tp}(d_\lambda)$ for any realization d_τ of Σ_τ and d_λ of Σ_λ, and ii) if d_τ is a realization of Σ_τ then Γ is non-principal over $A\cup d_\tau$. By ii), for every $\tau\in{}^\omega 2$ there is a countable model $M_\tau(\supset A\cup d_\tau)$ omitting Γ. By i), for any M_τ there are at most countably many M_λ's isomorphic to M_γ. Thus there is an $X\subset{}^\omega 2$ with $|X|=2^{\aleph_0}$ such that $M_\tau(\tau\in X)$ are pairwise non-isomorphic. Hence we obtain $2^{\aleph_0}$ countable models omitting Γ. This completes the proof of the lemma. ■

DEFINITION 3 (see, e.g., [1]). T is said to be *trivial* it has the following property: for any three elements $a, b, c \in C$ and any set $A \subset C$, if a, b and c are pairwise independent over A then they are independent over A.

3. Theorem and Proof

THEOREM. *Let T be stable and trivial. Suppose that T has a model M such that*

(1) *M is minimal and non-prime;*

(2) *$U(a) \leqq 1$, for all $a \in M$.*

Then T has $2^{\aleph_0}$ minimal models.

PROOF. First we show the following claim:

CLAIM 1. *There are an element a of M and a finite subset F of M such that* $\mathrm{tp}(a/F)$ *is non-principal.*

PROOF. M is a non-prime model. So it is not atomic, hence there is a minimal finite subset E of M such that $\mathrm{tp}(E)$ is non-principal. Pick any element a of E. Let $F=E-\{a\}$. By the minimality of E $\mathrm{tp}(F)$ is principal, so $\mathrm{tp}(a/F)$ is non-principal.

Here we say that a set $D(\subset C)$ is s *minimal component* if d and d' are interalgebralc for any $d, d' \in D$. Let $C=\mathrm{acl}(a)-\mathrm{acl}(\emptyset)$ and $A=M-C$. Then C is a minimal component since $U(a)=1$.

CLAIM 2. *There are a finite subset F' of A and an atomless formula $\phi(y)$ over F' such that any realization d of ϕ is independent from A over F'.*

PROOF. Since M is a minimal model, by the Tarski-Vaught test, we can easily find an $L(A)$-formula $\phi(y, \bar{a})$ such that $\phi^M \subset C$. Let $F'=F \cup \bar{a}$. We notice that under the assumption (2), in M the general notion of independence coincides algebraic independence. So C and A are independent by using the triviality of T. First we will show that ϕ is atomless over F'. If not, there is a complete formula $\phi'(y)$ over F' such that $\phi'^C \subset \phi^C$. Then ϕ' is realized by some element e of C. On the other hand, by claim 1, $\mathrm{typ}(e/F)$ is non-principal. Thus using the Open Map Theorem we obtain that $\mathrm{tp}(e/F')$ is non-principal, which contradicts that ϕ' is complete. Hence ϕ is atomless over F'. Next we show that any realization d of ϕ is independent from A over F'. Let d be any realizatoin of ϕ. Take any formula $\theta(y) \in \mathrm{tp}(d/A)$. Then $\phi(y) \wedge \theta(y)$

is consistent. Notice that $\psi^M \subset C$. So we can pick a realization d' of θ in C. Now $\mathrm{tp}(d'/A)$ does not fork over F' since C and A are independent. Hence θ does not fork over F'. It follows that $\mathrm{tp}(d/A)$ does not fork over F'.

Define $\Gamma(x, y)=\{x$ and y are not interalgebraic$\}\cup\{x\neq c: c\in A\}\cup\{y\neq c: c\in A\}$. Γ is non-principal over F' because our model $M(\supset F')$ omits it. From claim 2 it follows that Γ and ψ satisfy the assumptions of the lemma. So we get the followıng claim (Note that F' is finite):

CLAIM 3. *There are pairwise non-isomorphic countable models* $M_\tau(\tau<2^{\aleph_0})$ *omitting* Γ.

CLAIM 4. *Each* M_τ *is a minimal model.*

PROOF. Since M_τ omits Γ and contains A, there is a minimal component D such that $M_\tau=D\cup A$. Suppose that M_τ is not minimal. Then there is a proper sucset B of A such that $D\cup B$ is an elementary submodel of M_τ. So we can pick a minimal component $E\subset A-B$. First, by the minimality of M there is an $L(M-E)$-formula $\psi(x, \bar{b})$ such that ψ^M is contained in E. Hence $\psi^B=\emptyset$. By the triviality of T, E and $\bar{b}$ are independent, so ψ does not fork over $\emptyset$. Thus ψ is realized by the model $D\cup B$. We have therefore $\psi^D\neq\emptyset$. Next, by the minimality of M, there is an $L(A)$-formula $\varphi(x, \bar{a})$ such that φ^M is contained in C. So φ^{M_τ} is contained in D. Hence $\varphi^D\neq\emptyset$. Note that any two elements of D are interalgebraic. Hence we can assume that there is an element $d\in C$ which realizes both φ and ψ. In particular we have $M\models(\exists x)(\varphi(x, \bar{a})\wedge\psi(x, \bar{b}))$. This contradicts that C and E are disjoint. Hence M_τ is minimal.

By claim 3, 4, we obtain $2^{\aleph_0}$ minimal models. This completes the proof of the theorem. ∎

REMARKS. (1) It is known that a theory of one unary function symbol f is stable and trivial (see e. g. [5]). Moreover a minimal model of ouch a theory has minimum Lascar rank. This can be shown as follows: Pick any element a of a minimal model of the theory. Let $\mathrm{tp}(a/B)$ be a forking extension of $\mathrm{tp}(a)$. Then by Lemma 1 in [5], there is an element b of B which is contained in the *connected component* $C(a)$ of a, where $C(a)=\{x:\exists n, m<\omega[f^n(a)=f^m(x)]\}$. On the other hand we see that each connected component in a minimal model is a minimal component in our language (see Lamma 3.1 in [3]). Therefore

$C(a)$ is a minimal component, so a and b are interalgebraic. Thus tp(a/B) is algebraic. Hence $U(a) \leqq 1$. It follows that our theorem is a generalization of Marcus' one.

(2) The theory T_0 (see Introduction) satisfies the assumption of our theorem, i. e. it is stable and trivial, and has a minimal non-prime model with minimum Lascar rank.

(3) In [4] Shelah has shown that any κ with $1 \leqq \kappa \leqq \aleph_0$ there is a complete theory, with no prime model, and exactly κ minimal models. Theories he gave are stable, trivial and have a minimal non-prime model. But all minimal models of them have the Lascar rank 2. This shows that the condition (2) of our theorem is essential.

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