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FIRST ORDER DEFORMATIONS OF CONES OVER PROJECTIVELY NORMAL HYPERELLIPTIC CURVES

By

Jiryo KOMEDA

Introduction

Let X be a smooth, proper algebraic variety over an algebraically closed field k, and let L be an ample, projectively normal invertible sheaf on X. $\varphi_L: X\rightarrow P_k^n$ denotes the embedding of X corresponding to the complete linear system $|L|$, and C denotes the affine cone over $\varphi_L(X)$. We denote by D_C the deformation functor of C from the category of artin local k-algebras with residue field k to the category of sets. Then since the affine ring of C has a natural grading, the k-vector space $T_C^1=D_C(k[\varepsilon])$ of first order deformations of C has a natural graded structure

$$T_C^1=\bigoplus_{\nu\in Z} T_C^1(\nu)\,.$$

When $\dim X\geqq 2$, Schlessinger [6] showed that if L is sufficiently ample on X, then $T_C^1(\nu)=0$ for all $\nu\neq 0$. In the case of $\dim X=1$, Mumford [2] proved that $T_C^1(\nu)=0$ for all $\nu\neq 0$, if X is non-hyperelliptic of genus$\geqq 3$ and if L is sufficiently ample on X. Moreover, he showed that if X and L are respectively the rational curve P^1 and the invertible sheaf $\mathcal{O}_{P^1}(d)$ for $d\geqq 3$, then $T_C^1(\nu)=0$ for all $\nu\neq -1$ and $\dim_k T_C^1(-1)=2d-4$. Applying Mumford's techniques to the case of elliptic curves, Pinkham [3] showed that if X is an elliptic curve and L is an invertible sheaf on X of degree $d\geqq 5$, then $T_C^1(\nu)=0$ for $\nu>0$, $\dim_k T_C^1(0)=1$, $\dim_k T_C^1(-1)=d$ and $T_C^1(\nu)=0$ for $\nu\leqq -2$. Moreover, he gave a complete description of T_C^1 in the following cases:

1) $X=P^1$ and $L=\mathcal{O}_{P^1}(d)$ for $d\geqq 3$,

2) X is the elliptic curve $Z_0^3+Z_1^3+Z_2^3=0$ in P^2 and $L=\mathcal{O}_{P^2}(2)|_X$.

In this paper we shall compute the dimension of the k-vector space of first order deformations of the affine cone C over a projectively normal hyperelliptic curve of genus $g\geqq 2$. In the case $g\geqq 3$, our results are only partial. Our main theorem is the following: *let the characteristic of k be different from* 2 *and let*

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X be a hyperelliptic curve of genus $g\geqq 2$ whose function field $\boldsymbol{K}(X)=k(x, y)$ is the extension of $k(x)$ determined by the equation

$$y^2=x(x-1)(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_{2g-1}),$$

where $\alpha_1, \alpha_2, \cdots, \alpha_{2g-1}$ are $2g-1$ distinct elements of k different from 0 and 1. Let $\pi: X\to \boldsymbol{P}^1$ be the morphism corresponding to the inclusion map $k(x)\to k(x, y)$ and let L be the sheaf $\mathcal{O}_X(dQ_\infty)$ where Q_∞ is the branch point $\pi^{-1}((0, 1))$ on X. If $d\geqq \mathrm{Max}\{4g-3, 2g+3\}$, then

1) $T_C^1=T_C^1(-1)\oplus T_C^1(0)$,

2) $\dim_k T_C^1(-1)\leqq(g-1)g(g+1)$ *and* $\dim_k T_C^1(0)=4g-3$.

In the case $g=2$, we get more explicit results: if $d\geqq 7$, then

1) $T_C^1=T_C^1(-1)\oplus T_C^1(0)$,

2) $\dim_k T_C^1(-1)=6$ *and* $\dim_k T_C^1(0)=5$,

and if $d=6$ (resp. 5), then

1) $T_C^1=T_C^1(-2)\oplus T_C^1(-1)\oplus T_C^1(0)$,

2) $\dim_k T_C^1(-2)=1$ (*resp.* 2), $\dim_k T_C^1(-1)=6$ *and* $\dim_k T_C^1(0)=5$.

In particular, in the cases $d=5$, 6 and 7, we will give a k-basis for T_C^1 explicitly.

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Notation.

Throughout this paper we will use the following notation without further warning.

We denote by k an algebraically closed field and by $\boldsymbol{P}^n$ the n-dimensional projective space over k. Moreover, we denote by X a smooth, proper algebraic variety over k and by L an ample, projectively normal invertible sheaf on X. $\varphi_L: X\to \boldsymbol{P}^n$ denotes the embedding of X corresponding to the complete linear system $|L|$ and C denotes the affine cone over $\varphi_L(X)$. In this paper $\varphi_L(X)$ is always identified with X through φ_L. We denote by $\mathcal{N}_X$ the normal sheaf of X, by $\mathcal{T}_X$ the tangent sheaf of X and by Ω_X the canonical sheaf on X. For any $\mathcal{O}_X$-module F and any $\nu\in\boldsymbol{Z}$, we write $F(\nu)=F\otimes L^\nu$. $\chi(X, F)$ denotes the Euler characteristic of the sheaf F on X and $h^i(X, F)$ denotes the dimension of the k-vector space $H^i(X, F)$. For a scheme C over a field K, we denote by T_C^1 the K-vector space of first order deformations of C.

1. First order deformations of cones over projectively normal curves.

We first recall the natural grading on T_C^1 due to Schlessinger [6] in terms of cohomology on X. We have the standard exact sequences of $\mathcal{O}_X$-modules:

$$0 \longrightarrow \mathcal{O}_X \longrightarrow L^{\oplus(n+1)} \longrightarrow \mathcal{T}_{P^n}|_X \longrightarrow 0 \tag{1.1}$$

and

$$0 \longrightarrow \mathcal{T}_X \longrightarrow \mathcal{T}_{P^n}|_X \longrightarrow \mathcal{N}_X \longrightarrow 0\,. \tag{1.2}$$

Then Schlessinger showed that the following sequence is exact.

$$\sum_{\nu=-\infty}^{\infty} H^0(X, L^{\nu+1})^{\oplus(n+1)} \longrightarrow \sum_{\nu=-\infty}^{\infty} H^0(X, \mathcal{N}_X(\nu)) \longrightarrow T_C^1 \longrightarrow 0\,.$$

This gives a natural grading on T_C^1 by

$$T_C^1(\nu) = \mathrm{coker}\,(H^0(X, (L^{\nu+1})^{\oplus(n+1)}) \longrightarrow H^0(X, \mathcal{N}_X(\nu)))\,. \tag{1.3}$$

In this section, we are concerned with the 1-demensional case. g and d are respectively the genus of X and the degree of L. The following remark is due to Mumford [2].

REMARK 1.1. If the degree d of L is larger than $4g-4$, then we have $T_C^1(\nu)=0$ for all $\nu>0$.

In the case $\nu=0$, using the standard exact sequences (1.1) and (1.2) we can compute the dimension of the k-vector space $T_C^1(0)$.

PROPOSITION 1.2. *If $g \geqq 2$ and $d \geqq 2g-1$, then we get* $\dim_k T_C^1(0)=4g-3$.

PROOF. The exact sequences (1.1) and (1.2) induce the long exact sequences

$$0 \longrightarrow H^0(\mathcal{O}_X) \longrightarrow H^0(L^{\oplus(n+1)}) \xrightarrow{f} H^0(\mathcal{T}_{P^n}|_X) \longrightarrow H^1(\mathcal{O}_X) \longrightarrow$$
$$H^1(L^{\oplus(n+1)}) \longrightarrow H^1(\mathcal{T}_{P^n}|_X) \longrightarrow 0$$

and

$$0 \longrightarrow H^0(\mathcal{T}_X) \longrightarrow H^0(\mathcal{T}_{P^n}|_X) \xrightarrow{h} H^0(\mathcal{N}_X) \longrightarrow H^1(\mathcal{T}_X) \longrightarrow H^1(\mathcal{T}_{P^n}|_X) \longrightarrow$$
$$H^1(\mathcal{N}_X) \longrightarrow 0\,,$$

respectively. Two equalities $H^0(\mathcal{T}_X)=0$ and $H^1(L^{\oplus(n+1)})=0$ come from our assumption $g \geqq 2$ and $d \geqq 2g-1$. Since the dimension of $\mathrm{Im}\,f$ is $(n+1)^2-1$, we get $h^0(\mathcal{N}_X)=(n+1)^2+4g-4$. The formula (1.3) leads us to the following commutative diagram:

$$\begin{array}{ccccccc} H^0(L^{\oplus(n+1)}) & \longrightarrow & H^0(\mathcal{N}_X) & \longrightarrow & T^1_C(0) & \longrightarrow & 0 \\ & f\searrow & \nearrow h & & & & \\ & & H^0(\mathcal{T}_{P^n}|_X) & & & & \end{array}$$

such that the row is exact and h is injective where f and h are the maps in the above long exact sequences. Therefore we get

$$\dim_k T^1_C(0)=h^0(\mathcal{N}_X)-\dim_k \operatorname{Im} f=4g-3\,. \qquad \text{Q.E.D.}$$

Now we recall Mumford's results in [2]. Let $p_i: X\times X\to X$ be the i-th projection for $i=1, 2$, and let Δ be the diagonal of $X\times X$. Mumford showed that

$$T^1_C(\nu)=\operatorname{coker}(H^0(X, R^1p_{2*}[p_1^*(\Omega_X\otimes L^{-1})]\otimes L^{\nu+1})\longrightarrow$$
$$H^0(X, R^1p_{2*}[p_1^*(\Omega_X\otimes L^{-1})\otimes\mathcal{O}_{X\times X}(2\Delta)]\otimes L^{\nu+1}))\,,$$

for all $\nu\in\boldsymbol{Z}$. If $d\geqq 2g+1$, then $R^0p_{2*}[p_1^*(\Omega_X\otimes L^{-1})]=0$ and $R^0p_{2*}[p_1^*(\Omega_X\otimes L^{-1})\otimes\mathcal{O}_{X\times X}(2\Delta)]=0$, hence using the Leray spectral sequence for p_2 we get

$$T^1_C(\nu)=\operatorname{coker}(H^1(X\times X,\ p_1^*(\Omega_X\otimes L^{-1})\otimes p_2^*L^{\nu+1})\xrightarrow{\alpha(\nu)}$$
$$H^1(X\times X,\ p_1^*(\Omega_X\otimes L^{-1})\otimes p_2^*L^{\nu+1}\otimes\mathcal{O}_{X\times X}(2\Delta)))\,,$$

for all $\nu\in\boldsymbol{Z}$.

LEMMA 1.3. *If $g\geqq 1$ and $d\geqq 2g+1$, then $\alpha(\nu)$ is injective for all $\nu\leqq -1$, where $\alpha(\nu)$ is the map in the above formula.*

PROOF. It suffices to check that the composite of the maps

$$H^0(X, (L^{\nu+1})^{\oplus(n+1)})\longrightarrow H^0(X, \mathcal{T}_{P^n}|_X\otimes L^\nu)\longrightarrow H^0(X, \mathcal{N}_X(\nu))$$

is injective for all $\nu\leqq -1$. The standard exact sequences (1.1) and (1.2) induce exact sequences

$$H^0(X, L^\nu)\longrightarrow H^0(X, (L^{\nu+1})^{\oplus(n+1)})\longrightarrow H^0(X, \mathcal{T}_{P^n}|_X\otimes L^\nu)$$

and

$$H^0(X, \mathcal{T}_X(\nu))\longrightarrow H^0(X, \mathcal{T}_{P^n}|_X\otimes L^\nu)\longrightarrow H^0(X, \mathcal{N}_X(\nu))\,,$$

respectively. If $g\geqq 1$ and $d\geqq 2g+1$, we get $H^0(X, \mathcal{T}_X(\nu))=0$ for all $\nu\leqq -1$. On the other hand, for any $\nu\leqq -1$ we have $H^0(X, L^\nu)=0$. Hence the map

$$H^0(X, (L^{\nu+1})^{\oplus(n+1)})\longrightarrow H^0(X, \mathcal{N}_X(\nu))$$

is injective. Q.E.D.

Manipulating the Leray spectral sequence for p_1, we can give an upper bound for $\dim_k T_C^1(-1)$ when X is hyperelliptic.

PROPOSITION 1.4. *If X is a hyperelliptic curve of genus $g \geqq 2$ and if the degree d of L is larger than or equal to $3g-1$, then we have*

$$\dim_k T_C^1(-1) \leqq (g-1)g(g+1).$$

In particular, if $g=2$ and $d \geqq 5$ then we get

$$\dim_k T_C^1(-1)=6.$$

PROOF. Let $\mathcal{O}_\Delta$ be the cokernel of the natural inclusion map $\mathcal{O}_{X\times X}(-\Delta) \to \mathcal{O}_{X\times X}$. For any $\nu \in \boldsymbol{Z}$ we have an exact sequence of $\mathcal{O}_{X\times X}$-modules

$$\begin{aligned} 0 \longrightarrow p_1^*(\Omega_X \otimes L^{-1}) \otimes \mathcal{O}((\nu-1)\Delta) \longrightarrow p_1^*(\Omega_X \otimes L^{-1}) \otimes \mathcal{O}(\nu\Delta) \longrightarrow \\ p_1^*(\Omega_X \otimes L^{-1}) \otimes \mathcal{O}(\nu\Delta) \otimes \mathcal{O}_\Delta \longrightarrow 0. \end{aligned} \tag{1.4}$$

Using this exact sequence (1.4) inductively, we see

$$\begin{aligned} \chi(X\times X,\ p_1^*(\Omega_X \otimes L^{-1}) \otimes \mathcal{O}(g\Delta) = \chi(X\times X,\ p_1^*(\Omega_X \otimes L^{-1})) + \\ \sum_{\nu=1}^{g} \chi(X\times X,\ p_1^*(\Omega_X \otimes L^{-1}) \otimes \mathcal{O}(\nu\Delta) \otimes \mathcal{O}_\Delta). \end{aligned}$$

Since we have

$$\begin{aligned} H^i(X\times X,\ p_1^*(\Omega_X \otimes L^{-1}) \otimes \mathcal{O}(\nu\Delta) \otimes \mathcal{O}_\Delta) = H^i(\Delta,\ p_1^*(\Omega_X \otimes L^{-1})|_\Delta \otimes \mathcal{O}(\nu\Delta)|_\Delta) \\ = H^i(X,\ \Omega_X^{1-\nu} \otimes L^{-1}), \end{aligned}$$

we get

$$\begin{aligned} \chi(X\times X,\ p_1^*(\Omega_X \otimes L^{-1}) \otimes \mathcal{O}(\nu\Delta) \otimes \mathcal{O}_\Delta) = \chi(X,\ \Omega_X^{1-\nu} \otimes L^{-1}) \\ = -2(g-1)\nu + g - d - 1 \end{aligned} \tag{1.5}$$

for all $\nu \geqq 1$. Since the Künneth formula shows

$$\chi(X\times X,\ p_1^*(\Omega_X \otimes L^{-1})) = (g-1)(d+1-g),$$

combining this with (1.5) we get

$$\chi(X\times X,\ p_1^*(\Omega_X \otimes L^{-1}) \otimes \mathcal{O}(g\Delta)) = -(g-1)g(g+1) - (d+1-g).$$

On the other hand, we have

$$\chi(X\times X,\ p_1^*(\Omega_X \otimes L^{-1}) \otimes \mathcal{O}(g\Delta)) = -h^1(X\times X,\ p_1^*(\Omega_X \otimes L^{-1}) \otimes \mathcal{O}(g\Delta)).$$

Indeed, $p_{2*}[p_1^*(\Omega_X \otimes L^{-1}) \otimes \mathcal{O}(g\Delta)]=0$, because by the assumption $d \geqq 3g-1$ one sees that for any closed point x of X

$$H^0(p_2^{-1}(x),\ (p_1^*(\Omega_X \otimes L^{-1}) \otimes \mathcal{O}(g\Delta))|_{p_2^{-1}(x)}) = H^0(X,\ \Omega_X \otimes L^{-1} \otimes \mathcal{O}_X(gx)) = 0.$$

Hence we get

$$H^0(X\times X,\ p_1^*(\Omega_X\otimes L^{-1})\otimes\mathcal{O}(g\Delta))=H^0(X,\ p_{2*}[p_1^*(\Omega_X\otimes L^{-1})\otimes\mathcal{O}(g\Delta)])=0\,.$$

Moreover, the support of $R^1p_{1*}[p_1^*(\Omega_X\otimes L^{-1})\otimes\mathcal{O}(g\Delta)]$ is a finite set, because we have

$$\dim_k R^1p_{1*}[p_1^*(\Omega_X\otimes L^{-1})\otimes\mathcal{O}(g\Delta)]_x\otimes(\mathcal{O}_{X,x}/m_{X,x})=$$
$$h^1(p_1^{-1}(x),\ (p_1^*(\Omega_X\otimes L^{-1})\otimes\mathcal{O}(g\Delta))\,|_{p_1^{-1}(x)})=h^1(X,\ \mathcal{O}_X(gx))=$$
$$h^0(X,\ \Omega_X\otimes\mathcal{O}_X(-gx))=\begin{cases}1 & \text{if } x \text{ is a branch point for the double covering } \pi: X\longrightarrow \boldsymbol{P}^1.\\ 0 & \text{otherwise.}\end{cases}$$

Therefore we get

$$H^1(X,\ R^1p_{1*}[p_1^*(\Omega_X\otimes L^{-1})\otimes\mathcal{O}(g\Delta)])=0\,.$$

Let us consider the Leray spectral sequence

$$E_2^{p,q}=H^p(X,\ R^qp_{1*}[p_1^*(\Omega_X\otimes L^{-1})\otimes\mathcal{O}(g\Delta)])\Rightarrow$$
$$H^{p+q}(X\times X,\ p_1^*(\Omega_X\otimes L^{-1})\otimes\mathcal{O}(g\Delta))=E^{p+q}$$

for p_1 and $p_1^*(\Omega_X\otimes L^{-1})\otimes\mathcal{O}(g\Delta)$. Since X is 1-dimensional we get $E_2^{2,0}=E_2^{0,2}=0$. Hence by the exact sequence of terms of low degree we have $E^2=0$. Since $E^i=0$ for $i=0$ or $i\geqq 2$, we get

$$\chi(X\times X,\ p_1^*(\Omega_X\otimes L^{-1})\otimes\mathcal{O}(g\Delta))=-h^1(X\times X,\ p_1^*(\Omega_X\otimes L^{-1})\otimes\mathcal{O}(g\Delta))\,.$$

Now the exact sequence (1.4) induces the following exact sequence:

$$H^0(X\times X,\ p_1^*(\Omega_X\otimes L^{-1})\otimes\mathcal{O}(\nu\Delta)\otimes\mathcal{O}_\Delta)\longrightarrow H^1(X\times X,\ p_1^*(\Omega_X\otimes L^{-1})\otimes\mathcal{O}((\nu-1)\Delta))\longrightarrow$$
$$H^1(X\times X,\ p_1^*(\Omega_X\otimes L^{-1})\otimes\mathcal{O}(\nu\Delta))\,.$$

Since for any $\nu\geqq 1$

$$H^0(X\times X,\ p_1^*(\Omega_X\otimes L^{-1})\otimes\mathcal{O}(\nu\Delta)\otimes\mathcal{O}_\Delta)=H^0(X,\ \Omega_X^{1-\nu}\otimes L^{-1})=0\,,$$

we get

$$h^1(X\times X,\ p_1^*(\Omega_X\otimes L^{-1})\otimes\mathcal{O}(2\Delta))\leqq h^1(X\times X,\ p_1^*(\Omega_X\otimes L^{-1})\otimes\mathcal{O}(g\Delta))\,.$$

By Lemma 1.3, we see

$$\dim_k T_C^1(-1)=h^1(X\times X,\ p_1^*(\Omega_X\otimes L^{-1})\otimes\mathcal{O}(2\Delta))-h^1(X\times X,\ p_1^*(\Omega_X\otimes L^{-1}))$$
$$\leqq -\chi(X\times X,\ p_1^*(\Omega_X\otimes L^{-1})\otimes\mathcal{O}(g\Delta))-h^1(X,\ \Omega_X\otimes L^{-1})=(g-1)g(g+1)\,.$$

If $g=2$ and $d\geqq 5$, then the above inequality is obviously an equality and we get $\dim_k T_C^1(-1)=6$. Q. E. D.

2. The proof of $T_C^1(-2)=0$ using the equations defining hyperelliptic curves.

In this section, first, when a homogeneous ideal I of a polynomial ring P and its generators are explicitly given, a natural grading on the space T_C^1 of first order deformations of $C=\operatorname{Spec} P/I$ is defined. Specifically, let $P=K[X_0, X_1, \cdots, X_n]$ be a polynomial ring over a field K, and let $I\subset P$ be an ideal generated by homogeneous elements $f_i(1\leqq i\leqq N)$ of degree d_i. We set $B=P/I$ and $C=\operatorname{Spec} B$. Then we will apply the definition of a natural grading on T_C^1 obtained by Pinkham [3] to the above case.

REMARK 2.1. Let $R=\{(r_1, \cdots, r_N)\in P^N \mid \sum_{i=1}^{N} r_i f_i=0\}$ be the relation P-module for $f_1, \cdots, f_N$ and let $R_1, \cdots, R_M$ be a system of generators of the P-module R. For a fixed N-tuple $(g_1, \cdots, g_N)\in P^N$, a map θ' from I to B defined by $\sum_{i=1}^{N} h_i f_i \mapsto \sum_{i=1}^{N} h_i g_i+I$ with $h_1, \cdots, h_N\in P$ is well-defined if and only if $R_j\cdot {}^t(g_1, \cdots, g_N)\equiv 0 \bmod I$ for all $j=1, \cdots, M$. In this case, a homomorphism $\theta: I/I^2\to B$ of B-modules is determined by $\sum_{i=1}^{N} h_i f_i+I^2\mapsto \sum_{i=1}^{N} h_i g_i+I$, and any element of $\operatorname{Hom}_B(I/I^2, B)$ is given in this manner.

Now we introduce a grading for the B-module $\operatorname{Hom}_B(I/I^2, B)$.

DEFINITION 2.2. Let us take $\theta\in\operatorname{Hom}_B(I/I^2, B)$ such that $\theta(f_i+I^2)=g_i+I$ with homogeneous elements g_i for all $i=1, \cdots, N$. Now we set $\mu_i=+\infty$ if $g_i\in I$ and $\mu_i=\deg g_i$ if $g_i\notin I$. Then $(\mu_1, \cdots, \mu_N)$ depends only on θ and does not depend on the choice of the set $\{g_i\}$ of homogeneous elements. We grade elements of $\operatorname{Hom}_B(I/I^2 B)$ as follows: the above element θ is *homogeneous of degree* ν if $\mu_i=\nu+d_i$ or $+\infty$ for any $i=1, \cdots, N$. Then ν depends only on θ and does not depend on the choice of the system $\{f_i\}$ of homogeneous generators of the ideal I. This defines a structure of a graded B-module on $\operatorname{Hom}_B(I/I^2, B)$.

REMARK 2.3. Let D be the B-submodule of $\operatorname{Hom}_B(I/I^2, B)$ generated by the homomorphisms $d_{(0)}, d_{(1)}, \cdots, d_{(n)}$ of B-modules where $d_{(l)}: I/I^2\to B$ is defined by sending $h+I^2\to\partial h/\partial X_l+I$ with $h\in I$ for all $l=0, 1, \cdots, n$. Then we have the exact sequence of K-vector spaces ([5]):

$$0\longrightarrow D\longrightarrow \operatorname{Hom}_B(I/I^2, B)\xrightarrow{\Phi} T_C^1\longrightarrow 0. \tag{2.1}$$

Since the $d_{(l)}$ are homogeneous of degree -1, the above exact sequence (2.1) defines a natural grading on T_C^1 as follows:

DEFINITION 2.4. For any $\nu \in \boldsymbol{Z}$, we denote by $T_C^1(\nu)$ the image of the ν-th graded piece of $\mathrm{Hom}_B(I/I^2, B)$ by the homomorphism Φ in (2.1). This gives a grading on T_C^1.

By this definition we see easily:

REMARK 2.5. If $d=\mathrm{Max}\{d_i | i=1, \cdots, N\}$, then we get $T_C^1(\nu)=0$ for all $\nu<-d$.

REMARK 2.6 This grading is of course the same one which was obtained by Schlessinger in the projectively normal case and which was described in section 1.

Henceforth we specialize ourselves to the situation where C is the affine cone over a projectively normal hyperelliptic curve X over k. We are in the following situation:

NOTATION 2.7. Let the characteristic of k be different from 2, let X be a hyperelliptic curve of genus $g \geqq 2$ whose function field $K(X)=k(x, y)$ is the extension of $k(x)$ determined by the equation

$$y^2=x(x-1)(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_{2g-1}),$$

where $\alpha_1, \alpha_2, \cdots, \alpha_{2g-1}$ are $2g-1$ distinct elements of k different from 0 and 1, and let $\pi: X \to \boldsymbol{P}^1$ be the morphism corresponding to the inclusion map $k(x) \to k(x, y)$. L is the invertible sheaf $\mathcal{O}_X(dQ_\infty)$ with $d \geqq 2g+1$ where Q_∞ is the branch point $\pi^{-1}((0, 1))$ on X. Since $\dim_k \Gamma(X, L)=d+1-g$, we see easily that $y, xy, \cdots, x^{[(d-2g-1)/2]}y, 1, x, \cdots, x^{[d/2]}$ form a k-basis of $\Gamma(X, L)$, where [ ] is the Gauss symbol. $\varphi_L: X \to \boldsymbol{P}^{d-g}=\mathrm{Proj}\, k[X_0, X_1, \cdots, X_{d-g}]$ is the embedding such that $L \cong \varphi_L^*(\mathcal{O}_{\boldsymbol{P}^{d-g}}(1))$ and that the sections $y, xy, \cdots, x^{[(d-2g-1)/2]}y, 1, x, \cdots, x^{[d/2]}$ correspond to $\varphi_L^*(X_0), \varphi_L^*(X_1), \cdots, \varphi_L^*(X_{d-g})$ respectively under this isomorphism. Let I be the largest homogeneous ideal defining the subvariety $\varphi_L(X)$ of $\boldsymbol{P}^{d-g}$ with the decomposition $I=\bigoplus_{\nu \geqq 0} I_\nu$ into the direct sum of homogeneous pieces. If we set

$$x(x-1)(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_{2g-1})=\sum_{i=1}^{2g+1} a_i x^i,$$

then we have $a_{2g+1}=1$ and $a_1=\alpha_1\alpha_2\cdots\alpha_{2g-1}\neq 0$.

Now we give a system of generators of the ideal I. If we apply the results in [1] and [4] to our case, we see:

REMARK 2.8. For any $d \geqq 2g+2$, the ideal I is generated by I_2 and $\dim_k I_2 =(d^2-(2g+1)d+g^2-g)/2$.

PROPOSITION 2.9. *Set* $f_{ij}=X_iX_j-X_{i+1}X_{j-1}$ *for* $0\leqq i\leqq d-g-2$ *and* $i+2\leqq j\leqq d-g$, *and set* $l=[(d-2g-1)/2]$. *If* $d\geqq 2g+2$, *then the ideal* I *is generated by*

(A) $f_{ij}(0\leqq i\leqq l-2,\ i+2\leqq j\leqq l)$,

(B) $f_{ij}(l+1\leqq i\leqq d-g-2,\ i+2\leqq j\leqq d-g)$,

(C) $f_{ij}(0\leqq i\leqq l-1,\ l+2\leqq j\leqq d-g)$,

(D) $g_m(0\leqq m\leqq d-(2g+2))$, *where for* $0\leqq m\leqq l-g-1$

$$g_m=X_0X_m-X_{l+1}\sum_{i=1}^{2g+1}a_iX_{i+m+l+1},$$

for $l-g\leqq m\leqq l$

$$g_m=X_0X_m-X_{d-g}\sum_{i=l-m+g+1}^{2g+1}a_iX_{i+m-g+(1-(-1)^d)/2}-X_{l+1}\sum_{i=1}^{l-m+g}a_iX_{i+m+l+1},$$

for $l+1\leqq m\leqq l+g-1$

$$g_m=X_{m-l}X_l-X_{d-g}\sum_{i=l-m+g+1}^{2g+1}a_iX_{i+m-g+(1-(-1)^d)/2}-X_{l+1}\sum_{i=1}^{l-m+g}a_iX_{i+m+l+1},$$

and for $l+g\leqq m\leqq d-(2g+2)$

$$g_m=X_{m-l}X_l-X_{d-g}\sum_{i=1}^{2g+1}a_iX_{i+m-g+(1-(-1)^d)/2}.$$

PROOF. Trivial relations among $y, xy, \cdots, x^{[(d-2g-1)/2]}y, 1, x, \cdots, x^{[d/2]}$ induce (A), (B) and (C). The equation $y^2=\sum_{i=1}^{2g+1}a_ix^i$ induces (D). It is easy to check that the above polynomals are linearly independent and that the number of them is equal to $(d^2-(2g+1)d+g^2-g)/2$. By Remark 2.8, they generate the ideal I. Q. E. D.

Applying Definition 2.4 to our case, we get the following:

PROPOSITION 2.10. (0) *If* $d\geqq 2g+2$, *then we have* $T_C^1(\nu)=0$ *for all* $\nu\leqq -3$.

(1) *Moreover if* $d\geqq 2g+3$, *then we get* $T_C^1(-2)=0$.

PROOF. If one combines Remark 2.8 with Remark 2.5, $T_C^1(\nu)$ is zero for all $\nu\leqq -3$. In the proof of (1) we will use the notation in Proposition 2.9. Since

$$R_{i,j}^k=X_{j-1}f_{k,i+1}-X_if_{k,j}+X_kf_{i,j}=0$$

for all $0\leqq k<i<j-1\leqq d-g-1$, we have the following relations:

a) for $0\leqq i\leqq l-2$ and $i+2\leqq j\leqq l$

$$R_{j-1,l+2}^i=X_{l+1}f_{i,j}-X_{j-1}f_{i,l+2}+X_if_{j-1,l+2}=0,$$

b) for $l+1\leqq i\leqq d-g-2$ and $i+2\leqq j\leqq d-g$

$$R^0_{i,j}=X_{j-1}f_{0,i+1}-X_if_{0,j}+X_0f_{i,j}=0,$$

c) for $0\leqq i\leqq l-1$ and $l+2\leqq j\leqq d-g-1$

$$R^i_{j-1,d-g}=X_{d-g-1}f_{i,j}-X_{j-1}f_{i,d-g}+X_if_{j-1,d-g}=0.$$

Moreover, we have the relations

$$T^0_{m,m+1}=-X_1g_m+X_0g_{m+1}-X_0f_{0,m+1}+X_{l+1}\sum_{i=1}^{2g+1}a_if_{0,i+m+l+2}=0$$

for all $0\leqq m\leqq l-g-2$. Similarly, we get the relations $T^0_{m,m+1}$ for all $l-g-1\leqq m\leqq d-2g-3$. We write an element $r\in P^N$ as

$$(r_{ij}(0\leqq i\leqq l-2,\ i+2\leqq j\leqq l;\ l+1\leqq i\leqq d-g-2,\ i+2\leqq j\leqq d-g;$$
$$0\leqq i\leqq l-1,\ l+2\leqq j\leqq d-g),\ r_m(0\leqq m\leqq d-2g-2)),$$

where $P=k[X_0, X_1, \cdots, X_{d-g}]$ and $N=(d^2-(2g+1)d+g^2-g)/2$. For any relation $H=\sum h_{ij}f_{ij}+\sum h_mg_m=0$, we call $h=(h_{ij};h_m)$ the element of P^N corresponding to the relation H. Let r^k_{ij} (resp. $t^0_{m,m+1}$) be the element of P^N corresponding to the relation $R^k_{i,j}$ (resp. $T^0_{m,m+1}$).

For any element $\xi\in T^1_C(-2)$, there exists a homogeneous element $\theta\in \mathrm{Hom}_B(I/I^2, B)$ of degree -2 such that ξ is the image of θ by the homomorphism Φ in (2.1) Since elements of $\mathrm{Hom}_B(I/I^2, B)$ are given by N-tuples of elements of B, which we write as row vectors, we can think of θ as $(c_{ij}+I;c_m+I)$ whose entries are homogeneous of degree 0. If we set $c=(c_{ij};c_m)\in P^N$, then by Remark 2.1 we get the following:

$$r^i_{j-1,l+2}\cdot{}^tc\equiv \bmod I \text{ for } 0\leqq i\leqq l-2 \text{ and } i+2\leqq j\leqq l,$$

$$r^0_{i,j}\cdot{}^tc\equiv 0 \bmod I \text{ for } l+1\leqq i\leqq d-g-2 \text{ and } i+2\leqq j\leqq d-g,$$

$$r^i_{j-1,d-g}\cdot{}^tc\equiv 0 \bmod I \text{ for } 0\leqq i\leqq l-1 \text{ and } l+2\leqq j\leqq d-g-1,$$

$$t^0_{m,m+1}\cdot{}^tc\equiv 0 \bmod I \text{ for } 0\leqq m\leqq d-2g-3.$$

Hence we get $c_{ij}=0$ and if $d\geqq 2g+3$ we have $c_m=0$. Therefore θ is the zero map, that is to say, $\xi=0$. Hence we have $T^1_C(-2)=0$. Q.E.D.

By Remark 1.1 and Propositions 1.2, 1.4 and 2.10, we get

THEOREM 2.11. *Let the notation be as in* Notation 2.7. *If* $d\geqq \mathrm{Max}\{4g-3, 2g+3\}$, *then we have*

1) $T^1_C=T^1_C(-1)\oplus T^1_C(0)$,

2) $\dim_k T^1_C(-1)\leqq(g-1)g(g+1)$ *and* $\dim_k T^1_C(0)=4g-3$.

In particular, if $g=2$ *we get* $\dim_k T^1_C(-1)=6$.

REMARK 2.12. In the case $g \geqq 3$, the estimate $\dim_k T_C^1(-1) \leqq (g-1)g(g+1)$ given in Theorem 2.11 is not sharp. For example, if $g=3$ and $d=9$, 10, by calculation we get $\dim_k T_C^1(-1)=8<24$.

3. A k-basis for T_C^1 in the cases $g=2$ and $d=5$, 6, 7.

In this section we use the notation in Notation 2.7 and assume that $g=2$. Looking at a k-basis for T_C^1 we compute its dimension in the cases $d=5$, 6.

3.1. *A system of generators of the ideal I.*

1) The case $d=5$. I is generated by f_1, f_2 and f_3, where

$$f_1=-X_0^2X_1+X_2X_3^2+a_4X_1X_3^2+a_3X_1X_2X_3+a_2X_1^2X_3+a_1X_1^2X_2\,,\ f_2=X_1X_3-X_2^2\,,$$

$$f_3=X_0^2X_2-X_3^3-a_4X_2X_3^2-a_3X_1X_3^2-a_2X_1X_2X_3-a_1X_1^2X_3\,.$$

If we apply Proposition 2.9 to the cases $g=2$ and $d=6$, 7, we get the following:

2) The case $d=6$. I is generated by f_1, f_2, f_3 and f_4, where

$$f_1=X_1X_3-X_2^2\,,\ f_2=X_1X_4-X_2X_3\,,\ f_3=X_2X_4-X_3^2\,,$$

$$f_4=X_0^2-X_3X_4-a_4X_2X_4-a_3X_1X_4-a_2X_1X_3-a_1X_1X_2\,.$$

3) The case $d=7$. I is generated by $f_1, f_2, \cdots, f_8$, where

$$f_1=X_2X_4-X_3^2\,,\ f_2=X_2X_5-X_3X_4\,,\ f_3=X_3X_5-X_4^2\,,\ f_4=X_0X_3-X_1X_2\,,$$

$$f_5=X_0X_4-X_1X_3\,,\ f_6=X_0X_5-X_1X_4\,,$$

$$f_7=X_0^2-X_4X_5-a_4X_3X_5-a_3X_2X_5-a_2X_2X_4-a_1X_2X_3\,,$$

$$f_8=X_0X_1-X_5^2-a_4X_4X_5-a_3X_3X_5-a_2X_2X_5-a_1X_2X_4\,.$$

We set $N=3$ if $d=5$ and $N=(d^2-5d+2)/2$ if $d=6$, 7. Let $R=\{(g_1, \cdots, g_N) \in P^N \mid \sum_{i=1}^{N} g_if_i=0\}$ be the relation P-module, where P is the polynomial ring $k[X_0, X_1, \cdots, X_{d-2}]$ over k. Then elementary computations show the following:

3.2. *A system of generators of the relation P-module.*

1) The case $d=5$. r_1 and r_2 form a system of generators of R, where

$$r_1=(X_2,\ X_3^2+a_3X_1X_3+a_1X_1^2,\ X_1),\ r_2=(X_3,\ X_0^2-a_4X_3^2-a_2X_1X_3,\ X_2)\,.$$

2) The case $d=6$. $r_1, \cdots, r_5$ form a system of generators of R, where

$$r_1=(X_3,\ -X_2,\ X_1,\ 0),\ r_2=(X_4,\ -X_3,\ X_2,\ 0),\ r_3=(-f_4,\ 0,\ 0,\ f_1)\,,$$

$$r_4=(0,\ -f_4,\ 0,\ f_2),\ r_5=(0,\ 0,\ -f_4,\ f_3)\,.$$

3) The case $d=7$. $r_1, \cdots, r_{12}$ form a system of generators of R, where

$$r_1=(X_4, -X_3, X_2, 0, \cdots, 0),\ r_2=(X_5, -X_4, X_3, 0, \cdots, 0),$$
$$r_3=(X_0, 0, 0, X_3, -X_2, 0, 0, 0),\ r_4=(X_1, 0, 0, X_4, -X_3, 0, 0, 0),$$
$$r_5=(0, X_0, 0, X_4, 0, -X_2, 0, 0),\ r_6=(0, X_1, 0, X_5, 0, -X_3, 0, 0),$$
$$r_7=(0, 0, X_0, 0, X_4, -X_3, 0, 0),\ r_8=(0, 0, X_1, 0, X_5, -X_4, 0, 0),$$
$$r_9=(0, 0, 0, a_3X_5, a_1X_2+a_4X_5, a_2X_2+X_5, -X_1, X_0),$$
$$r_{10}=(a_1X_2+a_4X_5, a_2X_2+X_5, 0, X_0, 0, 0, -X_3, X_2),$$
$$r_{11}=(-a_3X_5, 0, a_2X_2+X_5, 0, X_0, 0, -X_4, X_3),$$
$$r_{12}=(0, -a_3X_5, -a_1X_2-a_4X_5, 0, 0, X_0, -X_5, X_4).$$

Let $\Phi: \mathrm{Hom}_{P/I}(I/I^2, P/I)\to T_C^1$ be the homomorphism in (2.1). Elements of $\mathrm{Hom}_{P/I}(I/I^2, P/I)$ are N-tuples of elements of P/I, which we write as row vectors. Then using Remark 1.1 and Definition 2.4, it is easy, albeit tedious, to check the following:

3.3. *A k-basic for* T_C^1.

1) The case $d=5$. We have $T_C^1=T_C^1(-2)\oplus T_C^1(-1)\oplus T_C^1(0)$. The images of θ_1 and θ_2 by Φ form a k-basis of $T_C^1(-2)$. The images of $X_0\theta_1$, $X_1\theta_1$, $X_2\theta_1$, $X_3\theta_1$, $X_1\theta_2$, and θ_3 by Φ form a k-basis of $T_C^1(-1)$. The images of $X_0X_3\theta_1$, $X_1X_2\theta_1$, $X_1X_3\theta_1$, $X_2X_3\theta_1$ and $X_0\theta_3$ by Φ form a k-basis of $T_C^1(0)$. Here

$$\theta_1=(-X_2, 0, X_3),\ \theta_2=(X_1, 0, -X_2),\ \theta_3=(0, X_2, -X_0^2+a_4X_3^2+a_2X_1X_3).$$

Therefore we get $\dim_k T_C^1=13$.

2) The case $d=6$. We have $T_C^1=T_C^1(-2)\oplus T_C^1(-1)\oplus T_C^1(0)$. The image of θ_1 forms a k-basis of $T_C^1(-2)$. The images of $X_1\theta_1$, $X_2\theta_1$, $X_3\theta_1$, $X_4\theta_1$, θ_2 and θ_3 form a k-basis of $T_C^1(-1)$. The images of $X_1X_3\theta_1$, $X_2X_3\theta_1$, $X_3^2\theta_1$, $X_0\theta_2$ and $X_0\theta_3$ form a k-basis of $T_C^1(0)$. Here

$$\theta_1=(0, 0, 0, 1),\ \theta_2=(X_1, 0, -X_3, 0),\ \theta_3=(X_2, X_3, 0, 0).$$

Therefore we get $\dim_k T_C^1=12$.

3) The case $d=7$. We have $T_C^1=T_C^1(-1)\oplus T_C^1(0)$. The images of θ_1, θ_2, θ_3, θ_4, θ_5 and θ_6 form a k-basis of $T_C^1(-1)$. The images of $X_1\theta_2$, $X_1\theta_6$, $X_2\theta_6$, $X_3\theta_6$ and $X_4\theta_6$ form a k-basis of $T_C^1(0)$. Here

$$\theta_1=(X_3, 0, -X_5, 0, X_1, 0, a_4X_5, -a_1X_3),\ \theta_2=(0, X_3, X_4, 0, 0, X_1, X_5, -a_2X_3),$$
$$\theta_3=(0, \cdots, 0, X_0, X_1),\ \theta_4=(0, \cdots, 0, X_2, X_3),\ \theta_5=(0, \cdots, 0, X_3, X_4),$$
$$\theta_6=(0, \cdots, 0, X_4, X_5).$$

Therefore we get $\dim_k T^1_C = 11$.

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Institute of Mathematics
University of Tsukuba
Ibaraki, Japan