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ON RINGS WITH FINITE SELF-INJECTIVE DIMENSION II

(Dedicated to Professor Goro Azumaya on his 60th birthday)

By

Yasuo IWANAGA

For a module M over a ring R (with an identity), pd(M) and id(M) denote the projective and injective dimension of M, respectively. In the previous paper [5] and [6], we called a (left and right) noether ring R *n-Gorenstein* if id$({}_RR)\leqq n$ and id$(R_R)\leqq n$ for an $n\geqq 0$, and *Gorenstein* if R is n-Gorenstein for some n. This note is concerned with two subjects on Gorenstein rings. In § 1, we consider the modules of finite projective or injective dimension over a Gorenstein ring and, first, show that the finiteness of projective dimension coincides with one of injective dimension. Then it follows that the highest finite projective (or injective) dimension is n for modules over an n-Gorenstein ring and, next, such modules over an artinian Gorenstein ring are investigated. Finally, we present some example to compare with Auslander's definition of an n-Gorenstein ring.

In § 2, for a Gorenstein ring R, we consider a quasi-Frobenius extension of R and show it also is a Gorenstein ring. Further we generalize [3, Corollary 8 and 8′] to the case of a quasi-Frobenius extension. Also an example concerning with a maximal quotient ring of a Gorenstein ring is presented.

1. Modules of finite projective or injective dimension

We start with the next proposition which states [4, Korollar 1.12] and [7, Corollary 5] more precisely:

PROPOSITION 1. *For a noether ring R,*

$$\text{id}(R_R)=\sup\{\text{flat dim}(E)\,;\ {}_RE \textit{ is an injective left } R\textit{-module.}\}\,.$$

PROOF. By [2, Chap. VI, Proposition 5.3],

$$(*)\qquad \text{Tor}_i^R(A_R,\ {}_RE)\cong\text{Hom}_R(\text{Ext}_R^i(A_R,\ {}_RR_R),\ {}_RE)$$

for any finitely generated right R-module A_R, injective left R-module ${}_RE$ and $i>0$.

First assume id$(R_R)=n<\infty$, then $\text{Ext}_R^{n+1}(A,R)=0$ for any finitely generated

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A_R and so $\mathrm{Tor}^R_{n+1}(A, E)=0$ for any injective ${}_RE$. Further, for any X_R, we can represent $X=\varinjlim A_\alpha$ such that each A_α is finitely generated and hence

$$\mathrm{Tor}^R_{n+1}(X_R, {}_RE)\cong\varinjlim \mathrm{Tor}^R_{n+1}(A_\alpha, E)=0\,.$$

Therefore flat dim $(E)\leqq n$.

Conversely, if flat dim $(E)\leqq n<\infty$ for any injective ${}_RE$, (*) induces

$$\mathrm{Hom}_R(\mathrm{Ext}^{n+1}_R(A, R), E)\cong\mathrm{Tor}^R_{n+1}(A, E)=0$$

for any finitely generated A_R. Now then, by taking ${}_RE$ as an injective cogenerator, it holds that $\mathrm{Ext}^{n+1}_R(A, R)=0$ for any finitely generated A_R and hence id $(R_R)\leqq n$.

The following was shown in [5] and [6] under certain assumption on the dominant dimension, but now we can release this assumption and include completely the commutative case.

THEOREM 2. *For an n-Gorenstein ring R and an R-module M, the following are equivalent*:

(1) pd $(M)<\infty$, (2) pd $(M)\leqq n$, (3) id $(M)<\infty$, (4) id $(M)\leqq n$.

PROOF. Since the implications (1)⇒(2) and (2)⇒(4) are proved in [1] and [5], respectively, we prove only (3)⇒(2).

Let

$$0\longrightarrow M\overset{f_0}{\longrightarrow} E_0\overset{f_1}{\longrightarrow} E_1\longrightarrow\cdots\overset{f_m}{\longrightarrow} E_m\longrightarrow 0$$

be an injective resolution of M and $K_{i-1}=\ker(f_i)$ $(i=1, \cdots, m)$, then in the exact sequence

$$0\longrightarrow K_{m-1}\longrightarrow E_{m-1}\longrightarrow E_m\longrightarrow 0\,,$$

if pd (E_{m-1}), pd $(E_m)\leqq n$, then pd $(K_{m-1})\leqq n$ by [5, Lemma 4]. For an arbitrary i, in the exact sequence

$$0\longrightarrow K_{i-1}\longrightarrow E_{i-1}\longrightarrow K_i\longrightarrow 0\,,$$

if pd (K_i), pd $(E_{i-1})\leqq n$, then pd $(K_{i-1})\leqq n$ and therefore pd $(M)=$pd $(K_0)\leqq n$ by the induction. Thus, it is enough to show pd $(E)\leqq n$ for any injective left module ${}_RE$.

Now, since flat dim $(E)\leqq n$ by Proposition 1, let

$$0\longrightarrow U_n\overset{f_n}{\longrightarrow} U_{n-1}\overset{f_{n-1}}{\longrightarrow}\cdots\overset{f_1}{\longrightarrow} U_0\longrightarrow E\longrightarrow 0$$

be a resolution of ${}_RE$ by flat modules U_i $(i=0, 1, \cdots, n)$ and $C_{i-1}=\mathrm{cok}(f_i)$ $(i=1, \cdots, n)$, then pd $(U_i)<\infty$ for $i=0, 1, \cdots, n$ by [7, Proposition 6]. First, from the exact sequence

$$0 \longrightarrow U_n \longrightarrow U_{n-1} \longrightarrow C_{n-1} \longrightarrow 0$$

with $\mathrm{pd}(U_n)$, $\mathrm{pd}(U_{n-1})<\infty$, it follows that $\mathrm{pd}(C_{n-1})<\infty$. For an arbitrary i, in the exact sequence

$$0 \longrightarrow C_{i+1} \longrightarrow U_i \longrightarrow C_i \longrightarrow 0\,,$$

if $\mathrm{pd}(C_{i+1})$, $\mathrm{pd}(U_i)<\infty$, then it follows that $\mathrm{pd}(C_i)<\infty$ and hence $\mathrm{pd}(E)=\mathrm{pd}(C_0)<\infty$ by the induction, which is equivalent to $\mathrm{pd}(E)\leqq n$ by the implication (1)$\Rightarrow$(2).

From Theorem 2, we are interested in modules M satisfying $\mathrm{pd}(M)=n$ or $\mathrm{id}(M)=n$ over an n-Gorenstein ring. Thus we next consider such modules.

For a module M, we define $\mathrm{E}^i(M)$ for $i\geqq 0$ as the $(i+1)$-th term in a minimal injective resolution of M and $\mathrm{E}(M)=\mathrm{E}^0(M)$, i.e.

$$0 \longrightarrow M \longrightarrow \mathrm{E}^0(M) \longrightarrow \cdots \longrightarrow \mathrm{E}^i(M) \longrightarrow \cdots$$

is a minimal injective resolution of M. Dually, if M has a minimal projective resolution, we define $\mathrm{P}^i(M)$ for $i\geqq 0$, similarly.

THEOREM 3. *Let R be an artinian n-Gorenstein ring,* $0\rightarrow {}_RR\rightarrow E_0\rightarrow\cdots\rightarrow E_n\rightarrow 0$ *a minimal injective resolution for ${}_RR$ and ${}_RM$ a left R-module.*

(1) *If* $\mathrm{id}(M)=n$, *then* $\mathrm{id}(M)=\mathrm{pd}(\mathrm{E}^n(M))=n$ *and, for any direct summand* ${}_RE$ *of* $\mathrm{E}^n(M)$, $\mathrm{pd}(E)=n$.

If $\mathrm{pd}(M)=n$, *then* $\mathrm{id}(\mathrm{P}^n(M))=\mathrm{pd}(M)=n$ *and, for any direct summand* ${}_RP$ *of* $\mathrm{P}^n(M)$, $\mathrm{id}(P)=n$.

In particular, $\mathrm{id}(\mathrm{P}^n(E_n))=\mathrm{pd}(E_n)=n$ *provided* $\mathrm{id}({}_RR)=n$.

(2) *If* $\mathrm{pd}(M)=n$, *then* $\mathrm{E}^n\mathrm{P}^n(M)$ *is isomorphic to a direct summand of a direct sum of copies of* E_n.

Especially, $\mathrm{E}^n\mathrm{P}^n(E_n)$ *is isomorphic to a direct summand of* E_n.

PROOF. (1) Suppose $\mathrm{id}(M)=n$ and ${}_RE$ an indecomposable summand of $\mathrm{E}^n(M)$, then since E is of the form $\mathrm{E}(S)$ for some simple module ${}_RS$, the exact sequence

$$0 \longrightarrow {}_RS \longrightarrow {}_R\mathrm{E}(S) \longrightarrow {}_R\mathrm{E}(S)/S \longrightarrow 0$$

induces

$$\mathrm{Ext}_R^n(\mathrm{E}(S), M) \longrightarrow \mathrm{Ext}_R^n(S, M) \longrightarrow \mathrm{Ext}_R^{n+1}(\mathrm{E}(S)/S, M) \quad \text{(exact)}.$$

Here, $\mathrm{Ext}_R^{n+1}(\mathrm{E}(S)/S, M)=0$ but $\mathrm{Ext}_R^n(S, M)\neq 0$ by [6, Lemma 1] since ${}_RS$ is monomorphic to $\mathrm{E}^n(M)$, and hence $\mathrm{Ext}_R^n(\mathrm{E}(S), M)\neq 0$. So $\mathrm{pd}(\mathrm{E}(S))\geqq n$ implies $\mathrm{pd}(\mathrm{E}(S))=n$ by Theorem 2.

Next, assume $\mathrm{pd}(M)=n$ and ${}_RP$ an indecomposable summand of $\mathrm{P}^n(M)$, then for any simple homomorphic image ${}_RS$ of P, the exact sequence

$$0 \longrightarrow {}_RK \longrightarrow {}_RP \longrightarrow {}_RS \longrightarrow 0$$

induces

$$\mathrm{Ext}_R^n(M, P) \longrightarrow \mathrm{Ext}_R^n(M, S) \longrightarrow \mathrm{Ext}_R^{n+1}(M, K) \quad \text{(exact)}.$$

Now, since $\mathrm{Ext}_R^{n+1}(M, K)=0$ but $\mathrm{Ext}_R^n(M, S)\neq 0$ by the dual of [6, Lemma 1], $\mathrm{Ext}_R^n(M, P)\neq 0$ and hence $\mathrm{id}(P)=n$ again by Theorem 2.

(2) Decompose ${}_RR$ into projective indecomposables, then for any projective indecomposable ${}_RP$ with $\mathrm{id}(P)=n$, $\mathrm{E}^n(P)$ is isomorphic to a direct summand of E_n. On the other hand, if $\mathrm{pd}(M)=n$, $\mathrm{id}(\mathrm{P}^n(M))=n$ by (1) and hence $\mathrm{E}^n\mathrm{P}^n(M)$ is isomorphic to a summand of a direct sum of copies of E_n.

COROLLARY 4. *Let R be an n-Gorenstein ring with* $\mathrm{dom}\cdot\mathrm{dim}_R R\geqq n$ *and assume ${}_RM$ a left R-module with* $\mathrm{id}(M)=n$, *then* $\mathrm{E}^n(M)$ *is isomorphic to a direct summand of a direct sum of copies of E_n.*

Now we present an example which seems itself interesting.

EXAMPLE. Let R be an artinian Gorenstein ring with $\mathrm{id}({}_RR)=n$ and $0\to {}_RR\to E_0\to \cdots \to E_n\to 0$ a minimal injective resolution of ${}_RR$, then we see from Theorem 3 that E_n has the largest projective dimension n. Here, we give an example of n-Gorenstein ring R with $\mathrm{pd}(E_0)=\cdots=\mathrm{pd}(E_n)=n$, which shows that our definition of an n-Gorenstein ring is different from Auslander's one.

Let k be a field and R a subalgebra of $(k)_8$, all 8×8 matrices over k, having $\{c_{11}+c_{88}, c_{22}+c_{55}, c_{33}+c_{44}, c_{66}, c_{77}, c_{21}, c_{31}, c_{32}, c_{54}, c_{86}, c_{87}\}$ as a k-basis where c_{ij} is a matrix unit in $(k)_8$. Then $\mathrm{id}({}_RR)=\mathrm{id}(R_R)=2$, i.e. R is 2-Gorenstein, $\mathrm{gl}\cdot\mathrm{dim}\, R=\infty$ and $\mathrm{pd}(E_0)=\mathrm{pd}(E_1)=\mathrm{pd}(E_2)=2$. Further any left R-module of projective dimension$=2$ is a summand in a direct sum of copies of $E_0\oplus E_1\oplus E_2$.

2. A quasi-Frobenius extension of a Gorenstein ring

For rings $R\subseteqq T$, T/R is called a *left quasi-Frobenius* (=QF) *extension* if ${}_RT$ is finitely generated projective and ${}_TT_R$ is isomorphic to a direct summand in a direct sum of copies of ${}_T\mathrm{Hom}_R({}_RT_T, {}_RR_R)_R$. A *quasi-Frobenius extension* is a left and right quasi-Frobenius extension. See [9] for details.

In this section we show a QF extension of a Gorenstein ring is also a Gorenstein ring. First we observe the following.

Let R, T be rings and F: ${}_R\boldsymbol{M}\to {}_T\boldsymbol{M}$ a functor of the category of left R-modules to one of left T-modules, which satisfies the condition:

1) F is exact,

2) if ${}_RE$ is injective, so is ${}_TF(E)$,
then $\mathrm{id}({}_TF(M)) \leqq \mathrm{id}({}_RM)$ for any left R-module ${}_RM$. Further if

3) F preserves an essential monomorphism
is satisfied, $\mathrm{id}({}_TF(M)) = \mathrm{id}({}_RM)$ for any ${}_RM$.

The next is a generalization of [3, Corollary 8] to a quasi-Frobenius extension and concerns with the case of a Gorenstein order [10, Lemma 5].

PROPOSITION 5. *Let T be a left quasi-Frobenius extension of a ring R and ${}_RM$ a left R-module, then*

$$\mathrm{id}({}_TT \otimes_R M) \leqq \mathrm{id}({}_RM).$$

PROOF. By [2, VI Proposition 5.2],

$$\begin{aligned} {}_TT \otimes_R M &\cong {}_TT \otimes_R \mathrm{Hom}_R({}_RR_R, {}_RM) \\ &\cong {}_T\mathrm{Hom}_R({}_R\mathrm{Hom}_R({}_TT_R, {}_RR_R)_T, {}_RM). \end{aligned}$$

Here, T_R is projective by [9, Satz 2] and since $\mathrm{Hom}_R({}_TT_R, R_R)_T$ is projective ([9, Satz 2]), ${}_TT \otimes_R E$ is injective for any injective left R-module ${}_RE$. Therefore the functor ${}_TT \otimes_R - : {}_R\boldsymbol{M} \to {}_T\boldsymbol{M}$ satisfies the conditions 1)—2) and so

$$\mathrm{id}({}_TT \otimes_R M) \leqq \mathrm{id}({}_RM).$$

The following should be compared with [9, satz 3].

COROLLARY 6. *A quasi-Frobenius extension of an n-Gorenstein ring also is an n-Gorenstein ring.*

In connection with [1, Example (2)] and [3, Corollary 8′], we state the following.

PROPOSITION 7. (1) *Let T be a left quasi-Frobenius extension of a ring R and suppose T_R a generator, then*

$$\mathrm{id}({}_TT \otimes_R M) = \mathrm{id}({}_RM)$$

for any left R-module M and especially $\mathrm{id}({}_TT) = \mathrm{id}({}_RR)$.

Moreover, for a finite group G and a ring R,

$$\mathrm{id}({}_{R[G]}R[G]) = \mathrm{id}({}_RR).$$

(2) *Let T be a quasi-Frobenius extension of a ring R and suppose ${}_RT$ (or T_R) a generator, then*

$$\mathrm{id}({}_TT) = \mathrm{id}({}_RR) \quad \textit{and} \quad \mathrm{id}(T_T) = \mathrm{id}(R_R).$$

RROOF. (1) Let $F = T \otimes_R - : {}_R\boldsymbol{M} \to {}_T\boldsymbol{M}$, then F satisfies the conditions 1)—3) for T_R is a progenerator by [9, Satz 2].

(2) Let $F=\mathrm{Hom}_R({}_RT_T, -): {}_R\boldsymbol{M}\to{}_T\boldsymbol{M}$, then ${}_RT$ is a progenerator and so $\mathrm{id}({}_T\mathrm{Hom}_R({}_RT_T, {}_RR))=\mathrm{id}({}_TF({}_RR))=\mathrm{id}({}_RR)$. Now, since T/R is a left (resp. right) quasi-Frobenius extension, ${}_T\mathrm{Hom}_R({}_RT_T, {}_RR)$ is a generator (resp. finitely generated projective) and therefore $\mathrm{id}({}_T\mathrm{Hom}_R({}_RT_T, {}_RR))=\mathrm{id}({}_TT)$. Also $\mathrm{id}(T_T)=\mathrm{id}(R_R)$ follows from (1).

REMARK. In Proposition 7, if we replace a ring T by an R-module and its endomorphism ring, then we obtain the following.

Let R be a ring, ${}_RP$ a projective left R-module, $T=\mathrm{End}_R(P)$ and assume P_T flat, then the functor $F=\mathrm{Hom}_R({}_RP_T, -): {}_R\boldsymbol{M}\to{}_T\boldsymbol{M}$ satisfies 1)—2) by [2, VI Proposition 5.1] and hence $\mathrm{id}({}_TF(P))\leqq\mathrm{id}({}_RP)$. Observing this fact,

(i) Let R be a left noether ring, ${}_RP$ a projective generator and $T=\mathrm{End}_R(P)$, then $\mathrm{id}({}_TT)\leqq\mathrm{id}({}_RR)$. Therefore it follows immediately that an endomorphism ring of a faithful finitely generated projective module over a quasi-Frobenius ring also is a quasi-Frobenius ring. (Curtis and Morita)

(ii) If rings R and T are Morita equivalent, then $\mathrm{id}({}_RR)=\mathrm{id}({}_TT)$ and $\mathrm{id}(R_R)=\mathrm{id}(T_T)$.

Now, if rings R and T are Morita equivalent, there exists a finitely generated projective generator (i.e. progenerator) ${}_RP$ and $T\cong\mathrm{End}_R(P)$. However, if we delete that ${}_RP$ is a generator, it happens that R is Gorenstein but T is not and we see also that faithfulness in Curtis-Morita theorem above is necessary. For example, let R be a self-basic serial ring and $R=Re_1\oplus Re_2\oplus Re_3$ a decomposition into primitive left ideals such that $|Re_1|=|Re_2|=|Re_3|=5$ and Re_1 (resp. Re_2) is epimorphic to Ne_2 (resp. Ne_3) where N is the radical of R. Then R is a quasi-Frobenius ring, but $\mathrm{id}({}_{eRe}eRe)$ is infinite for $e=e_1+e_2$.

Finally we state an example concerning with a maximal quotient ring of a Gorenstein ring.

EXAMPLE. It is easily seen that a classical quotient ring or more generally a flat epimorphic extension of a Gorenstein ring also is a Gorenstein ring, but it is not known yet that a maximal quotient ring of a Gorenstein ring is also so. (See [11] in the special case.) Here we present an example of a Gorenstein ring R whose left maximal quotient ring Q has $\mathrm{id}({}_QQ)>\mathrm{id}({}_RR)$.

Let k be a field, R a subalgebra of $(k)_5$ whose k-basis consists of $c_{11}+c_{55}$, $c_{22}+c_{44}$, c_{33}, c_{31}, c_{32}, c_{54} and Q_l (resp. Q_r) a left (resp. right) maximal quotient ring of R. Then R is 1-Gorenstein, $\mathrm{id}({}_{Q_l}Q_l)=2$ and Q_r is a quasi-Frobenius ring.

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Institute of Mathematics
University of Tsukuba
Ibaraki, 305 Japan