TSUKUBA J. MATH.
Vol. 14 No. 2 (1990), 263–273

VANISHING OF HOCHSCHILD'S COHOMOLOGIES $H^i(A\otimes A)$ AND GRADABILITY OF A LOCAL COMMUTATIVE ALGEBRA A

By

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0. Introduction.

In [8] Nakayama conjectured that a finite dimensional algebra R with an infinite dominant dimension is selfinjective. As such an algebra R is isomorphic to an endomorphism ring of a generator-cogenerator over an algebra A, Tachikawa [10] has shown that the Nakayama's conjecture is reduced to the following conjectures (i) and (ii): For a finite dimensional algebra A over a field K,

(i) A is selfinjective if Hochschild's cohomological groups $H^i(A\otimes_K A)\cong \mathrm{Ext}_A^i(D(A), A)=0$ for $i\geqq 1$, where $D(A)=\mathrm{Hom}_K(A, K)$.

(ii) An A-module X is projective if A is selfinjective and if $\mathrm{Ext}_A^i(X, X)=0$ for $i\geqq 1$.

It is to be noted here that the Nakayama's conjecture is true if and only if both the conjectures (i) and (ii) are true.

For the conjecture (ii) there have been already several interesting results by Hoshino [6] and Schulz [9]. In [7] Hoshino applied Wilson's therem to settle the conjecture (i) for algebras A's with cube zero radicals, because in this case both A's and the corresponding endomorphism rings R's are positively $\mathbf{Z}$-graded.

This paper concerns with the conjecture (i) for local commutative algebras. In §1 we provide a theorem that for a local (not necessarily commutative) algebra A, $R=\mathrm{End}_A(A\oplus D(A))$ is positively $\mathbf{Z}$-graded if and only if so is A. It is proved in §2 that local algebras with quartic zero radicals such that they are homomorphic images of polynomial ring $K[x, y]$ over an algebraically closed field K are positively $\mathbf{Z}$-graded, and applying Wilson's theorem we can prove that conjecture (i) is true for such algebras. In §3 we shall give, however, a not positively $\mathbf{Z}$-graded commutative local algebra, which is a homomorphic image of the polynomial ring $K[x, y, z]$ with quartic zero radical.

Received March 22, 1989. Revised December 4, 1989.

1. Preliminary

Let R be an finite dimensional algebra over a field K. Let

$$0 \longrightarrow R \longrightarrow E_1 \longrightarrow E_2 \longrightarrow \cdots \longrightarrow E_n \longrightarrow \cdots \tag{1}$$

be a minimal injective resolution of the right R-module R.

In [2] Auslander and Reiten introduced the generalized Nakayama conjecture: Every simple R-module appears as a submodule of some E_n in (1). We shall say dom dim $R_R \geqq n$ (resp. $=\infty$) if E_j are projective R-modules for all $j<n+1$ (resp. all $j>0$) in (1).

In [8] Nakayama conjectured that R is selfinjective if dom dim $R_R=\infty$. The Nakayama conjecture is true if the generalized Nakayama conjecture is true, because the injective envelope of any simple right R-module S is projective, if dom dim $R_R=\infty$.

In [11] Wilson proved that the generalized Nakayama conjecture is true for positively $\mathbf{Z}$-graded algebras.

Suppose dom dim $R_R \geqq 2$. It is well known that there exists a minimal faithful left R-module which is a projective and injective left ideal Re for an idempotent e. Further $R \cong \mathrm{End}_{eRe}Re$ and Re is a generator-cogenerator as a right eRe-module. Cf. [10]. Conversely for any algebra A and for a generator-cogenrator X_A, dom dim $\mathrm{End}_A X \geqq 2$. This connection between A and $\mathrm{End}_A X$ plays an important role in this paper. In our context $\mathrm{End}_A X$ is selfinjective iff A is selfinjective.

A *graded* algebra is an algebra A together with a vector space decomposition $A=\bigoplus_{k\in\mathbf{Z}} A_k$ such that $A_i A_j \subset A_{i+j}$.

Since A is a finite dimensional algebra, $A_k=0$ for $|k| \gg 0$. We will consider positively $\mathbf{Z}$-*graded* algebras, that is, graded algebras with $A_k=0$ if $k<0$. We will further assume rad $A=\bigoplus_{k\geqq 1} A_k$. Thus we will write $A=\bigoplus_{k\geqq 0} A_k$.

A *graded right A-module* is a module M together with a vector space decomposition $M=\bigoplus_{k\in\mathbf{Z}} M_k$ such that $M_i A_j \subset M_{i+j}$. Notice that we are allowing negative gradings on our modules. If $L=\bigoplus_{k\in\mathbf{Z}} L_k$ is another graded A-module, we define a degree i morphism to be an A-homomorphism $f: M \to L$ such that $f(M_k) \subset L_{i+k}$. It is to be noted that for a graded A-module M the degrees of morphisms make $\mathrm{End}_A M$ be a (not necessarily positively) $\mathbf{Z}$-graded algebra (see [4, §2]).

The i-th shift $\sigma(i)(M)$ of $M=\bigoplus_{k\in\mathbf{Z}} M_k$ is defined to be a graded A-module $L=\bigoplus_{k\in\mathbf{Z}} L_k$ such that $L_k=M_{k-i}$.

THEOREM 1.1. *Let A be a local algebra, $D(A)=\mathrm{Hom}_K(A, K)$ the injective* *cogenerator as a right A-module and R the endomorphism ring of $A\oplus D(A)$. Then R is positively $\mathbf{Z}$-graded iff so is A. Here it is to be noted that the grading of A is one induced from the grading of R and the grading of R is one induced by the degrees of morphisms in* $\mathrm{End}_A(A\oplus D(A))$.

PROOF. "Only if" part. Let $R=\oplus_{k=0}^{n} R_k$ and e a projection: $A\oplus D(A)\to A$. Since rad $R=\oplus_{k=1}^{n} R_k$, there is an idempotent f of R such that $f\cong e$ and we have that $A\cong eRe$ is isomorphic to a positively $\mathbf{Z}$-graded algebra $fRf=\oplus_{k=0}^{n}(fRf)_k$ with $(fRf)_k=fR_kf$.

"If" part. Let $A=\oplus_{k=0}^{n} A$. Then $D(A)$ is gradable such that $D(A)_{-k}=D(A_k)$ for $n\geqq k\geqq 0$. If A is selfinjective, i.e. $A\cong D(A)$, then $R\cong\begin{pmatrix} A & A \\ A & A\end{pmatrix}$ and R has a grading with $R_k\cong\begin{pmatrix} A_k & A_k \\ A_k & A_k\end{pmatrix}$. So we may assume that A is not selfinjective.

By using the n-th shift $\sigma(n)$ we obtain a new grading of $D(A)$ such that $D(A)=(D(A))_0\oplus(D(A))_1\oplus\cdots\oplus(D(A))_n$, where $(D(A))_i\cong D(A_{n-i})$, $0\leqq i\leqq n$.

Now

$$R\cong\mathrm{End}_A(A\oplus D(A))\cong\begin{pmatrix} \mathrm{Hom}_A(A, A) & \mathrm{Hom}_A(D(A), A) \\ \mathrm{Hom}_A(A, D(A)) & \mathrm{Hom}_A(D(A), D(A))\end{pmatrix}$$

and it is clear that $\mathrm{Hom}_A(A, A)\cong\mathrm{Hom}_A(D(A), D(A))\cong A$, $\mathrm{Hom}_A(A, D(A))\cong D(A)$ and degrees of morphisms define naturally non-negative $\mathbf{Z}$-gradings of $\mathrm{Hom}_A(A, A)$, $\mathrm{Hom}_A(D(A), D(A))$ and $\mathrm{Hom}_A(A, D(A))$ which are respectively identical with A, A and $D(A)$.

Next for the $\mathbf{Z}$-grading $\mathrm{Hom}_A(D(A), A)=\oplus_{i\in\mathbf{Z}}\mathrm{Hom}_{\mathrm{gr}A}(D(A), \sigma(-i)A)$, we want to notice here that the degree of any morphism from $D(A)$ to A is at least one. This fact will be proved by induction on n as follows: If $n=1$, then $A=A_0\oplus A_1$ and $(D(A))_0\cong D(A_1)$, $(D(A))_1\cong D(A_0)$. Hence it is clear that $-1\leqq$degree of $\phi\leqq 1$ for $\phi\in\mathrm{Hom}_A(D(A), A)$. But since $D(A_0)$ is the socle of $D(A)$ and A is not selfinjective, ϕ is not a monomorphism and the degree ϕ must be 1. Assume that for any grading $B=B_0\oplus B_1\oplus\cdots\oplus B_r$, $r<n$, the degree of $\varphi\geqq 1$ for $\varphi\in\mathrm{Hom}_B(D(B), B)$ and suppose the degree of $\phi=i\leqq 0$ for $\phi\in\mathrm{Hom}_A(D(A), A)$. In the case $i=0, 0\neq\phi(D(A)_0)\subset A_0$ and A_0 is considered to be a division algebra. Hence $\phi(D(A)_n)\supset\phi(D(A)_0A_n)=A_0A_n=A_n$ and ϕ must be a monomorphism. Then similarly as in $n=1$ this contradicts to that A is not selfinjective. Next assume $i\leqq -1$. Then $0=\phi(D(A)_0)=\phi(D(A_n))$. Hence ϕ is considered to be a homomorphism of $D(A_{n-1}\oplus A_{n-2}\oplus\cdots\oplus A_0)$ to A and $A_{n-1}\oplus A_{n-2}\oplus\cdots\oplus A_0$ can be cosidered as a grading of A/A_n. Let $\rho: D(A/A_n)\to A/A_n$ be the composition of ϕ and the canonical homomorphism from A to A/A_n. Then we know that the degree of $\rho\leqq -1$ but this contradicts to the assumption

of induction.

Let us denote the gradings of $\mathrm{Hom}_A(A, A)$, $\mathrm{Hom}_A(D(A), D(A))$, $\mathrm{Hom}_A(A, D(A))$ and $\mathrm{Hom}_A(D(A), A)$ by

$$\mathrm{Hom}_A(A, A)=\oplus_{i=0}^{n} E_i^{(1,1)}, \qquad \mathrm{Hom}_A(D(A), D(A))=\oplus_{i=0}^{n} E_i^{(2,2)},$$

$$\mathrm{Hom}_A(A, D(A))=\oplus_{i=0}^{n} E_i^{(2,1)} \text{ and } \mathrm{Hom}_A(D(A), A)=\oplus_{i=0}^{n} E_i^{(1,2)}.$$

Now we can introduce a positive $\mathbf{Z}$-grading of R by

$$R_{2k}=\begin{pmatrix} E_k^{(1,1)} & 0 \\ 0 & E_k^{(2,2)} \end{pmatrix}, \qquad R_{2k+1}=\begin{pmatrix} 0 & E_{k+1}^{(1,2)} \\ E_k^{(2,1)} & 0 \end{pmatrix}.$$

Because

$$R_{2k+1}R_{2j+1}=\begin{pmatrix} E_{k+1}^{(1,2)}E_j^{(2,1)} & 0 \\ 0 & E_k^{(2,1)}E_{j+1}^{(1,2)} \end{pmatrix}\subset\begin{pmatrix} E_{k+j+1}^{(1,1)} & 0 \\ 0 & E_{k+j+1}^{(2,2)} \end{pmatrix}=R_{2(k+j+1)},$$

$$R_{2k}R_{2j+1}=\begin{pmatrix} 0 & E_k^{(1,1)}E_{j+1}^{(1,2)} \\ E_k^{(2,2)}E_j^{(2,1)} & 0 \end{pmatrix}\subset\begin{pmatrix} 0 & E_{k+j+1}^{(1,2)} \\ E_{k+j}^{(2,1)} & 0 \end{pmatrix}=R_{2(k+j)+1}$$

and

$$R_{2k+1}R_{2j}=\begin{pmatrix} 0 & E_{k+1}^{(1,2)}E_j^{(2,2)} \\ E_k^{(2,1)}E_j^{(1,1)} & 0 \end{pmatrix}\subset\begin{pmatrix} 0 & E_{k+j+1}^{(1,2)} \\ E_{k+j}^{(2,1)} & 0 \end{pmatrix}=R_{2(k+j)+1}.$$

Since a commutative algebra is a direct sum of local algebras we have immediately

COROLLARY 1.2. *Let A be a commutative algebra. Then* $\mathrm{End}_A(A\oplus D(A))$ *is positively $\mathbf{Z}$-graded if and only if so is A.*

THEOREM 1.3. *Let A be a positively $\mathbf{Z}$-graded local algebra. If* $\mathrm{Ext}_A^i(D(A), A)=0$ *for all $i\geqq 1$, then A is selfinjective.*

PROOF. Suppose that A_A is not selfinjective and $\mathrm{Ext}_A^i(D(A), A)=0$ for all $i\geqq 1$. Let

$$0\longrightarrow A\oplus D(A)\longrightarrow E_0\longrightarrow E_1\longrightarrow\cdots\longrightarrow E_n\longrightarrow\cdots$$

be a minimal injective resolution of $A\oplus D(A)$ as a right A-module. Denote $\mathrm{End}_A(A\oplus D(A))$ by R. Since $E_i\in \mathrm{Add}\text{-}D(A)$, $D(A)$ is a direct summand of $A\oplus D(A)$ and since $\mathrm{Ext}_A^i(D(A), A)=0$ for all $i\geqq 1$, we have the following injective resolution of R_R:

$$0\longrightarrow R\longrightarrow H_0\longrightarrow H_1\longrightarrow\cdots\longrightarrow H_n\longrightarrow\cdots,$$

where $H_i\cong\mathrm{Hom}_A({}_R(A\oplus D(A))_A, E_i)$ and H_i are projective and injective right R-modules.

On the other hand, by Theorem 1.1 R is positively $\mathbf{Z}$-graded. Hence by Wilson's theorem R is selfinjective. However this implies that A is selfinjective and a contradiction.

PROPOSITION 1.4. *Let A be a positively $\mathbf{Z}$-graded local algebra and R the endomorphism ring of right A-module $A \oplus D(A)$. Then Nakayama conjecture is true for R.*

§2. Local Commutative Graded Algebras

Throughout this section K is assumed to be an algebraically closes field of characteristic zero. The following Lemma 2.1 and Proposition 2.2 are well known, cf. [1] and [4, V. 3.9.5], but for the sake of reader's convenience, we shall write elementary proofs.

LEMMA 2.1. *A commutative K-algebra A is local if and only if A is a homomorphic image of $K[x_1, x_2, \cdots, x_m]/I^n$, where I is the ideal of the polynomial ring $K[x_1, x_2, \cdots, x_m]$ of variables $x_1, x_2, \cdots, x_m$, which is generated by $x_1, x_2, \cdots, x_m$.*

PROOF. Let J be the radical of a local commutative algebra A and $J^n=0$. Then there are ring-homomorphisms $\alpha\colon K[X_1, X_2, \cdots, X_m] \to A$ and $\beta\colon A \to A/J \cong K$. Put $\beta\alpha(X_i)=a_i$. Then $\beta\alpha(X_i-a_i)=0$ and hence $\alpha(X_i-a_i)\in J$. Therefore $\alpha((X_i-a_i)^n)=(\alpha(X_i-a_i))^n=0$ and hence $(X_i-a_i)^n \in \operatorname{Ker}\alpha$. Now we can take $x_i=X_i-a_i$.

For $f(x, y)\in K[x, y]$ we shall denote by $f_t(x, y)$ the homogeneous term of $f(x, y)$ of degree t.

PROPOSITION 2.2. *Let $f(x, y)$ be a polynomial in $K[x, y]$ such that $f(x, y)=\sum_{t\geq 2} f_t(x, y)$ with the non-zero homogeneous term $f_2(x, y)=ax^2+bxy+cy^2$ of degree 2, $I=(x, y)$ and $A=K[x, y]/(I^n, f(x, y))$, $n\geqq 3$. Then A is isomorphic to a local algebra $K[X, Y]/(L^n, g(X, Y))$ such that $L=(X, Y)$ and $g(X, Y)=XY$ or X^2-Y^p, $p>2$.*

PROOF. Assume $a\neq 0$. Then $ax^2+bxy+cy^2=a(x-\alpha y)(x-\beta y)$ for some $\alpha, \beta \in K$.

Case (1): $\alpha\neq\beta$. As we can consider $x-\alpha y$ and $x-\beta y$ as new parametess of $K[x, y]$ we can take $f_2(x, y)=xy$. On the other hand, in the case (2): $\alpha=\beta$, by replacing $x-\alpha y$ with x we can take $f_2(x, y)=x^2$. Further it is easily seen that the above context for $f_2(x, y)$ are valid even if $a=0$.

At first we shall proceed the proof for the Case (1) by induction on n: we can replace xy with $f(x, y)-xy \bmod I^n$ and after repetitions of such rearrangements we obtain an expression of $f(x, y)$ which excludes terms $x^i y^j$, $i, j \geqq 1$ and $ij>1$. So if $n=4$ we may assume that $f(x, y)=xy+ax^3+by^3$. Put $X=x+by^2$ and $Y=y+ax^2$. Then $XY=xy+ax^3+by^3 \bmod I^4$. Since X and $Y \in \mathrm{rad} A \setminus \mathrm{rad}^2 A$, we can take X and Y as new parameters and we have $A \cong K[X, Y]/((X, Y)^4, XY)$.

Assume $n>4$. Applying the assumption of induction to $K[x, y]/(I^{n-1}, f(x, y))$ we can take $f(x, y) \equiv xy+ax^{n-1}+by^{n-1} \bmod I^n$. Similarly as in the case $n=4$, putting $X=x+by^{n-2}$ and $Y=y+ax^{n-2}$ we can take X an Y as new parameters and we conclude $A \cong K[X, Y]/(L^n, XY)$.

Now we shall begin the proof of the Case (2). First we can replace x^2 with $f(x, y)-x^2 \bmod I^n$, which is a sum of homogeneous terms of degrees >2. And by repetitions of such rearrangements we may assume that terms $x^i y^j$, $i>1, j>0$ do not appear in $f(x, y)$. Hence if $n=4$, $f(x, y) \equiv x^2+ay^3+bxy^2 \bmod I^4$. Then $f(x, y) \equiv (x+(1/2)by^2)^2+ay^3 \bmod I^4$. So replacing parameters x and y with $X=x+(1/2)by^2$ and $Y=-a^{1/3}y$ respectively, we have $A \cong K[X, Y]/((X, Y)^4, X^2-Y^3)$.

Assume $n>4$. Applying the assumption of induction to $K[x, y]/(I^{n-1}, f(x, y))$ we can take $f(x, y)=x^2-y^p+ay^{n-1}+bxy^{n-2}$, $3 \leqq p<n$. Then $f(x, y) \equiv (x+(1/2)by^{n-2})^2-(y-(1/p)ay^{n-p})^p \bmod I^n$ and we can replace parameters x and y with $X=x+(1/2)by^{n-2}$ and $Y=y-(1/p)ay^{n-p}$ respectively. Therefore $A \cong K[X, Y]/(L^n, X^2-Y^p)$. This completes the proof.

It should be noted that $K[x, y]/((x, y)^n, xy)$, $n \geqq 3$, is biserial in the sense of Fuller [3]. On the other hand, $K[x, y]/((x, y)^4, x^2-y^3)$ has a unique maximal serial ideal, i.e., a serial ideal which contains every non-simple serial ideal.

PROPOSITION 2.3. *Let A be a local commutative algebra as in Proposition 2.2. Then A is positively* $\mathbf{Z}$*-graded.*

PROOF. Denote by $\bar{u}$ the residue class of $K[x, y]/((x, y)^n, xy)$ (resp. $K[x, y]/((x, y)^n, x^2-y^p)$ which contains $u \in K[x, y]$. It is easily seen that $K[x, y]/((x, y)^n, xy)=\oplus_{i=0}^{n-1} A_i$, where $A_0=\overline{K}$ and $A_i=\overline{Kx^i+Ky^i}$, $i>0$, gives a positive $\mathbf{Z}$-grading. On the other hand, according to $p (p<n)$ is odd or even we have the following positive $\mathbf{Z}$-gradings of $K[x, y]/((x, y)^n, x^2-y^p))$ respectively:

$$K[x, y]/((x, y)^n, x^2-y^p))=B_0 \oplus \oplus_{i=0}^{n-2} B_{p+2i} \oplus \oplus_{i=0}^{n-1} B_{2j},$$

where $B_0=\overline{K}$, $B_{p+2i}=\overline{Kxy^i}$ and $B_{2j}=\overline{Ky^j}$, and

$$K[x, y]/((x, y)^n, x^2-y^p))=B_0\oplus\bigoplus_{i=0}^{q-1}B_i\oplus\bigoplus_{j=0}^{n-q-1}B_{q+j}\oplus\bigoplus_{k=n-q}^{n-2}B_k,$$

where $p=2q$, $B_0=\overline{K}$, $B_i=\overline{Ky^i}$, $B_{q+j}=\overline{Kxy^j+Ky^{q+j}}$ and $B_k=\overline{Ky^k}$.

If $p\geqq n$, $K[x, y]/((x, y)^n, x^2)=\bigoplus_{i=0}^{n-1}C_i$ where $C_0=\overline{K}$ and $C_i=\overline{Kxy^{i-1}+Ky^i}$, $i>0$, gives a positively $\mathbf{Z}$-grading.

COROLLARY 2.4. *A homomorphic image of $K[x, y]/(x, y)^4$ is positively $\mathbf{Z}$-graded.*

PROOF. $K[x, y]/(x, y)^4=\overline{K}\oplus\overline{(Kx+Ky)}\oplus\overline{(Kx+Ky)}^2\oplus\overline{(Kx+Ky)}^3$ is a positive $\mathbf{Z}$-grading of $K[x, y[/(x, y)^4$. If $g(x, y)=\sum_{i=1}^3 g_i(x, y)$ with $g_1(x, y)\neq 0$, then $K[x, y]/((x, y)^4, g(x, y))$ is uniserial and clearly its homomorphic image is positively $\mathbf{Z}$-graded. Therefore by Proposition 2.3 it is enough to consider homomorphic images of $K[x, y]/((x, y)^4, f(x, y))$, where $f_0(x, y)=f_1(x, y)=0$ and $f_2(x, y)+f_3(x, y)=xy$, x^2-y^3 or x^2. However if $b\neq 0$ in the below, the ideal of $K[x, y]/((x, y)^4, x^2-y^3)$ (resp. $K[x, y]/((x, y)^4, x^2))$ generated by $\overline{(axy+by^2+cxy^2+dy^3)}$ contains $\overline{(x, y)}^3=\mathrm{rad}^3(K[x, y]/((x, y)^4, x^2-y^3))$ (resp. $\mathrm{rad}^3(K[x, y]/((x, y)^4, x^2)))$. Hence $K[x, y]/((x, y)^4, x^2-y^3, axy+by^2+cxy^2+dy^3)$ (resp. $K[x, y]/((x, y)^4, x^2, axy+by^2+cxy^2+dy^3))$ with $b\neq 0$ has a cube zero radical and consequently is positively $\mathbf{Z}$-graded. Similarly, if $ab\neq 0$, the ideal generated by $\overline{ax^2+by^2+cx^3+dy^3}$ contains $\overline{(x, y)}^3=\mathrm{rad}^3(K[x, y]/(x, y)^4, xy))$. Hence $K[x, y]/((x, y)^4, xy, ax^2+by^2+cx^3+dy^3)$, with $ab\neq 0$, has a cube zero radical and consequently positively $\mathbf{Z}$-graded. Further positive $\mathbf{Z}$-gradings of $K[x, y]/((x, y)^4, xy, ax^i+by^j)$, $3\geqq i\geqq 2, 3\geqq j\geqq 2$, are induced by one of $K[x, y]/((x, y)^4, xy)$, if $ab=0$. Also a positive $\mathbf{Z}$-grading of $K[x, y]/((x, y)^4, xy, ax^3+by^3)$, $ab\neq 0$, is induced by one of $K[x, y]/((x, y)^4, xy)$. Since both $K[x, y]/((x, y)^4, xy, ax^2+cx^3+dy^3)$ and $K[x, y]/((x, y)^4, xy, by^2+cx^3+dy^3)$ are isomorphic to $K[x, y]/((x, y)^4, x^2-y^3, a'xy)$, $a'\in K$, we return to check the positive $\mathbf{Z}$-gradability of $K[x, y]/((x, y)^4, x^2-y^3, axy^i-by^3)$, $2\geqq i\geqq 1$. But in the case $i=1$ and $ab\neq 0$, it is isomorphic to $K[x, y]/((x, y)^4, b'xy, x^2-y^3)$ with $b'(\neq 0)\in K$ because $axy-by^3=(ax-by^2)y$ and we can take $ax-by^2$ and y as new parameters. So the grading is induced by one of $K[x, y]/((x, y)^4, x^2-y^3)$. Further in the case $i=2$ and $ab=0$, the grading is induced by $K[x, y]/((x, y)^4, x^2-y^3)$. For $K[x, y]/((x, y)^4, x^2-y^3, axy^2-by^3)$ with $ab\neq 0$, by taking $X=x-(a/2b)y^2$ and $Y=y$ as new parameters we have $K[x, y]/((x, y)^4, x^2-y^3, axy^2-by^3)\cong K[X, Y]/((X, Y)^4, X^2, aXY^2-bY^3)$ and so the grading of A is induced by $K[X, Y]/((X, Y)^4, X^2)=\overline{K}\oplus\overline{KX+KY}\oplus\overline{KXY+KY^2}\oplus\overline{KXY^2+KY^3}$. For homomorphic images of $K[x, y]/((x, y)^4, x^2)$ it remains to check the positively $\mathbf{Z}$-gradability of $K[x, y]/((x, y)^4, axy-by^3)$ with $b\neq 0$, but it is isomorphic to a positively **Z**-graded algebra $K[x, y]/((x, y)^4, x^2, axy)$. Now by the analogous discussion we know that the grading of any homomorphic image of all local algebra considered above is induced by one of $K[x, y]/((x, y)^4, xy)$, $K[x, y]/((x, y)^4, x^2-y^3)$ or $K[x, y]/((x, y)^4, x^2)$. This completes the proof.

By Theorem 1.3 we have immediately

THEOREM 2.5. *The conjecture* (i) *is true for homomorphic images of* $K[x, y]$ *with quartic zero radicals.*

Similarly we know that the conjecture (i) is true for local algebras $K[x, y]/((x, y)^n, f(x, y))$, where $n\geqq 4$ and $f(x, y)=ax^2+bxy+cy^2+dx^3+\cdots$, provided at least one of a, b, c is nonzero. It seems to be of interest that those local algebras correspond to Arnol'd's normal forms A_t, $t<n$, of functions in the neighborhood of a simple critical point. We are indebted to Drs. K. Watanabe and M. Tomari for drawing our attention to these facts. (cf. [1]).

§3. Example of Local Commutative Algebra Which Is Not Gradable

As our proof in §2 is effective for positively **Z**-graded local algebras it is important to assure the existence of a local commutative algebra which is not positively **Z**-graded. The following Proposition provides the example.

PROPOSITION 3.1. *Let* $A=K[x_1, x_2, x_3]/((x_1, x_2, x_3)^4, x_1x_2-x_3^3, x_2x_3-x_1^3, x_3x_1-x_2^3)$. *Then* A *is not positively* **Z**-*graded.*

PROOF. Suppose that $A=A_0\oplus A_1\oplus A_2\oplus\cdots\oplus A_q$ is a positive **Z**-grading such that $\mathrm{rad}\,A=A_1\oplus A_2\oplus\cdots\oplus A_q$. Let us denote by $\overline{f(x, y, z)}$ an element of A which is the residue class containing $f(x, y, z)\in K[x, y, z]$. Then $A=K\bar{1}=K\bar{x}_1+K\bar{x}_2+K\bar{x}_3+K\bar{x}_1^2+K\bar{x}_2^2+K\bar{x}_3^2+K\bar{x}_1^3+K\bar{x}_2^3+K\bar{x}_3^3$, $\mathrm{rad}A=K\bar{x}_1+K\bar{x}_2+K\bar{x}_3+K\bar{x}_1^2+K\bar{x}_2^2+K\bar{x}_3^2+K\bar{x}_1^3+K\bar{x}_2^3+K\bar{x}_3^3$, $\mathrm{rad}^2A=K\bar{x}_1^2+K\bar{x}_2^2+K\bar{x}_3^2+K\bar{x}_1^3+K\bar{x}_2^3+K\bar{x}_3^3$ and $\mathrm{rad}^3A=\mathrm{soc}A=K\bar{x}_1^3\oplus K\bar{x}_2^3\oplus K\bar{x}_3^3$. Since $\dim_K(\mathrm{rad}A\setminus\mathrm{rad}^2A)=3$, there exists $\alpha_i\in\mathrm{rad}A\setminus\mathrm{rad}^2A$, $i=1, 2, 3$ and positive integers n_1, n_2, n_3 such that $\alpha_1\in A_{n_1}$, $\alpha_2\in A_{n_2}$, $\alpha_3\in A_{n_3}$ with $n_1\leqq n_2\leqq n_3$ and α_i, $i=1, 2, 3$, are K-linearly independent. For the simplicity we shall abbreviate from now $\bar{x}_i$ to x_i, $i=1, 2, 3$.

Then we have

$$\begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{pmatrix}=(a_{ij})\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}+(b_{ij})\begin{pmatrix}x_1^2\\ x_2^2\\ x_3^2\end{pmatrix}+(c_{ij})\begin{pmatrix}x_1^3\\ x_2^3\\ x_3^3\end{pmatrix},\ a_{ij}, b_{ij}, c_{ij}\in K,$$

$i, j=1, 2, 3$ and $\Delta=\det(a_{ij})\neq 0$.

At first we shall notice that $n_1=n_2=n_3$ is impossible. Let $n=n_1$ and $\begin{pmatrix}\alpha_1'\\ \alpha_2'\\ \alpha_3'\end{pmatrix}=(a_{ij})^{-1}\begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{pmatrix}$. Then $\alpha_i'\in A_n$, $i=1, 2, 3$, and $\alpha_i'=x_i+\alpha_i''$ with $\alpha_i''\in\mathrm{rad}^2 A$, $i=1, 2, 3$. Therefore $0\neq\alpha_1'\alpha_2'=x_1x_2+cx_1^3+dx_2^3$ with $c, d\in K$. However $\alpha_1'\alpha_2'\in A_{2n}$ but $x_1x_2+cx_1^3+dx_2^3=x_3^3+cx_1^3+dx_2^3\in A_{3n}$ because $\alpha_i'^3=x_i^3$, $i=1, 2, 3$, this is a contradiction.

Now assume that $n_1<n_2<n_3$. It is clear that $0\neq\alpha_i^3\in A_{n_i}^3\subset\mathrm{soc}\,A$, $i=1, 2, 3$. Since $\dim_K\ \mathrm{soc}\,A=3$ and $A_{n_i}^3\subset A_{3n_i}$, $i=1, 2, 3$, $A_{n_1}^3\oplus A_{n_2}^3\oplus A_{n_3}^3=\mathrm{soc}\,A$. By the assumption it holds that $3n_1<n_1+2n_3<n_2+2n_3<3n_3$ and $3n_1<3n_2<n_2+2n_3<3n_3$.

Further we make an assumption (a): $n_1+2n_3\neq 3n_2$. Since $A_{n_1}A_{n_3}^2\subset A_{n_1+2n_3}\cap\mathrm{soc}\,A$ and $A_{n_2}A_{n_3}^2\subset A_{n_2+2n_3}\cap\mathrm{soc}\,A$, $\alpha_1\alpha_3^2=\sum_{i=1}^3 a_{1i}a_{3i}^2x_i^3\in A_{n_1}A_{n_2}^2=0$ and $\alpha_2\alpha_3^2=\sum_{i=1}^3 a_{2i}a_{3i}^2x_i^3\in A_{n_2}A_{n_3}^2=0$. It follows that $a_{1i}a_{3i}^2=a_{2i}a_{3i}^2=0$, $i=1, 2, 3$. Further from $3n_1<2n_1+n_2<3n_2$ we similarly obtain $\alpha_1^2\alpha_2\in A_{n_1}^2A_{n_2}=0$ and consequently $a_{1i}^2a_{2i}=0$, $i=1, 2, 3$. Therefore $a_{1i}\neq 0$ implies $a_{2i}=a_{3i}=0$. Also $a_{2i}\neq 0$ implies $a_{3i}=0$. Then (a_{ij}) must be a monomial matrix because $\Delta=\det(a_{ij})\neq 0$. So we have $0\neq\alpha_1\alpha_2=c\alpha_3^3\in\mathrm{soc}\,A$ for some $c\in K$. But this implies $0\neq A_{n_1}A_{n_2}\cap A_{n_3}^3\subset A_{n_1+n_2}\cap A_{3n_3}$. But $n_1+n_2=3n_3$ contradicts to $n_1<n_2<n_3$.

Now we make another assumption (b): $n_1+2n_3=3n_2$. In this case it holds that $3n_1<2n_1+n_2<2n_1+n_3<n_2+2n_3<3n_3$, $2n_1+n_2<3n_2<n_2+2n_3<3n_3$ and $2n_1+n_3\neq 3n_2$ because $n_1+2n_3=3n_2$ and $n_1<n_2<n_3$. Then $A_{n_1}^2A_{n_2}\subset A_{2n_1+n_2}\cup\mathrm{soc}\,A$, $A_{n_1}A_{n_3}\subset A_{2n_1+n_3}\cap\mathrm{soc}\,A$, and $A_{n_2}A_{n_3}^2\subset A_{n_2+2n_3}\cap\mathrm{soc}\,A$ and they induce $\alpha_1^2\alpha_2=\alpha_1^2\alpha_2=\alpha_1^2\alpha_3=\alpha_2\alpha_3^2=0$. Hence we have $a_{1i}^2a_{2i}=a_{1i}^2a_{3i}=a_{2i}a_{3i}^2=0$, $i=1, 2, 3$, and we arrive at the same contradiction as in the case (a).

Assume now that $n_1<n_2=n_3$. And at first assume further α_2^3 and α_3^3 are K-linearly independent. Then it holds that $3n_1<2n_1+n_1+2n_2<3n_2$ and $n_{n_1}^3\oplus A_{n_1}^2A_{n_2}\oplus A_{n_1}A_{n_3}^2\oplus A_{n_2}^3\subset\mathrm{soc}\,A$. So it follows that $A_{n_1}^2A_{n_2}=A_{n_1}A_{n_3}^2=0$ and hence $a_{1i}^2a_{2i}=a_{1i}a_{3i}^2=0$, $i=1, 2, 3$. If $a_{11}\neq 0$, then $a_{21}=a_{31}=0$.

Further suppose one of a_{12} or a_{13} is nonzero. Then $\Delta=0$. Therefore $a_{21}=a_{31}=a_{12}=a_1{}^8=0$ and $\begin{vmatrix}a_{22} & a_{23}\\ a_{32} & a_{33}\end{vmatrix}\neq 0$.

So we have

$$\alpha_1=a_{11}x_1+b_{11}x_1^2+b_{12}x_2^2+b_{13}x_3^2+\beta_1',$$

$$\alpha_2=a_{22}x_2+a_{23}x_3+b_{21}x_1^2+b_{22}x_2^2+b_{23}x_3^2+\beta_2'$$

and

$$\alpha_3=a_{32}x_2+a_{33}x_3+b_{31}x_1^2+b_{32}x_2^2+b_{33}x_3^2+\beta_3',$$

where $\beta_i'\in\mathrm{rad}^3A$, $i=1, 2, 3$.

Then we have $x_2=d_{22}\alpha_2+d_{23}\alpha_3+x_2'$ and $x_3=d_{32}\alpha_2+d_{33}\alpha_3+x_3'$ with $x_2', x_3'\in \mathrm{rad}^2A$ and $(d_{ij})=\begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33}\end{pmatrix}^{-1}$, $i, j=2, 3$ and hence $0\neq A_{n_1}^3\cap A_{n_2}^2$ and $3n_1=2n_2$ because $a_{11}^{-3}\alpha_1^3=x_2x_3\in A_{n_1}^3$ and $(d_{22}\alpha_2+d_{23}\alpha_3)(d_{32}\alpha_2+d_{33}\alpha_3)\in A_{n_2}^2$ and $x_2'(d_{22}\alpha_2+d_{23}\alpha_3)+(d_{32}\alpha_2+d_{33}\alpha_3)x_3'\in Kx_2^3\oplus Kx_3^3=A_{n_2}^3$.

However either $\alpha_1\alpha_2=a_{11}b_{21}x_1^3+(b_{12}a_{22}a_{11}a_{23})x_2^3+(a_{11}a_{22}+b_{13}a_{23})x_3^3$ or $\alpha_1\alpha_3=a_{11}b_{31}x_1^3+(b_{12}a_{32}+a_{11}a_{33})x_2^3+(a_{11}a_{32}+b_{13}a_{33})x_3^3$ is nonzero, for otherwise $a_{11}=0$ and it contradicts to our assumption. As they belong to both $A_{n_1}A_{n_2}$ and soc $A=A_{n_1}^3\oplus A_{n_2}^3$, we have $n_1+n_2=3n_1$, i.e. $n_2=2n_1$. But this is also impossible because $3n_1=2n_2$. As in the case where $a_{12}\neq 0$ or $a_{13}\neq 0$ we arrive at a similar contradiction. We can proceed our proof to the next case where α_2^3 and α_3^3 are K-linearly dependent. Then since $a_{21}^3/a_{31}^3=a_{22}^3/a_{32}^3=a_{23}^3/a_{33}^3$, we have $a_{31}=\omega_1a_{21}$, $a_{32}=\omega_2a_{22}$ and $a_{33}=\omega_3a_{23}$, where ω_i, $i=1, 2, 3$, are cube roots of unit. (It is to be noted that this case does not occur if the characteristic of K is 3).

Now the inequality $3n_1<2n_1+n_2<n_1+2n_2<3n_2$ induces either $A_{n_1}^2A_{n_2}=0$ or $A_{n_1}A_{n_2}^2=0$. Then according to them we have either $a_{11}^2a_{21}=a_{12}^2a_{22}=a_{13}^2a_{23}=0$ and $\mathrm{soc}A=A_{n_1}A_{n_2}^2\oplus A_{n_1}^3\oplus A_{n_2}^3$, or $a_{11}a_{21}^2=a_{12}a_{22}^2=a_{13}a_{23}^2=0$ and $\mathrm{soc}A=A_{n_1}^2A_{n_2}\oplus A_{n_1}^3\oplus A_{n_2}^3$.

Assume $a_{11}\neq 0$. Then $a_{21}=0$. And both a_{12} and $a_{13}=0$; otherwise, $a_{12}\neq 0$ or $a_{13}\neq 0$ implies $\Delta=0$. Therefore we have $\alpha_1=a_{11}x_1+\gamma_1'$, $\alpha_2=a_{22}x_2+a_{23}x_3+\gamma_3'$ and $\alpha_3=\omega_2a_{22}x_2+\omega_3a_{23}x_3+\gamma_2'$, where $\gamma_i'\in\mathrm{rad}^2A$, $i=1, 2, 3$. Then similarly as in the preceding case, from the assumption $a_{11}\neq 0$ and $\begin{vmatrix} a_{22} & a_{23} \\ \omega_2a_{22} & \omega_3a_{23}\end{vmatrix}\neq 0$ we have $2n_2=3n_1$, and either $\alpha_1\alpha_2\neq 0$ or $\alpha_1\alpha_3\neq 0$. The later fact induces that $A_{n_1}A_{n_2}\cap(A_{n_1}A_{n_2}^2\oplus A_{n_1}^3\oplus A_{n_2}^3)\neq 0$ or $A_{n_1}A_{n_2}\cap(A_{n_1}^2A_{n_2}\oplus A_{n_1}^3\oplus A_{n_2}^3)\neq 0$, and it follows that $n_1+n_2=3n_1$, but this contradicts to $2n_2=3n_1$. In the case where $a_{12}\neq 0$ or $a_{13}\neq 0$, we also arrive at a similar contradiction.

Now it remains to prove thae $n_1=n_2<n_3$ does not occur. In this case $A_{n_1}^3\oplus A_{n_3}^3=\mathrm{soc}\,A$ and α_1^3 and α_2^3 are K-linearly independent, and the inequality $3n_1<2n_1+n_3<n_1+2n_3<3n_3$ implies $A_{n_1}^2A_{n_3}=0$ and $A_{n_1}A_{n_3}^2=0$. Thus we have $a_{11}^2a_{31}=a_{12}^2a_{32}=a_{13}^2a_{33}=0$ and $a_{11}a_{31}^2=a_{12}a_{32}^2=a_{13}a_{33}^2=0$. Then similarly as in the case where $n_1<n_2=n_3$ and α_2^3 and α_3^3 are K-linearly independent, we arrive at a similar contradiction.

It is to be noted that for this example our conjecture (i) is true.

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