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ANALYSIS OF THE ACTION OF A PSEUDODIFFERENTIAL OPERATOR OVER $(C_{\Omega|X})_{T_M^*X}$

By

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Abstract. Let M be a real analytic manifold, $\Omega\subset M$ an open set, X a complexification of M, P a pseudodifferential operator on X.

Using the action of P over holomorphic functions on suitable domains of X, by [B-S], and the theory of representation of microfunctions at the boundary $(C_{\Omega|X})_{T_M^*X}$, by [S-Z], [Z], we show that P defines in a natural manner a sheaf morphism of $(C_{\Omega|X})_{T_M^*X}$. Let us note that the hypotheses on $\partial\Omega$ are here weaker than in [K 2] where $\partial\Omega$ is supposed to be analytic.

We also easily prove that P is an isomorphism of $(C_{\Omega|X})_{T_M^*X}$ out of $T_M^*X\cap\mathrm{char}(P)$ (both by composition rule and by non-characteristic deformation).

We shall apply the method of this paper in our forthcoming work on regularity at the boundary for solutions of P (cf. [D'A-Z]).

0. Preliminaries.

Let M be a C^{ω}-manifold, X a complexification of M. We denote by T^*M, T^*X the cotangent bundles to M, X, and by T_M^*X the conormal bundle to M in X; in particular we denote by T_X^*X the zero section of T^*X. We set $\dot{T}^*X=T^*X\setminus T_X^*X$.

For subsets S, $V\subset X$ one denotes by $C(S, V)$ the normal cone to S along V and by $N(S)$ the normal cone to S in X; these are objects of TX (cf. [K-S]).

We will denote by $\mathcal{B}_M$ (resp. $C_{M|X}$) the sheaf of hyperfunctions on M (resp. of microfunctions).

Let Ω be an open subset of M and let $C_{\Omega|X}$ be the complex of sheaves defined in [S] (cf. also [K 1]). In this paper we shall assume that:

(1) Ω is C^{ω}-convex.

(2) $H^0(C_{\Omega|X})=(C_{\Omega|X})_{T_M^*X}$.

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(One can prove that if Ω has a C^2-boundary then (1) and (2) are satisfied.)

Let γ be an open subset of $\bar{\Omega}\times_M T_M X$ with convex conic fibers; a domain $U\subset X$ is said to be an *Ω-tuboid* with profile γ iff $C(X\setminus U, \bar{\Omega})\cap\gamma_1=\emptyset$ for some open set $\gamma_1\subset TX$ with convex conic fibers such that $\bar{\gamma}_1\supset\sigma(N(\Omega))$, $\rho(\gamma_1)\supset\gamma$. Here

$$T_M X \xleftarrow{\rho} M\times_X TX \xleftarrow{\sigma} TM$$

are the canonical maps.

If one chooses coordinates $x\in M$, $z=x+\sqrt{-1}y\in X$ then U is an Ω-tuboid with profile γ iff for every $\gamma'\Subset\gamma$ there exists $\varepsilon=\varepsilon_{\gamma'}$ such that

$$U\supset\{(x, y)\in\Omega\times_M\gamma';\ |y|<\varepsilon\,\mathrm{dist}(x, \partial\Omega)\wedge 1\}.$$

(Here we identify $T_M X\cong X$ in local coordinates.)

For example if $\Omega=\{x_1>0\}$ and $\gamma=\bar{\Omega}\times\{y_n>0\}$, the set

$$U=\{(x, y)\in X;\ x\in\Omega,\ y_n x_1>y'^2\}$$

is an Ω-tuboid with profile γ.

We now recall how sections of $(\mathcal{C}_{\Omega|X})_{T_M^*X}$ can be represented as boundary values of holomorphic functions (cf. [S-Z], [Z]).

Take $f\in\pi_*\Gamma_{\gamma^{\circ a}}((\mathcal{C}_{\Omega|X})_{T_M^*X})(S)$, S a ball in a local chart $M\cong\boldsymbol{R}^n$, $\pi\colon T_M^*X\to M$ the projection, $\gamma\subset\bar{\Omega}\times_M T_M X$ open with convex conic fibers over S and with $\pi(\gamma)=S\cap\bar{\Omega}$. Then one can write f as the boundary value $b(F)$, $F\in\mathcal{O}_X(U)$, U being both an Ω-tuboid with profile γ and a domain of holomorphy. At this subject we refer the reader to [Z]. Note here that the results of [S-Z], [Z] concerning the representation of the stalks $(\pi_*\Gamma_{\gamma^{\circ a}}((\mathcal{C}_{\Omega|X})_{T_M^*X}))_x$, $x\in\pi(\gamma)$, easily extend to global sections over vectors spaces.

For $f\in\Gamma_\Omega(\mathcal{B}_M)_{\mathring{x}}$, $\mathring{x}\in\partial\Omega$, one denotes by $SS_\Omega(f)$ the support of f identified to a section of $(\mathcal{C}_{\Omega|X})_{T_M^*X}$ in $\pi^{-1}(\mathring{x})$. On account of the above characterization one proves that given $(\mathring{x}, \sqrt{-1}\mathring{\eta})\in T_M^*X$ one has $(\mathring{x}, \sqrt{-1}\mathring{\eta})\notin SS_\Omega(f)$ iff $f=\sum b(F_j)$ with F_j holomorphic in U_j, U_j Ω-tuboid whose profile γ_j verifies $\sqrt{-1}\mathring{\eta}\notin((\gamma_j)_{\mathring{x}})^{*a}$.

One also gets the following decomposition of microsupport.

Let $f\in\Gamma_\Omega(\mathcal{B}_M)(S)$ and decompose $\dot{\pi}^{-1}(S)\cap SS_\Omega(f)\subset\bigcup_j\gamma_j^{*a}$, $((\gamma_j^{*a})_x$ closed proper convex $\forall x\in\pi(\gamma_j)=S\cap\bar{\Omega})$. Then one can write $f=\Sigma b(F_j)$, $F_j\in\mathcal{O}_X(U_j)$, U_j Ω-tuboid (of holomorphy) with profile γ_j.

Let $P\in\mathcal{E}_{X,t^*}$ be a pseudodifferential operator of order m defined in a neighborhood of a point $t^*\in\dot{T}_M^*X$. Fix a system of coordinates near t^*: $x=(x_1, \cdots, x_n)\in M\cong\boldsymbol{R}^n$, $z=x+\sqrt{-1}y\in X\cong\boldsymbol{C}^n$, $(z;\zeta)\in T^*X\cong\boldsymbol{C}^n\times\boldsymbol{C}^n$, $\zeta=\xi+\sqrt{-1}\eta$, $(x;\sqrt{-1}\eta)\in T_M^*X\cong\boldsymbol{R}^n\times\sqrt{-1}\boldsymbol{R}^n$, $t^*=(\mathring{x};\sqrt{-1}\mathring{\eta})$, $\mathring{\eta}=(0, \cdots, 0, 1)$.

One can write the symbol of P as

$$\sum_{-\infty}^{m} P_l(z, \zeta), \qquad P_l \in \mathcal{O}_{T^*X}(\tilde{U}\times\tilde{W}),$$

$\tilde{U}\times\tilde{W}$ open subset of T^*X, $\tilde{W}\supset\{\zeta: |\zeta_i|\leqq k_0|\zeta_n|, i=1, \cdots, n-1\}$, $\tilde{U}\ni\mathring{x}$, P_l homogeneous in ζ of degree l,

$$\sup_{(z\in\tilde{U},\, |\zeta_i|\leqq k_0|\zeta_n|)} |\zeta|^l |P_{-l}(z, \zeta)| \leqq M_0^{l+1} l!\,.$$

In [B-S] Bony and Schapira have shown that, under these conditions, if one fixes a complex hyperplane $\Sigma=\{z: \langle z, \mathring{\eta}\rangle=\mathring{x}_n+\sqrt{-1}\varepsilon\}$ it is possible to define an "action" for P over holomorphic functions on suitable domains.

DEFINITION 0.1. Let $k>0$. An open convex subset U of $\boldsymbol{C}^n$ is said to be $k-\Sigma$-*plat* if:

$$\forall z\in U,\ \forall\tilde{z}\in\Sigma \text{ such that } |z_n-\tilde{z}_n|\geqq k|z_i-\tilde{z}_i|,$$
$$i=1, \cdots, n-1 \text{ we have } \tilde{z}\in U\cap\Sigma\,.$$

We refer the reader to [B-S] for the definition of the operator P_Σ over holomorphic functions defined in domains $k_0-\Sigma-$plat with diameter $\leqq 1/M_0$.

1. Definition of the action.

Let $P\in\mathcal{E}_X(V)$ be a pseudodifferential operator on an open set $V\subset\dot{T}^*_M X$.

Our aim is to define an action for P over sections of $(\mathcal{C}_{\Omega|X})_{T^*_M X}$. More precisely, if α is the map

$$\alpha: \pi^{-1}(\Gamma_\Omega(\mathcal{B}_M)) \longrightarrow (\mathcal{C}_{\Omega|X})_{T^*_M X}\,,$$

we will show how P operates on $\alpha(\pi^{-1}(\Gamma_\Omega(\mathcal{B}_M)))|_V$.

To this end we will proceed in several steps.

First we will make the operator act on cohomology classes of holomorphic functions defined on Ω-tuboids with prescribed profile. Since each germ of $\alpha(\pi^{-1}(\Gamma_\Omega(\mathcal{B}_M)))$ is represented as boundary value of a holomorphic function defined on a domain as above we can interpret the previous action as an action over $(\alpha(\pi^{-1}(\Gamma_\Omega(\mathcal{B}_M))))_{(x, \sqrt{-1}\eta)}$, for every $(x, \sqrt{-1}\eta)\in V$.

Finally we will show how P operates on $\alpha(\pi^{-1}(\Gamma_\Omega(\mathcal{B}_M)))|_V$ glueing up the actions over each fiber.

Let $(\mathring{x}, \sqrt{-1}\mathring{\eta})\in V$ with $\mathring{x}\in\partial\Omega$.

We fix a system of local coordinates so that $\mathring{\eta}=(0, \cdots, 0, 1)$, $\Omega=\{x\,;\,\rho(x)>0\}$ (where $\rho(x)=x_1-\varphi(x_2, \cdots, x_n)$ for a convex function φ), and the symbol of P is defined on a set $\tilde{U}\times\tilde{V}$ as in the previous section.

Let

$$S=\{x\,;\,|x-\mathring{x}|<r\},$$

with $r<1/M_0$ such that $S\subset\tilde{U}$ and let Γ be an open proper convex cone of $\boldsymbol{R}^n$ such that

$$\Gamma^{*a}\subset\{\eta\,;\,|\eta_i|<k\eta_n,\ i=1,\cdots,n-1\},\quad k<\frac{k_0}{5}.$$

For $b>0$ let

$$U=U_{\Gamma,b}=((S\cap\Omega)+\sqrt{-1}\,\Gamma)\cap\{z\,;\,|y|<b\rho(x)\}\,. \tag{1.1}$$

Let $F\in\mathcal{O}_X(U)$; we shall now define how P operates on $f=\alpha(b(F))$.

For $\bar{x}\in\partial\Omega\cap S$, $\eta\in\Gamma^{*a}$, $|\eta|=1$, $\nu\in N_+$, let

$$\Sigma=\Sigma_{\bar{x},\eta,\nu}=\left\{z\,;\,\langle z-\bar{x},\,\eta\rangle=\sqrt{-1}\frac{1}{\nu}\right\},$$

$$\Omega_{\bar{x}}=\bar{x}+\left\{x\,;\,x_1>\frac{1}{k}|x'|\right\};$$

let $\rho_{\bar{x}}$ be the function defined by $\Omega_{\bar{x}}=\{x\,;\,\rho_{\bar{x}}>0\}$. Observe that $\Omega\cap S=\bigcup_{\bar{x}}\Omega_{\bar{x}}\cap S$ for k small.

Consider

$$U_{\bar{x}}=U_{\bar{x},\Gamma,b}=((S\cap\Omega_{\bar{x}})+\sqrt{-1}\,\Gamma)\cap\{z\,;\,|y|<b\rho_{\bar{x}}(x)\},$$

$$U_{\bar{x},\nu}=\{z\,;\,|z_n-\tilde{z}_n|\geq k|z_i-\tilde{z}_i|,\ \tilde{z}\in\Sigma_{\bar{x},\mathring{\eta},\nu}\Longrightarrow\tilde{z}\in U\cap\Sigma_{\bar{x},\mathring{\eta},\nu}\},$$

a k-$\Sigma_{\bar{x},\mathring{\eta},\nu}$-plat set.

One can rewrite

$$U_{\bar{x},\nu}=\left\{z=(z',\,z_n)\in X\,;\,\left|z_n-\bar{x}_n-\frac{\sqrt{-1}}{\nu}\right|<k\ \mathrm{dist}\left(\left(z',\,\bar{x}_n+\frac{\sqrt{-1}}{\nu}\right),\,\Sigma_{\bar{x},\mathring{\eta},\nu}\setminus U\right)\right\},$$

and hence deduce at once the convexity of $U_{\bar{x},\nu}$.

REMARK 1.2. $U_{\bar{x},\nu}$ is $(1/2(k_0-k)-|\eta-\mathring{\eta}|)$-$\Sigma_{\bar{x},\mathring{\eta},\nu}$-plat (due to $|\eta-\mathring{\eta}|<k$, $k<k_0/5$) thus $U_{\bar{x},\nu}$ is (k_0-k)-$\Sigma_{\bar{x},\eta,\nu}$-plat and moreover it contains the largest $1/2(k_0-k)$-$\Sigma_{\bar{x},\eta,\nu}$-plat set V such that

$$V\cap\Sigma_{\bar{x},\eta,\nu}=U_{\bar{x},\nu}\cap\Sigma_{\bar{x},\eta,\nu}\,.$$

One has

LEMMA 1.3. *For every $S'\Subset S$ there exists $b'=b'_{S'}<b$ and for every $\Gamma'\Subset\Gamma$ there exists $b''=b''_{\Gamma'}$ so that for every $\bar{x}$:*

(1) $\bigcup_{\nu}(U_{\bar{x},\nu}\cap U_{\bar{x}})\supset(\Omega_{\bar{x}}+\sqrt{-1}\,\Gamma)\cap\{z\,;\,y_n<b'\rho_{\bar{x}}(x)\}\cap B'$,

(2) $\bigcup_{\nu}(U_{\bar{x},\nu}\cap\{z: y_n<1/\nu\})\supset\{z\,;\,x\in\Omega_{\bar{x}},\ y\in-b''\rho_{\bar{x}}(x)\mathring{\eta}+\Gamma',\ y_n<b'\rho_{\bar{x}}(x)\}\cap B'$,

where $B'=S'+\sqrt{-1}\boldsymbol{R}^n$.

PROOF. (1) can be easily proved by taking b' such that $\bigcup_{\nu}(U_{\bar{x},\nu}\cap U_{\bar{x}})\cap\{z;\, x\in S',\, y_n<b'\rho_{\bar{x}}(x)\}$ is a convex set. As for (2) one sees that for some b'' one has that for every z in the right hand side of (2) there exists ν such that:

$$|w_n-z_n|>k|w'-z'|,\qquad w\in\Sigma_{\bar{x},\mathring{\eta},\nu}\Longrightarrow w\in U_{\bar{x}}\cap\Sigma_{\bar{x},\mathring{\eta},\nu}=U_{\bar{x},\nu}.$$

To this end b'' has to be chosen small with respect to ε where ε is such that

$$\Gamma'^{*a}\subset\{\eta;\,|\eta'|<(k-\varepsilon)\eta_n\}. \qquad ■$$

In particular $\bigcup_{\nu}(U_{\bar{x},\nu}\cap U_{\bar{x}})$ is a $\Omega_{\bar{x}}$-tuboid with profile $\bar{\Omega}_{\bar{x}}+\sqrt{-1}\Gamma$ and $\bigcup_{\nu}(U_{\bar{x},\nu}\cap\{z:\, y_n<1/\nu\})$ is a $\Omega_{\bar{x}}$-tuboid with profile $\bar{\Omega}_{\bar{x}}+\sqrt{-1}\boldsymbol{R}^n$.

REMARK 1.4. We also observe that $(U_{\bar{x},\nu+1}\cap U_{\bar{x},\nu})$ is k-$\Sigma_{x,\eta,\nu}$-plat and

$$(U_{\bar{x},\nu+1}\cap U_{\bar{x},\nu})\cap\Sigma_{\bar{x},\mathring{\eta},\nu+1}=(U_{\bar{x}}\cap U_{\bar{x},\nu})\cap\Sigma_{\bar{x},\mathring{\eta},\nu+1}.$$

Set

$$\tilde{U}_{\bar{x},\nu}=U_{\bar{x},\nu}\cap\{z;\, y\in -b''\rho_{\bar{x}}(x)\mathring{\eta}+\Gamma',\, y_n<b'\rho_{\bar{x}}(x)\},$$

and note that, by a suitable choice of b', b'', $\tilde{U}_{\bar{x},\nu}\cap B'$ and $\bigcup_{\nu}\tilde{U}_{\bar{x},\nu}\cup B'$ are Stein domains (cf. the statement before Remark 1.2). $P_{\Sigma_{\bar{x},\mathring{\eta},\nu}}F\in\mathcal{O}_X(U_{\bar{x},\nu}\cap U_{\bar{x}})$ as defined in [B-S]. By [B-S, Théorème 2.5.1], and by Remark 1.4, we get

$$P_{\Sigma_{\bar{x},\mathring{\eta},\nu}}F-P_{\Sigma_{\bar{x},\mathring{\eta},\nu'}}F\in\mathcal{O}_X(U_{\bar{x},\nu}\cap U_{\bar{x},\nu'}).\tag{1.5}$$

Put

$$H_{\nu,\nu'}=P_{\Sigma_{\bar{x},\mathring{\eta},\nu}}F-P_{\Sigma_{\bar{x},\mathring{\eta},\nu'}}F;$$

these functions satisfy

$$H_{\nu,\nu'}+H_{\nu',\nu}=0$$
$$H_{\nu,\nu'}+H_{\nu',\nu''}+H_{\nu'',\nu}=0.$$

Since $H^1(\bigcup_{\nu}\tilde{U}_{\bar{x},\nu}\cap B',\mathcal{O}_X)=0$ then there exists $G_\nu\in\mathcal{O}_X(\tilde{U}_{\bar{x},\nu}\cap B')$ such that $G_\nu-G_{\nu'}=H_{\nu,\nu'}$ in $\tilde{U}_{\bar{x},\nu}\cap\tilde{U}_{\bar{x},\nu'}\cap B'$.

Summarizing up, to every $F\in\mathcal{O}_X(U)$, we can associate a function

$$P_{\Sigma_{\bar{x},\mathring{\eta}}}F\in\mathcal{O}_X(\cup_\nu\tilde{U}_{\bar{x},\nu}\cap U_{\bar{x}}\cap B')\tag{1.6}$$

setting

$$P_{\Sigma_{\bar{x},\mathring{\eta}}}F|_{\tilde{U}_{\bar{x},\nu}\cap U_{\bar{x}}\cap B'}=P_{\Sigma_{\bar{x},\mathring{\eta},\nu}}F-G_\nu|_{\tilde{U}_{\bar{x},\nu}\cap U_{\bar{x}}\cap B'}.$$

This function satisfies

$$P_{\Sigma_{\bar{x},\mathring{\eta}}}F=P_{\Sigma_{\bar{x},\mathring{\eta},\nu}}F\ \text{ in }\ H^1_{X\setminus U_{\bar{x}}}(\tilde{U}_{\bar{x},\nu}\cap B',\mathcal{O}_X)\xleftarrow{\ \sim\ }\frac{\Gamma(\tilde{U}_{\bar{x},\nu}\cap U_{\bar{x}}\cap B',\mathcal{O}_X)}{\Gamma(\tilde{U}_{\bar{x},\nu}\cap B',\mathcal{O}_X)}.$$

REMARK 1.7.

(i) Let $\{\mu_\nu\}$, $\mu_\nu \searrow 0$, be another sequence and let $P'_{\Sigma_{\bar{x}},\mathring{\eta}}F$ be a function defined as in (1.6) (with $1/\nu$ replaced by μ_ν). Similarly as before we get

$$P'_{\Sigma_{\bar{x}},\mathring{\eta}}F - P_{\Sigma_{\bar{x}},\mathring{\eta}}F \in \mathcal{O}_X(\cup_\nu(U_{\bar{x},\nu} \cup U_{\bar{x},\mu_\nu}))\,.$$

On the other hand, since

$$N(\cup_\nu(\tilde{U}_{\bar{x},\nu} \cap \tilde{U}_{\bar{x},\mu_\nu})) \cap \sigma((S' \cap \bar{\Omega}_{\bar{x}}) \times_M \dot{T}M) \neq \emptyset\,.$$

then $\alpha(b(P_{\Sigma_{\bar{x}},\mathring{\eta}}F)) = \alpha(b(P'_{\Sigma_{\bar{x}},\mathring{\eta}}F))$ in $\pi^{-1}(\Omega_{\bar{x}} \cap S')$.

(ii) In the same way one proves that $\alpha(b(P_{\Sigma_{\bar{x}},\mathring{\eta}}F))|_{\pi^{-1}(\bar{\Omega}_{\bar{x}} \cap S')}$ does not depend neither on the choice of the constant b nor on the sets Γ, S (as long as the conditions of Remark 1.2 are satisfied).

And now we consider Ω instead of $\Omega_{\bar{x}}$.

Due to the convexity of Ω we observe that

$$\hat{U} = (\bigcup_{\nu,\, \bar{x} \in \partial\Omega \cap S}) \tilde{U}_{\bar{x},\nu} \cap B'$$

is still a Stein domain.

Reasoning as before we get

PROPOSITION 1.8. *To any $F \in \mathcal{O}_X(U)$ we can associate a holomorphic function*

$$P_{\Sigma_{\mathring{\eta}}}F \in \mathcal{O}_X(\hat{U} \cap U)\,,$$

such that

$$P_{\Sigma_{\mathring{\eta}}}F = P_{\Sigma_{\bar{x}},\mathring{\eta}}F \quad in \quad H^1_{X \setminus (\hat{U} \cap U)}(\cup_\nu \tilde{U}_{\bar{x},\nu} \cap B', \mathcal{O}_X)\,.$$

Note that, on the same line as Lemma 1.3, one proves that $\hat{U} \cap U$ is a Ω-tuboid with profile $\bar{\Omega} + \sqrt{-1}\Gamma$.

REMARK 1.9. By its very definition it is clear that $\alpha(b(P_{\Sigma_{\mathring{\eta}}}F))|_{\pi^{-1}(W)}$ (for $W \subset \Omega$) equals $P_{\Sigma_{\mathring{\eta}}}\alpha(b(F))|_{\pi^{-1}(W)}$ defined in [B-S].

The compatibility with the action of P on $\mathcal{C}_{M|X}$ is thus assured.

Let $f \in \alpha(\pi^{-1}\Gamma_\Omega(\mathcal{B}_M))(W)$, $W = S + \sqrt{-1}\,\mathrm{int}\,\Gamma^{*a}$; on account of the first section we can write $f = b(F)$, $F \in \mathcal{O}_X(U)$, U being an Ω-tuboid with profile $(S \cap \bar{\Omega}) + \sqrt{-1}\Gamma$, i.e. $\forall S' \Subset S$, $\forall \Gamma' \Subset \Gamma$, $\exists b$ so that $U \supset U'$ where

$$U' = ((S' \cap \Omega) + \sqrt{-1}\Gamma') \cap \{z : |y| < b\rho(x)\}, \tag{1.10}$$

According to Proposition 1.8 we can define an holomorphic function $P_{\Sigma_{\mathring{\eta}}}(F|_{U'})$ and by Remark 1.7 $P_{\Sigma_{\mathring{\eta}}}(F|_{U'})|_{S+\sqrt{-1}\,\mathrm{int}\,\Gamma^{*a}}$ does not depend on U'.

It does not depend on the choice of the representative F neither. In fact if $b(F-F')|_W=0$ then $b(F-F')=\Sigma_j b(F_j)$, $F_j\in\mathcal{O}_X(U'_j)$, $U'_j=((S'\cap\Omega)+\sqrt{-1}\Gamma'_j)\cap\{z: |y|<b_j\rho(x)\}$ where $\Gamma'_j\Subset\Gamma_j$ with $\Gamma^*_j\cap\text{int}\,\Gamma^*=\emptyset$.

Reasoning as in the proof of Proposition 1.8 we get

$$P_{\Sigma_{\mathring{\eta}}}F_j\in\mathcal{O}_X(U_j\cap\{z: |y|<b'_j\rho(x)\}\cap\{S''+\sqrt{-1}\boldsymbol{R}^n))$$

and thus

$$b(P_{\Sigma_{\mathring{\eta}}}F-P_{\Sigma_{\mathring{\eta}}}F')|_{S''+\sqrt{-1}(\boldsymbol{R}^n\setminus(\cup_j\Gamma'^{*a}_j))}=0\,;$$

for $S''+\sqrt{-1}(\boldsymbol{R}^n\setminus(\cup_j\Gamma'^{*a}_j))\nearrow W$ we have thus given an action of P over f.

REMARK 1.11. By similar arguments as before one gets:

$$SS_\Omega b(P_{\Sigma_{\mathring{\eta}}}F)\subset SS_\Omega b(F)\,.$$

And now we have to define an action over generic sections.

Given an open subset $V'\subset V\cap\dot{\pi}^{-1}(\bar{\Omega})$ and $f\in\alpha(\pi^{-1}\Gamma_\Omega(\mathcal{B}_M))(V')$, we can find an open covering $\{V_{t*}\}_{t*\in V'}$, $V_{t*}=S_{t*}+\sqrt{-1}\,\text{int}\,(\Gamma_{t*})^{*a}$ so that $f|_{V_{t*}}=b(F_{t*})$ with $F_{t*}\in\mathcal{O}_X(U'_{t*})$ (U'_{t*} as in (1.1) with S, Γ replaced by S'_{t*}, Γ'_{t*} respectively). Let $t^{*i}=(x^i,\sqrt{-1}\eta^i)$ $i=1, 2$; then

PROPOSITION 1.12. *$\alpha(b(P_{\Sigma_{\eta^1}}F_{t*1})-b(P_{\Sigma_{\eta^2}}F_{t*2}))=0$ in $V_{t*1}\cap V_{t*2}$.*

PROOF. Let $t^{*3}\in V_{t*1}\cap V_{t*2}$ then it is enough to show that

$$\alpha(b(P_{\Sigma_{\eta^1}}F_{t*1}))_{t*3}=\alpha(b(P_{\Sigma_{\eta^3}}F_{t*3}))_{t*3}\,. \tag{1.13}$$

First notice that it is possible to take $F_{t*1}=F_{t*3}$ due to Remark 1.7.

If $\eta^1=\eta^3$ (1.13) is then obvious.

If $x^1=x^3$ (1.13) follows from Remark 1.2, [B-S, Remarque 2.5.3] and from an analogous of Lemma 1.3. ■

We have then shown that the sections $\{b(P_{\Sigma_\eta}F_{t*})\}_{t*}$ define a section of $(\mathcal{C}_{\Omega|X})_{T^*_MX}(V')$ which, of course, will be denoted by Pf.

Let V be an open set of $\dot{T}^*_MX$ with proper convex hull, and let $f\in\Gamma(V',\alpha(\pi_*^{-1}(\Gamma_\Omega\mathcal{B}_M)))$, $V'\subset V$.

Then we can represent $f=\alpha(b(F))|_{V'}$, $F\in\mathcal{O}_X(U')$ where U' is a Ω-tuboid with profile $\text{int}\,(V'^{*a})$ (in fact one can find $\tilde{f}\in\mathcal{B}_M$ such that $SS\tilde{f}\subset\bar{V}'$ and $\alpha(\tilde{f}|_\Omega)|_{V'}-f=0$.)

Summarizing up the above results one gets:

THEOREM 1.14. *Let $P\in\mathcal{E}_X(V)$ then P is a sheaf endomorphism of $\alpha(\pi^{-1}\Gamma_\Omega\mathcal{B}_M)|_V$.*

COROLLARY 1.15. *Let $\partial\Omega$ be analytic; then P is a sheaf endomorphism of*

$(\mathcal{C}_{\Omega|X})_{T_M^*X}|_V$.

PROOF. Since $(\mathcal{C}_{\Omega|X})_{T_M^*X}$ is conically flabby (cf. [S-Z]), then $\alpha: \pi^{-1}\Gamma_\Omega\mathcal{B}_M \to (\mathcal{C}_{\Omega|X})_{T_M^*X}$ is surjective. ■

REMARK 1.16. In describing the action of P over $\mathcal{C}_{M|X}$ one can replace the language of P_Σ by the language of the γ-topology (cf. [K-S]).

Thus let X_γ be the space X endowed with the γ-topology and $\phi_\gamma: X \to X_\gamma$ the canonical map. Let $\Omega_1 \supset \Omega_0$ be two γ-open sets and set $V = \mathrm{int}(\Omega_1 \setminus \Omega_0) \times \mathrm{int}\,\gamma^{*a}$. We have:

(1) $\mathcal{O}_X \cong \phi_\gamma^{-1} R\Gamma_{\Omega_1 \setminus \Omega_0} R\phi_{\gamma*}\mathcal{O}_X$ in $D^b(X; V)$;

(2) $\phi_\gamma^{-1} R\Gamma_{\Omega_1 \setminus \Omega_0} R\phi_{\gamma*}\mathcal{O}_X$ is a $\Gamma(D \times \mathrm{int}\,\gamma^{*a}, \mathcal{E}_X)$-module ($D \subset X$ a γ-round set).

By applying $\mu hom(\mathbf{Z}_M, \cdot) \otimes \omega_{M|X}[n]$ to both sides of (1) one gets the conclusion.

According to a private communication by P. Schapira the same procedure could be applied for $\mathbf{Z}_M$ replaced by $\mathbf{Z}_\Omega$.

2. Elliptic regularity at the boundary.

Let P, Q be pseudodifferential operators in an open set $V \subset \dot{T}_M^*X$ and let $P \cdot Q$ denote their formal composition (cf. [B-S]).

If Q has negative order we have by [B-S, Proposition 2.1.2] that

$$P_{\Sigma_{\bar{x},\hat{\eta},\nu}} \circ Q_{\Sigma_{x,\sqrt{-1}\eta,\nu}} = (P \cdot Q)_{\Sigma_{\bar{x},\hat{\eta},\nu}} \qquad \text{in } \mathcal{O}_X$$

and hence

$$P \cdot Q = P \circ Q, \qquad \text{in } (\mathcal{C}_{\Omega|X})_{T_M^*X}. \tag{2.1}$$

In particular if P is a pseudodifferential operator of positive order whose principal symbol p never vanishes on V, we get:

$$P \text{ is an isomorphism of } \alpha(\pi^{-1}\Gamma_\Omega(\mathcal{B}_M))_V. \tag{2.2}$$

To this end one only needs to write $1 = P \cdot P^{-1}$ at any $t^* \in V$ and use (2.1) which is valid since P^{-1} is of negative order.

We remark that it would have been possible to get (2.2) without using (2.1).

Assume $p \neq 0$ in $\{z: |z - \bar{x}| < r\} \times \{\zeta: |\zeta_i| \leqq k_1 |\zeta_n|\}$. Take $k < \mathrm{int}\{k_0/3, k_1\}$ and let $S, \Gamma, U_{\bar{x}}, U_{\bar{x},\nu}$ be defined as in section 1. Recall in particular that $U_{\bar{x},\nu}$ and $U_{\bar{x},\nu} \cap U$ are $k - \Sigma_{\bar{x},\hat{\eta},\nu}$-plat, and recall that $N(\bigcup_\nu U_{\bar{x},\nu}) \cap \sigma((S' \cap \bar{\Omega}) \times_M \dot{T}M) \neq \emptyset$.

(i) For every $F \in \mathcal{O}_X(U)$ there exists $G_{\bar{x},\nu} \in \mathcal{O}_X(U_{\bar{x}} \cap U_{\bar{x},\nu})$ so that $P_{\Sigma_{\bar{x},\hat{\eta},\nu}} G_{\bar{x},\nu} = F$. In fact this solution exists in a neighborhood of $\Sigma_{\bar{x},\hat{\eta},\nu} \cap U_{\bar{x}}$ by Cauchy-Kovalevsky's theorem and then it extends to $U_{\bar{x}} \cap U_{\bar{x},\nu}$ by [B-S, Théorème 2.5.4].

(ii) By a similar argument

$$P_{\Sigma_{\bar{x},\hat{\eta},\nu}}G_{\bar{x},\nu}-P_{\Sigma_{\bar{x},\hat{\eta},\nu}}G_{\bar{x},\nu'}=0 \text{ implies } G_{\bar{x},\nu}-G_{\bar{x},\nu'}\in\mathcal{O}_X(U_{\bar{x},\nu}\cap U_{\bar{x},\nu'}\cap B')$$

Reasoning as in the first section we can find a function $G\in\mathcal{O}_X(U\cap B'\cap\{z: |y|<b'\rho(x)\})$ such that $G-G_{\bar{x},\nu}\in\mathcal{O}_X(\tilde{U}_{\bar{x},\nu}\cap B')$. In particular

$$P_{\Sigma_{\bar{x},\hat{\eta}}}G-F\in\mathcal{O}_X((\textstyle\bigcup_\nu\tilde{U}_{\bar{x},\nu})\cap B'),$$

with $\tilde{U}_{\bar{x},\nu}$ defined as in Remark 1.4.

(iii) If $G\in\mathcal{O}_X(U)$, $P_{\Sigma_{\bar{x},\hat{\eta}}}G\in\mathcal{O}_X(U')$, $U'\supset((S'\cap\Omega)+\sqrt{-1}\Gamma')\cap\{z: |y|<b'\rho(x)\}$ with $\Gamma'\supset\Gamma$, then $G\in\mathcal{O}_X(U')$. (Once more U' is chosen, without loss of generality, so that $U'\cap U_{\bar{x},\nu}$ is $k-\Sigma_{\bar{x},\hat{\eta},\nu}$--plat.)

Collecting these results (for different S, Γ), one gets (2.2).

In fact let $g\in\Gamma_{\bar{S}+\sqrt{-1}\bar{\Gamma}}(S\times_M T_M^*X, (\mathcal{C}_{\Omega|X})_{T_M^*X})$, $I=\text{int}\,\Gamma^{*a}$ (with S, Γ as above) and let $Pg=0$.

Hence $g=b(G)$, $G\in\mathcal{O}_X(U')$ (U' as in (1.12) for $S'\Subset S$, $\Gamma'\Subset\Gamma$), $Pg=\sum_j b(F_j)$, $F_j\in\mathcal{O}_X(U'_j)$ (U'_j as in (1.12) with $S'\Subset S$, $\Gamma'_j\Subset\Gamma_j$, $\Gamma_j^{*a}\cap I=\emptyset$).

One solves

$$P_{\Sigma_{\hat{\eta}}}G_j=F_j, \qquad G_j\in\mathcal{O}_X(U'_j\cap\{z: |y|<b'_j\rho(x)\}\cap B')$$

by (i)-(ii). This gives $b(P_{\Sigma_{\hat{\eta}}}(G-\sum_j G_j))=0$; hence by (iii) (applied with $\Gamma'=\boldsymbol{R}^n$) we get:

$$b(G-\sum_j G_j)=0$$

thus (2.2) is injective.

By (i)-(ii) the surjectivity follows at once.

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