

QUASI- G_δ -DIAGONALS AND WEAK σ -SPACES IN GO-SPACES

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1. Definitions and Preliminaries

In [5], topological spaces with various types of diagonals and weak σ -spaces are studied. In this paper these notions are extended and reflected upon GO-spaces.

Recall that a LOTS (=linearly ordered topological space) is a topological space whose topology agrees with the topology induced by some linear ordering $<$. A GO-space (=generalized ordered space) is a subspace of a LOTS. An equivalent way to obtain a GO-space X is to start with a linearly ordered set $(Y, <)$ and equip Y with any topology that contains the order topology and has a base consisting of open sets each of which are order-convex (where A is order-convex if given $x, y \in A$ with $x < y$, then $\{z : x < z < y\} \subset A$). It is then convenient to write

$$X = \text{GO}_Y(R, E, I, L),$$

where

- (i) $I = \{x \in X : \{x\} \text{ is open}\}$,
- (ii) $R = \{x \in X - I : [x, \rightarrow) \text{ is open}\}$,
- (iii) $L = \{x \in X - I : (\leftarrow, x] \text{ is open}\}$, and
- (iv) $E = X - (R \cup L \cup I)$

(where $[x, \rightarrow) = \{z \in X : x \leq z\}$ and $(\leftarrow, x]$ is similarly defined). The subscript Y will usually be omitted since it will be clear what it is.

Let Z^+ denote the set of natural numbers and let \mathbf{R} denote the real line with the usual order topology.

DEFINITION 1. Let $\mathcal{G} = \{\mathcal{G}_i : i \in Z^+\}$ be a set of collections of open subsets of a topological space X . If, for any distinct $x, y \in X$, there exists $n \in Z^+$ such that

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- (i) $x \in \text{St}(x, \mathcal{G}_n) \subset X - \{y\}$, then \mathcal{G} is a *quasi- G_δ -diagonal*,
- (ii) $x \in \text{St}(x, \mathcal{G}_n) \subset X - \{y\}$ and $\text{ord}(x, \mathcal{G}_n)$ is finite, then \mathcal{G} is a *θ -quasi- G_δ -diagonal*. Furthermore,
- (iii) if, for $x \in U$, where U is open in X , there exists $n \in \mathbb{Z}^+$ such that $x \in \text{St}(x, \mathcal{G}_n) \subset U$, then \mathcal{G} is a *quasi-development* for X and X is said to be *quasi-developable*.

It is clear that a quasi-developable space has a quasi- G_δ -diagonal and, in [1] it is shown that a quasi-developable space has a θ -quasi- G_δ -diagonal.

THEOREM 1 [6]. *If a LOTS X has a quasi- G_δ -diagonal, then X is quasi-developable.*

This theorem does not extend to the more general setting of a GO-space.

EXAMPLE 1. The Sorgenfrey Line $S = GO_{\mathbb{R}}(\mathbb{R}, \phi, \phi, \phi)$ is a hereditarily Lindelöf GO-space that is not quasi-developable. Since S has a weaker metric topology, it has a G_δ -diagonal.

DEFINITION 2. A collection \mathcal{C} of sets is *coherent* if whenever \mathcal{B} is a non-void proper subcollection of \mathcal{C} , then $(\cup \mathcal{B}) \cap (\cup (\mathcal{C} - \mathcal{B})) \neq \emptyset$. A collection \mathcal{C} is a *maximal coherent* subcollection of \mathcal{A} if \mathcal{C} is coherent and not a proper subcollection of another coherent subcollection of \mathcal{A} .

The following theorem is an adaption to GO-spaces of a result in [3]. See also [7, 3, 9]-

THEOREM 2. *If \mathcal{G} is a coherent collection of convex open subsets of a GO-space X , then there exist sequences $\langle x_i : i \in \mathbb{Z}^+ \rangle$ and $\langle y_i : i \in \mathbb{Z}^+ \rangle$ (each possibly finite) in $\cup \mathcal{G}$ such that*

- (i) $x_1 = y_1$,
- (ii) if $x_{k+1}(y_{k+1})$ is defined, then $x_k < x_{k+1}$ ($y_{k+1} < y_k$) and $x_{k+1} \notin \text{St}(x_k, \mathcal{G})$ ($y_{k+1} \notin \text{St}(y_k, \mathcal{G})$),
- (iii) $\text{St}(x_{k+1}, \mathcal{G}) \cap \text{St}(x_k, \mathcal{G}) \neq \emptyset$ ($\text{St}(y_{k+1}, \mathcal{G}) \cap \text{St}(y_k, \mathcal{G}) \neq \emptyset$), and
- (iv) $(\cup \{\text{St}(x_k, \mathcal{G}) : k \in \mathbb{Z}^+\}) \cup (\cup \{\text{St}(y_k, \mathcal{G}) : k \in \mathbb{Z}^+\}) = \cup \mathcal{G}$.

This result can be extended in a useful fashion.

THEOREM 3. *If \mathcal{G} is a coherent collection of convex open subsets of a GO-space X , then there is a collection \mathcal{H} of convex open subsets of X such that*

- (i) \mathcal{A} is countable, and
- (ii) for all $x \in \cup \mathcal{G}$, $x \in \text{St}(x, \mathcal{A}) \subset \text{St}(x, \mathcal{G})$.

PROOF. Let $\langle x_i \rangle$ and $\langle y_i \rangle$ be the sequences guaranteed by Theorem 2. If $\langle x_i \rangle$ is infinite, then choose $A_i, B_{i+1} \in \mathcal{G}$ such that $x_i \in A_i, x_{i+1} \in B_{i+1}$ and $A_i \cap B_{i+1} = \emptyset$. Let $\mathcal{A}^+ = \{A_i : i \in \mathbb{Z}^+\} \cup \{B_{i+1} : i \in \mathbb{Z}^+\}$. If $\langle x_i \rangle$ is finite, say $\langle x_i \rangle = \langle x_1, \dots, x_n \rangle$, then choose A_i, B_{i+1} for $i=1, \dots, n-1$ as above. Let $C_n = (x_n, \rightarrow) \cap (\cup \mathcal{G})$. Then let $\mathcal{A}^+ = \{A_i : i=1, \dots, n-1\} \cup \{B_{i+1} : i=1, \dots, n-1\} \cup \{C_n\}$. Argue using $\langle y_i \rangle$ in a similar fashion to obtain \mathcal{A}^- . Let $\mathcal{A} = \mathcal{A}^+ \cup \mathcal{A}^-$. It is clear that \mathcal{A} has the advertised properties.

2. Quasi- G_δ -diagonals in GO-spaces

THEOREM 4. *If a GO-space X has a quasi- G_δ -diagonal, then X has a quasi- G_δ -diagonal $\mathcal{K} = \{\mathcal{K}_n : n \in \mathbb{Z}^+\}$ such that the elements of each \mathcal{K}_n are order-convex and \mathcal{K}_n is a pairwise disjoint collection.*

PROOF. Let $\mathcal{G} = \{\mathcal{G}_n : n \in \mathbb{Z}^+\}$ be a quasi- G_δ -diagonal for X . No generality is lost if it is assumed that the elements of each \mathcal{G}_n are order-convex. For each $n \in \mathbb{Z}^+$, let $\{\mathcal{G}(n, \alpha) : \alpha \in \lambda_n\}$ be the set of maximal coherent subcollections of \mathcal{G}_n . For each $\alpha \in \lambda_n$, let $\mathcal{H}(n, \alpha) = \{H(n, \alpha, i) : i \in \mathbb{Z}^+\}$ be the collection obtained from $\mathcal{G}(n, \alpha)$ guaranteed by Theorem 3. Let $\mathcal{K}(n, i) = \{H(n, \alpha, i) : \alpha \in \lambda_n\}$. Then $\mathcal{K} = \{\mathcal{K}(n, i) : (n, i) \in \mathbb{Z}^+ \times \mathbb{Z}^+\}$ is the desired quasi- G_δ -diagonal.

COROLLARY 4.1. *A GO-space with a quasi- G_δ -diagonal has a θ -quasi- G_δ -diagonal.*

The following theorem extends a result in [6].

THEOREM 5. *If X is a GO-space with a quasi- G_δ -diagonal, then X is a first-countable space. Furthermore, X is quasi-developable at $E \cup I$.*

PROOF. Let $\{\mathcal{A}_n : n \in \mathbb{Z}^+\}$ be a quasi- G_δ -diagonal guaranteed by Theorem 4 for the GO-space $X = GO(R, E, I, L)$. If $x \in R$, let

$$U(x, n) = \text{St}(x, \mathcal{A}_n) \cap [x, \rightarrow) \quad \text{for } n \in \mathbb{Z}^+.$$

It follows that $\{U(x, n) : n \in \mathbb{Z}^+\}$ is a countable local base at x . A countable local base at $y, y \in L$ is similarly defined.

For each $n, m \in \mathbb{Z}^+$, let

$$\mathcal{G}(n, m) = \{H \cap H' : H \in \mathcal{H}_n, H' \in \mathcal{H}_m\}.$$

For each $x \in E$, $\{\text{St}(x, \mathcal{G}(n, m)) : (n, m) \in Z^+ \times Z^+\}$ is a countable local base at x . If $\mathcal{I} = \{x : x \in I\}$, then $\mathcal{I} \cup \{\mathcal{G}(n, m) : (n, m) \in Z^+ \times Z^+\}$ is a quasi-development for $E \cup I$.

Notice that the Sorgenfrey Line $S = GO(\mathbf{R}, \phi, \phi, \phi)$ is a GO-space with a G_δ -diagonal that is not quasi-developable. Thus, Theorem 5 cannot be extended.

THEOREM 6. *Let X be a GO-space with a quasi- G_δ -diagonal. If X is either connected or compact, then X is homeomorphic to a subspace of \mathbf{R} .*

PROOF. In either case X is a LOTS [6] and, hence, quasi-developable by Theorem 1. If X is connected, then the result follows from [3]. If X is compact, then X is a metrizable space [2]. Then it is a second-countable LOTS. Hence X has at most countably many jumps and no gaps. Extend X by filling each jump with an open unit interval and extending the given order in the natural fashion to obtain Y . It follows that Y is connected metrizable LOTS. Thus Y is homeomorphic to a subspace of \mathbf{R} . Since $X \subseteq Y$, it follows that X is homeomorphic to a subspace of \mathbf{R} .

3. Main theorem and its proof

DEFINITION 3. A space X is said to be a *weak σ -space* if X has a σ -disjoint network $\mathcal{M} = \cup \{\mathcal{M}_n : n \in Z^+\}$ such that each \mathcal{M}_n is a discrete collection of subsets of X in some open set containing $\cup \mathcal{M}_n$ (equivalently \mathcal{M}_n is discrete in $\cup \mathcal{M}_n$).

If X is a quasi-developable space, then X is a weak σ -space [1].

EXAMPLE 2. Let $Y = [0, \omega_1]$ be equipped with the usual order topology and $X = GO_Y(\phi, \phi, [0, \omega_1] - \{\omega_1\}, \{\omega_1\})$. If $\mathcal{M}_1 = \{\{\alpha\} : \alpha < \omega_1\}$ and $\mathcal{M}_2 = \{\{\omega_1\}\}$, then the network $\{\mathcal{M}_1, \mathcal{M}_2\}$ insures that X is a weak σ -space. Notice that X is not first-countable at ω_1 .

This example demonstrates the limits of the next theorem.

Let X^+ denote the order completion (the Dedekind compactification) of a GO-space X [8].

THEOREM 7. *If X is a first-countable GO-space, then X is quasi-developable if and only if X is a weak σ -space.*

PROOF. Let $\mathcal{M} = \cup \{\mathcal{M}_n : n \in Z^+\}$ be a σ -disjoint network for $X = GO(R, E, I, L)$ such that each \mathcal{M}_n is discrete in an open set B_n . For each $x \in X$, let $\{U(x, k) : k \in Z^+\}$ be a local base at x . For each $n \in Z^+$, let

$$R_n = \{x \in R : x \in M \subset [x, \rightarrow) \text{ for some } M \in \mathcal{M}_n\}.$$

It follows that R_n is discrete in B_n and $R = \cup R_n$. If $x \in R_n$, let $g'(x, n)$ be the order-convex component of $B_n - \{y \in R_n : y \neq x\}$ that contains x . Clearly $g'(x, n)$ is open. Let

$$g(x, n, k) = g'(x, n) \cap U(x, k) \cap [x, \rightarrow).$$

For each $n, k \in Z^+$, let

$$\mathcal{R}(n, k) = \{g(x, n, k) : x \in R_n\}.$$

Since R_n is discrete in B_n , it easily follows that $\mathcal{R}(n, k)$ is a pairwise disjoint collection of open sets and that $\mathcal{R} = \{\mathcal{R}(n, k) : (n, k) \in Z^+ \times Z^+\}$ is a quasi-development for R . Similarly define \mathcal{L} to be a quasi-development for L . Clearly $\mathcal{I} = \{\{x\} : x \in I\}$ is a quasi-development for I .

To construct a quasi-development for E , let $\mathcal{M}_n = \{M(n, \alpha) : \alpha \in \lambda(n)\}$ be discrete in the open set B_n . For each $\alpha \in \lambda(n)$, let

$$C(n, \alpha) = \{C(n, \alpha, \gamma) : \gamma \in \lambda(n, \alpha)\}$$

be the collection of order-convex components of

$$B_n - \cup \overline{\{M(n, \beta) : \beta \in \lambda(n), \beta \neq \alpha\}}$$

that intersect $M(n, \alpha)$. For each $\gamma \in \lambda(n, \alpha)$, let

$$M(n, \alpha, \gamma) = M(n, \alpha) \cap C(n, \alpha, \gamma).$$

For each $\gamma \in \lambda(n, \alpha)$,

(i) if $\inf_{x^+} M(n, \alpha, \gamma) = a^+ \notin M(n, \alpha, \gamma) \cap E$, then let

$$l(M(n, \alpha, \gamma), k) = (a^+, \rightarrow) \cap C(n, \alpha, \gamma) \quad \text{for each } k \in Z^+.$$

(ii) if $\inf_{x^+} M(n, \alpha, \gamma) = a \in M(n, \alpha, \gamma) \cap E$, then, by the first-countability of

X , there is an increasing sequence $\{x(n, \alpha, \gamma, k) : k \in Z^+\}$ of elements of $C(n, \alpha, \gamma)$ that converges to a . Let

$$l(M(n, \alpha, \gamma), k) = (x(n, \alpha, \gamma, k), \rightarrow) \cap C(n, \alpha, \gamma) \quad \text{for each } k \in Z^+.$$

(iii) if $\sup_{x^+} M(n, \alpha, \gamma) = e^+ \notin M(n, \alpha, \gamma) \cap E$, then let

$$r(M(n, \alpha, \gamma), k) = (\leftarrow, e^+) \cap C(n, \alpha, \gamma) \quad \text{for each } k \in Z^+.$$

(iv) if $\sup_{x^+} M(n, \alpha, \gamma) = e \in M(n, \alpha, \gamma) \cap E$, then choose a decreasing sequence $\{y(n, \alpha, \gamma, k) : k \in Z^+\}$ of elements of $C(n, \alpha, \gamma)$ that converges to e .

Let

$$r(M(n, \alpha, \gamma), k) = (\leftarrow, y(n, \alpha, \gamma, k)) \cap C(n, \alpha, \gamma) \quad \text{for each } k \in Z^+.$$

Finally, let

$$g(M(n, \alpha, \gamma), k) = l(M(n, \alpha, \gamma), k) \cap r(M(n, \alpha, \gamma), k) \quad \text{for each } k \in Z^+.$$

Notice that

- (a) $C(n, \alpha, \gamma) \cap (\cup \mathcal{M}_n) = M(n, \alpha, \gamma)$,
- (b) $g(M(n, \alpha, \gamma), k) \subset C(n, \alpha, \gamma)$ for all $k \in Z^+$, and
- (c) $g(M(n, \alpha, \gamma), k)$ is an open order-convex set.

Let $\mathcal{G}(n, k) = \{g(M(n, \alpha, \gamma), k) : \alpha \in \lambda(n), \gamma \in \lambda(n, \alpha)\}$ and $\mathcal{G} = \{g(n, k) : (n, k) \in Z^+ \times Z^+\}$. It follows that \mathcal{G} is a quasi-development for E . To see this, let $p \in E \cap U$, where U is an open set in X . There exist s and $t \in X$ such that $p \in (s, t) \subset U$. Since \mathcal{M} is a network, there exist $n \in Z^+$ and $\alpha \in \lambda(n)$ such that $p \in M(n, \alpha) \subset (s, t)$. There exists $\gamma \in \lambda(n, \alpha)$ such that $p \in M(n, \alpha, \gamma) \subset (s, t)$. Notice that $M(n, \alpha, \gamma) \subset C(n, \alpha, \gamma) \cap (s, t)$. Hence, depending on the endpoints of $M(n, \alpha, \gamma)$ in X^+ , there exists $k \in Z^+$ such that

$$\text{St}(p, \mathcal{G}(n, k)) = g(M(n, \alpha, \gamma), k) \subset (s, t).$$

Hence, $\mathcal{R} \cup \mathcal{L} \cup \mathcal{G} \cup \mathcal{U}$ is a quasi-development for X .

COROLLARY 7.1. *A GO-space is quasi-developable if and only if it is a weak σ -space with a quasi- G_δ -diagonal.*

4. Some other results

THEOREM 8. *If a GO-space X is a weak σ -space, then X is hereditarily paracompact.*

PROOF. Since a subspace of a GO-space is a GO-space and a subspace of a weak σ -space is a weak σ -space, it suffices to show that X is paracompact. Let $\mathcal{U} = \{U_\alpha : \alpha \in \lambda\}$ be an open cover of X and let $\mathcal{M} = \cup \{\mathcal{M}_n : n \in Z^+\}$ be a weak σ -structure on X . Let

$$\mathcal{C}_n = \{M : M \in \mathcal{M}_n \text{ and } M \subset U_\alpha \text{ for some } U_\alpha \in \mathcal{U}\}.$$

Then $\{\mathcal{C}_n : n \in Z^+\}$ is a discrete collection of subsets of the subspace $\cup \mathcal{C}_n$.

Therefore X is weakly θ -refinable [4] and hence paracompact.

DEFINITION 4. A space X is a $w\Delta$ -space if there is a set $\{G_n : n \in \mathbb{Z}^+\}$ of open covers of X such that, for each p in X , if $x_n \in \text{St}(p, G_n)$ for $n=1, 2, \dots$, then the sequence $\langle x_n \rangle$ has a cluster point.

COROLLARY 8.1. *If a first-countable GO-space X is a weak σ -space and a $w\Delta$ -space, then X is metrizable.*

PROOF. It follows from Theorem 7 that X is quasi-developable. Since X is paracompact by Theorem 8, the result follows from [2].

THEOREM 9. *Let X be a GO-space which is a weak σ -space. If X is either compact or connected, then X can be embedded in \mathbf{R} .*

PROOF. Since X is either compact or connected, it follows from Theorem 8 that X is first-countable. Hence X is quasi-developable by Theorem 7. Therefore, the result follows from Theorem 6.

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