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NON-TRIVIALITY OF CERTAIN FINITELY-PRESENTED GROUPS

By

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In this paper we shall prove the following theorem.

THEOREM. *Let p, q, r, s be integers greater than* 2. *Then, the group*

$$\langle a, b \mid a^{p}=b^{q}=(ab)^{r}=(a^{-1}b)^{s}=1\rangle$$

is non-abelian and hence non-trivial.

REMARK 1. The groups of this type were studied in [1], [2], [3]. But the above general theorem was not established.

REMARK 2. If one of p, q, r, s is 2 in the above group presentation, then there are many cases when the group becomes trivial.

PROOF OF THE THEOREM. We define matrices $A, B\in \mathrm{SL}(3, \boldsymbol{C})$ such that A and B do not commute and that $A^{p}=B^{q}=(AB)^{r}=(A^{-1}B)^{s}=E$.

Let ω_{p} be a primitive p-th root of 1, ω_{q} be a primitive q-th root of 1, ω_{r} be a primitive r-th root of 1, and ω_{s} be a primitive s-th root of 1. Since $p, q, r, s>2$, we have $\omega_{p}\neq\omega_{p}^{-1}$, $\omega_{q}\neq\omega_{q}^{-1}$, $\omega_{r}\neq\omega_{r}^{-1}$, $\omega_{s}\neq\omega_{s}^{-1}$.

Now let

$$A=\begin{pmatrix}1 & 0 & 0\\ 0 & \omega_{p} & 0\\ 0 & 0 & \omega_{p}^{-1}\end{pmatrix},\quad B=\begin{pmatrix}b_{11} & b_{12} & b_{13}\\ b_{21} & b_{22} & b_{23}\\ b_{31} & b_{32} & b_{33}\end{pmatrix},$$

where b_{ij}'s will be determined later. Obviously, $A^{p}=E$.

In order that $B^{q}=E$, it is sufficient that the characteristic polynomial $\chi_{B}(t)$ of B is $(t-1)(t-\omega_{q})(t-\omega_{q}^{-1})$, for it is a factor of $t^{q}-1$ so $t^{q}-1=f(t)\chi_{B}(t)$, for some polynomial $f(t)$ and hence $B^{q}-E=f(B)\chi_{B}(B)=0$ (the zero matrix).

From

$$\begin{aligned}\chi_{B}(t)&=t^{3}-(b_{11}+b_{22}+b_{33})t^{2}\\ &\quad+\left\{\begin{vmatrix}b_{22} & b_{23}\\ b_{32} & b_{33}\end{vmatrix}+\begin{vmatrix}b_{11} & b_{12}\\ b_{21} & b_{22}\end{vmatrix}+\begin{vmatrix}b_{11} & b_{13}\\ b_{31} & b_{33}\end{vmatrix}\right\}t-\begin{vmatrix}b_{11} & b_{12} & b_{13}\\ b_{21} & b_{22} & b_{23}\\ b_{31} & b_{32} & b_{33}\end{vmatrix}\\ &=t^{3}-(1+\omega_{q}+\omega_{q}^{-1})t^{2}+(1+\omega_{q}+\omega_{q}^{-1})t-1,\end{aligned}$$

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we have the equations:

(1) $b_{11}+b_{22}+b_{33}=1+\omega_q+\omega_q^{-1}$,

(2) $\begin{vmatrix} b_{22} & b_{23} \\ b_{32} & b_{33} \end{vmatrix}+\begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix}+\begin{vmatrix} b_{11} & b_{13} \\ b_{31} & b_{33} \end{vmatrix}=1+\omega_q+\omega_q^{-1}$,

(3) $\begin{vmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{vmatrix}=1$.

Similarly for

$$AB=\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ \omega_p b_{21} & \omega_p b_{22} & \omega_p b_{23} \\ \omega_p^{-1} b_{31} & \omega_p^{-1} b_{32} & \omega_p^{-1} b_{33} \end{pmatrix},$$

we have the equations:

(4) $b_{11}+\omega_p b_{22}+\omega_p^{-1} b_{33}=1+\omega_r+\omega_r^{-1}$,

(5) $\begin{vmatrix} b_{22} & b_{23} \\ b_{32} & b_{33} \end{vmatrix}+\omega_p\begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix}+\omega_p^{-1}\begin{vmatrix} b_{11} & b_{13} \\ b_{31} & b_{33} \end{vmatrix}=1+\omega_r+\omega_r^{-1}$.

And, for

$$A^{-1}B=\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ \omega_p^{-1} b_{21} & \omega_p^{-1} b_{22} & \omega_p^{-1} b_{23} \\ \omega_p b_{31} & \omega_p b_{32} & \omega_p b_{33} \end{pmatrix},$$

we have the equations:

(6) $b_{11}+\omega_p^{-1} b_{22}+\omega_p b_{33}=1+\omega_s+\omega_s^{-1}$,

(7) $\begin{vmatrix} b_{22} & b_{23} \\ b_{32} & b_{33} \end{vmatrix}+\omega_p^{-1}\begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix}+\omega_p\begin{vmatrix} b_{11} & b_{13} \\ b_{31} & b_{33} \end{vmatrix}=1+\omega_s+\omega_s^{-1}$.

We solve the equations (1)~(7). By the linear equations (1), (4), (6), b_{11}, b_{22}, b_{33} are uniquely determined since

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & \omega_p & \omega_p^{-1} \\ 1 & \omega_p^{-1} & \omega_p \end{vmatrix}=(\omega_p+1)(\omega_p-1)^3/\omega_p^2\neq 0.$$

Similarly, by the equations (2), (5), (7),

$$\begin{vmatrix} b_{22} & b_{23} \\ b_{32} & b_{33} \end{vmatrix}, \quad \begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix}, \quad \begin{vmatrix} b_{11} & b_{13} \\ b_{31} & b_{33} \end{vmatrix}$$

are uniquely determined and hence

(8) $b_{32}b_{23}=\alpha_1$, $b_{21}b_{12}=\alpha_2$, $b_{31}b_{12}=\alpha_3$

are uniquely determined.

Now, by (3),

$$b_{11}b_{22}b_{33}+b_{12}b_{23}b_{31}+b_{21}b_{32}b_{13}-b_{11}b_{32}b_{23}-b_{33}b_{21}b_{12}-b_{22}b_{31}b_{13}=1.$$

So,

(9) $b_{12}b_{23}b_{31}+b_{21}b_{32}b_{13}=\beta$

is uniquely determined.

In order to solve (8) and (9), first suppose that $\alpha_1=0$. Then, we put

$$b_{13}=1, \quad b_{31}=\alpha_3, \quad b_{21}=1, \quad b_{12}=\alpha_2, \quad b_{23}=0, \quad b_{32}=\beta.$$

Then, (8) and (9) are satisfied. Similarly for the case $\alpha_2=0$ or $\alpha_3=0$.

Next suppose that $\alpha_1\alpha_2\alpha_3\neq 0$. Then, $b_{12}b_{23}b_{31}\neq 0$ can be determined by the equation

$$(b_{12}b_{23}b_{31})^2-\beta(b_{12}b_{23}b_{31})+\alpha_1\alpha_2\alpha_3=0.$$

Then, we can take $b_{12}\neq 0$, $b_{23}\neq 0$, arbitrarily and if we put

$$b_{32}=\alpha_1/b_{23} \ (\neq 0), \quad b_{21}=\alpha_2/b_{12} \ (\neq 0), \quad b_{13}=\alpha_3/_{31} \ (\neq 0),$$

then (8) and (9) are satisfied.

In any case the equations (1)~(7) have solutions such that at least one of b_{13}, b_{21}, $b_{32}\neq 0$, which guarantees that $AB\neq BA$. Thus the group considered is non-abelian, and the proof is complete.

References

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