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A NOTE ON SYMMETRY OF PERPENDICULARITY IN A G-SPACE WITH NONPOSITIVE CURVATURE

By

Nobuhiro INNAMI

1. Introduction.

Let $\mathfrak{R}$ be a G-space. We denote by $m(a, b)$, for $a, b\in\mathfrak{R}$, a midpoint so that $am(a, b)=m(a, b)b=ab/2$. The G-space $\mathfrak{R}$ has "*nonpositive curvature*" if every point p has a neighborhood $S(p, \gamma_p)$, where $0<\gamma_p<\rho_1(p)$ (see [3] for definition of $\rho_1(p)$), such that for any three points a, b, c in $S(p, \gamma_p)$ the relation $2m(a, b)m(a, c)\leq bc$ holds, and R has "*negative curvature*" if $2m(a, b)m(a, c)<bc$ when a, b, c are not on one segment. Because a G-space $\mathfrak{R}$ with nonpositive curvature is finite-dimensional according to V. N. Berestovskii [1], and hence $\mathfrak{R}$ has "*domain invariance*" (see [4] p. 16), the universal covering space $\overline{\mathfrak{R}}$ of $\mathfrak{R}$ is straight by Busemann [3] p. 254. Moreover the spheres in $\overline{\mathfrak{R}}$ are convex. The straight line L in a G-space is called a "*perpendicular*" to the set M at f, if $f\in L\cap M$ and every point of L has f as a *foot* on M, i.e., $qf=qM$ for any $q\in L$. We say that perpendicularity between lines is *symmetric* if the following holds: if a straight line L is a perpendicular to a straight line G, then G is a perpendicular to L. We say that a set M of a G-space is *totally convex* if $p, q\in M$ implies that all geodesic curves from p to q are contained in M. If a closed set M of a G-space $\mathfrak{R}$ in which the spheres are convex is totally convex, then for each $p\in\mathfrak{R}$ there is a unique point $q\in M$ such that $pq=pM$. If the spheres of a straight G-space are convex, we denote by W_p the point set carring straight lines through $p\in K(q, \sigma):=\{r\,;qr=\sigma\}$ but not through any point $p'\in S(q, \sigma)=\{r\,;qr<\sigma\}$, which are called the supporting lines of $K(q, \sigma)$ at p. $K(q, \sigma)$ is "*differentiable at $p\in K(q, \sigma)$*" if no proper subset of the W_p decomposes the space.

In the present paper we prove

THEOREM 1. *Let $\mathfrak{R}$ be a G-space of nonpositive curvature. If the spheres in the universal covering space $\overline{\mathfrak{R}}$ of $\mathfrak{R}$ are differentiable and if perpendicularity between lines is symmetric in $\overline{\mathfrak{R}}$, then for every closed totally convex set M in $\mathfrak{R}$ the map $\rho:\mathfrak{R}\rightarrow M$ defined by sending $p\in\mathfrak{R}$ to the foot of p on M is distance non-increasing. Further, if $pq=\rho(p)\rho(q)\neq 0$, then $S:=\bigcup_{0\leq t\leq 1}T(p_t, q_t)$ is isometric to a trapezoid in a Minkowski plane, where $T(p_t, q_t)$ is the point set carring the segment from $p_t\in T(p, \rho(p))$ to $q_t\in T(q, \rho(q))$ with $pp_t:p_t\rho(p)=qq_t:q_t\rho(q)=t:(1-t)$.*

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In the class of Riemannian G-spaces of nonpositive curvature L. Bishop and B. O'Neill [2] have proved the theorem because our hypothesis is automatically satisfied in such a class. Our main purpose is to exhibit that the property as in the theorem depends only on differentiability of the spheres and symmetry of perpendicularity.

As an application we have

PROPOSITION 1. *Let $\mathfrak{R}$ be a G-space with nonpositive curvature. Suppose that the spheres in the universal covering space $\tilde{\mathfrak{R}}$ of $\mathfrak{R}$ are differentiable and that perpendicularity between lines in $\tilde{\mathfrak{R}}$ is symmetric. If $\mathfrak{R}$ contains a compact totally convex set M, then all free homotopy classes of closed curves in $\mathfrak{R}$ contains closed geodesics.*

Once we establish Theorem 1, the proof of Proposition 1 is done in the same way as in [2], because Busemann [3] has already given the requisite properties for the proof. So we omit the proof.

Lastly the author wishes the readers to accept the synthetic approach to differential geometry.

2. Lemmas and Proof of Theorem 1.

The following lemma is proved in [5].

LEMMA 1. *Let $\mathfrak{R}$ be a simply connected G-space with nonpositive curvature in which the spheres are differentiable. For any point p and for any representation $x(t)$, $-\infty<t<\infty$, of each geodesic with $p\notin x(\mathbf{R})$ if $px(t_0)=px(\mathbf{R})$, then $px(t)$ is differentiable at t_0 and $(px)'(t_0)=0$.*

Using Lemma 1 we prove

LEMMA 2. *Let $\mathfrak{R}$ be a simply connected G-space with nonpositive curvature in which the spheres are differentiable and let $x(t)$, $-\infty<t<\infty$, and $y(s)$, $-\infty<s<\infty$, be representations of geodesics in $\mathfrak{R}$. Then the length L of the segment $T(x(t_0), y(s_0))$ attains the distance between $x(\mathbf{R})$ and $y(\mathbf{R})$ if and only if each of $x(t_0)$ and $y(s_0)$ is the foot of the other on the geodesic containing one.*

PROOF. If one, say $x(t_0)$, is not the foot of the other, say $y(s_0)$, on $x(\mathbf{R})$, then there is a point $x(t_1)$ with $y(s_0)x(t_1)=y(s_0)x(\mathbf{R})<L$ and with $t_0\neq t_1$. This implies that $L>x(\mathbf{R})y(\mathbf{R})$. Thus the necessity is established.

Now, we suppose that each of $x(t_0)$ and $y(s_0)$ is the foot of the other on $x(\mathbf{R})$

or $y(\mathbf{R})$, i.e., $x(t_0)y(s_0)=x(t_0)y(\mathbf{R})=y(s_0)x(\mathbf{R})$. Further we suppose that $L\neq x(\mathbf{R})y(\mathbf{R})$, and hence there exists points $x(t_1)$ and $y(s_1)$ with $x(t_1)y(s_1)<L$. Choose parametrizations $x_1(t)$ and $y_1(t)$, $-\infty<t<\infty$, of $x(\mathbf{R})$ and $y(\mathbf{R})$ respectively in such a way that $x_1(t)=x((t_1-t_0)t+t_0)$ and $y_1(t)=y((s_1-s_0)t+s_0)$, $-\infty<t<\infty$. Then $x_1(t)y_1(t)=:f(t)$ is a convex function for t (see [3] p. 238). We show that $f'(0)=0$, and then from convexity of f $f(0)=x_1(0)y_1(0)=x(t_0)y(s_0)$ is a minimum of f, contradicting that $f(1)=x(t_1)y(s_1)<L=x(t_0)y(s_0)=f(0)$.

Let z be an interior point of the segment $T(x(t_0), y(s_0))$. Then both $x(t_0)$ and $y(s_0)$ are feet of z on $x(\mathbf{R})$ and $y(\mathbf{R})$ respectively. From Lemma 1 we have

$$\begin{aligned}
&\lim_{\substack{t\downarrow 0\\(t\uparrow 0)}} (f(t)-f(0))/t=\lim_{\substack{t\downarrow 0\\(t\uparrow 0)}} (x_1(t)y_1(t)-x_1(0)y_1(0))/t\\
&=\lim_{\substack{t\downarrow 0\\(t\uparrow 0)}} (x((t_1-t_0)t+t_0)y((s_1-s_0)t+s_0)-x(t_0)y(s_0))/t\\
&\underset{(\geqq)}{\leq} \lim_{\substack{t\downarrow 0\\(t\uparrow 0)}} (x((t_1-t_0)t+t_0)z-x(t_0)z)/t+\lim_{\substack{t\downarrow 0\\(t\uparrow 0)}} (y((s_1-s_0)t+s_0)z-y(s_0)z)/t\\
&=(t_1-t_0)\lim_{h\to 0} (x(h+t_0)z-x(t_0)z)/h+(s_1-s_0)\lim_{h\to 0} (y(h+s_0)z-y(s_0)z)/h=0.
\end{aligned}$$

Thus $0\leq \lim_{t\uparrow 0} (f(t)-f(0))/t \leq \lim_{t\downarrow 0} (f(t)-f(0))/t\leq 0$ from convexity of f.

We remark that if the spheres are not differentiable, then, in general, sufficiency does not hold. Such an example is found in Minkowski geometry (see [3]).

The following is the case where $\mathfrak{R}$ is simply connected.

PROPOSITION 2. *Let $\mathfrak{R}$ be a simply connected G-space with non-positive curvature in which the spheres are differentiable and in which perpendicularity between lines is symmetric. If M is a closed totally convex set in $\mathfrak{R}$, then the map $\rho:\mathfrak{R}\to M$ defined by sending $p\in\mathfrak{R}$ to the foot of p on M is distance nonincreasing. Further, if $pq=\rho(p)\rho(q)\neq 0$, then $S:=\bigcup_{0\leq t\leq 1}T(p_t, q_t)$ is isometric to a trapezoid in a Minkowski plane, where the segment $T(p_t, q_t)$ joins $p_t\in T(p, \rho(p))$ and $q_t\in T(q, \rho(q))$ with $pp_t:p_t\rho(p)=qq_t:q_t\rho(q)=t:(1-t)$.*

PROOF. Let p and q be any points in $\mathfrak{R}$. If $\rho(p)=\rho(q)$, then Proposition 2 is trivial. We assume that $\rho(p)\rho(q)=t_0>0$. Let $x(t)$, $-\infty<t<\infty$, be the representation of the geodesic which is determined by $x(0)=\rho(p)$ and $x(t_0)=\rho(q)$. Total convexity of M implies that $x([0, t_0])$ is contained in M. If $f_1=x(t_1)$ and $f_2=x(t_2)$ are the feet of p and q on $x(\mathbf{R})$ respectively, then $t_1\leq 0<t_0\leq t_2$ In fact, t_1 cannot be in $(0,\infty)$, because $x((0, t_0))\subset M$ and because $f(t):=px(t)$ is a strictly convex function for t (see [3] p. 240) which takes a minimum at $t=t_1$ or $f(t)=|t-t_1|$. By the same reasoning $t_0\leq t_2$.

We first treat the case where $p\notin x(\mathbf{R})$ and $q\notin x(\mathbf{R})$. Let $w(t)$, $-\infty<t<\infty$, and $z(t)$, $-\infty<t<\infty$, be the representations of geodesics in $\mathfrak{R}$ with $w(0)=x(t_1)$, $w(px(t_1))=p$, $z(0)=x(t_2)$ and $z(qx(t_2))=q$. Now we have only to prove that $w(\mathbf{R})z(\mathbf{R})=x(t_1)x(t_2)$, and then we have $pq\geq w(\mathbf{R})z(\mathbf{R})=x(t_1)x(t_2)=t_2-t_1\geq t_0=\rho(p)\rho(q)$. Since perpendicularity between lines is symmetric, $x(t_1)x(t_2)=x(t_1)z(\mathbf{R})=x(t_2)w(\mathbf{R})$ and therefore by Lemma 2 $x(t_1)x(t_2)=w(\mathbf{R})z(\mathbf{R})$.

If $p\in x(\mathbf{R})$ and $q\in x(\mathbf{R})$, then $pq=t_2-t_1\geq t_0=\rho(p)\rho(q)$. Hence we next treat the case where, for example, $p\in x(\mathbf{R})$ and $q\notin x(\mathbf{R})$. In this case we know from [3] p. 122 that the set W formed by the perpendiculars to $x(\mathbf{R})$ at $x(0)=\rho(p)$ of $x(\mathbf{R})$ decomposes the space into two arcwise connected sets. Because $\rho(p)$ and $\rho(q)$ are contained in distinct components of $\mathfrak{R}-W$, so are p and q. Thus there is a point $p'\in T(p,q)\cap W$. By Lemma 2 $pq\geq p'q\geq\rho(p)\rho(q)$.

On the second part of the statement, see [3] p. 241.

PROOF OF THEOREM 1. The idea of the proof is the same as the one in [2]. Let $\bar{M}\subset\bar{\mathfrak{R}}$, $\bar{p}\in\bar{\mathfrak{R}}$ and $\bar{q}\in\bar{\mathfrak{R}}$ lie over M, p and q respectively and let $\bar{\rho}$ be the map of $\bar{\mathfrak{R}}$ into $\bar{M}$ defined in Proposition 2. Clearly $\bar{M}$ is a closed totally convex set in $\bar{\mathfrak{R}}$. We first prove that $\rho\circ\pi=\pi\circ\bar{\rho}$ where π is the covering projection of $\bar{\mathfrak{R}}$ onto $\mathfrak{R}$. For any $\bar{p}\in\bar{\mathfrak{R}}$ if the segment $T(\bar{p},\overline{\rho\pi(\bar{p})})$ from $\bar{p}$ to $\overline{\rho\pi(\bar{p})}$ lies over $T(p,\rho(p))$, it is the distance minimizing segment from p to $\bar{M}$. If it is false, there is a point $\bar{\rho}(\bar{p})\neq\overline{\rho\pi(\bar{p})}$ with $\bar{p}\bar{\rho}(\bar{p})<\bar{p}\overline{\rho\pi(\bar{p})}$. Because π is distance nonincreasing, $pM<p\rho\pi(\bar{p})=p\rho(p)$, a contradiction.

For any $p\in\mathfrak{R}$ and $q\in\mathfrak{R}$ if a segment $T(\bar{p},\bar{q})$ lies over a segment $T(p,q)$, then $pq=\bar{p}\bar{q}\geq\bar{\rho}(\bar{p})\bar{\rho}(\bar{q})\geq\pi\bar{\rho}(\bar{p})\pi\bar{\rho}(\bar{q})=\rho\pi(\bar{p})\rho\pi(\bar{q})=\rho(p)\rho(q)$, and the equality $pq=\rho(p)\rho(q)$ holds if and only if $S:=\bigcup_{0\leq t\leq 1}T(p_t,q_t)$ is isometric to a trapezoid in a Minkowski plane.

Note. Professor Busemann informed the author that if perpendicularity between lines is symmetric in a straight space in which the spheres are convex, then the spheres are differentiable. So our assumption of the differentiability of the spheres in Theorem 1 and Proposition 1 is unnecessary. The author would like to express his thanks to Professor H. Busemann for his valuable information.

References

[1] Berestovskii, V.N., The finite-dimensionality problem for Busemann G-spaces, Siberian Math. J. 18, 1 (1977), 159-161.

[2] Bishop, R.L. and O'Neill, B., Manifolds of negative curvature, Trans. Am. Math. Soc. **145**, (1969), 1-49.

[3] Busemann, H., The geometry of geodesics, Academic Press, New York, 55.19

[4] Busemann, H., Recent synthetic differential geometry, Springer-Verlarg, Berlin, 1970.
[5] Innami, N., Axial and parabolic G-spaces, to appear.

Institute of Mathematics
University of Tsukuba
Sakura-mura Ibaraki
305 Japan