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ON THE NULLITIES OF KÄHLER C-SPACES IN $P_N(C)$

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Let M be a Kähler C-space which is holomorphically and isometrically imbedded in an N-dimensional complex projective space $P_N(\boldsymbol{C})$. Then M is a minimal submanifold of $P_N(\boldsymbol{C})$. Let $n_a(M)$ be the analytic nullity of M which was defined in [2]. We know that the nullity $n(M)$ of M is equal to $n_a(M)$ if M is a Hermitian symmetric space (Kimura [2]). In this note we prove that $n(M)=\mathrm{n}_a(M)$ for any Kähler C-space M.

By a theorem of Simons [5], the nullity of a Kähler submanifold coincides with the real dimension of the space of holomorphic sections of a normal bundle of the submanifold. Put $M=G/U$ where G is a complex semi-simple Lie group and U is a parabolic subgroup of G. By a result of Nakagawa and Takagi [4], we know that every imbedding of M in $P_N(\boldsymbol{C})$ is induced by a holomorphic linear representation of G. From this result we see that the normal bundle $N(M)$ over M is a homogeneous vector bundle.

We prove Theorem 1 which generalizes the generalized Borel-Weil theorem of Bott [1]. Applying the theorem to calculate the dimension of the space of holomorphic sections of $N(M)$ and prove that $n(M)=n_a(M)$.

The auther proved the above result before Proffesor Takeuchi gave another proof of it. His proof does not use Theorem 1 and is more simple than our proof (c.f. Takeuchi [6]).

§1. The generalization of Bott's result.

Let G be a simply connected compact semi-simple Lie group with Lie algebra $\mathfrak{g}$. Take a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. Denoto by Δ the root system of $\mathfrak{g}$ with respect to $\mathfrak{h}$. We fix a linear order on the real vector space spaned by the elements $\alpha\in\Delta$. Let Δ^+ (resp. Δ^-) be the set of all positive (resp. negative) roots. Let $\Pi=\{\alpha_1,\cdots,\alpha_l\}$ be the fundamental root system, where l is the rank of $\mathfrak{g}$ and Π_1 be a subsystem of Π. We put

$$\Delta_1=\{\alpha\in\Delta\,;\ \alpha=\sum_{i=1}^{l} m_i\alpha_i,\ m_j=0\quad\text{for any}\quad \alpha_j\notin\Pi_1\}$$

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$$\Delta(\mathfrak{n}^+)=\{\beta\in\Delta\,;\ \beta=\sum_{i=1}^{l} m_i\alpha_i,\ m_j>0 \quad \text{for some} \quad \alpha_j\notin\Pi_1\}$$

$$\Delta(\mathfrak{u})=\Delta_1\cup\Delta(\mathfrak{n}^+)\,.$$

Define Lie subalgebras $\mathfrak{g}_1$, $\mathfrak{n}^+$ and $\mathfrak{u}$ of $\mathfrak{g}$ by

$$\mathfrak{g}_1=\mathfrak{h}+\sum_{\alpha\in\Delta_1}\mathfrak{g}_\alpha$$

$$\mathfrak{n}^+=\sum_{\beta\in\Delta(\mathfrak{n}^+)}\mathfrak{g}_\beta$$

$$\mathfrak{u}=\mathfrak{h}+\sum_{\alpha\in\Delta(\mathfrak{u})}\mathfrak{g}_\alpha$$

where $\mathfrak{g}_\alpha$ is the root space corresponding to $\alpha\in\Delta$. Then $\mathfrak{g}_1$ (resp. $\mathfrak{n}^+$) is a reductive (resp. nilpotent) subalgebra of $\mathfrak{g}$ and $\mathfrak{u}=\mathfrak{g}_1+\mathfrak{n}^+$ (semi-direct). Let U be the connected Lie subgroup of G with Lie algebra $\mathfrak{u}$. Then U is a parabolic Lie subgroup of G, and $M=G/U$ is a Kähler C-space.

We denote by D (resp. D_1) the set of dominant integral forms of $\mathfrak{g}$ (resp. $\mathfrak{g}_1$). Let $\xi\in D_1$. Then there exists the irreducible representation $(\rho^1_{-\xi}, W_{-\xi})$ of $\mathfrak{g}_1$ with the lowest weight $-\xi$. We extend it to a representation of $\mathfrak{u}$ so that its restriction to $\mathfrak{n}^+$ is trivial, which will be denoted by $(\rho_{-\xi}, W_{-\xi})$. There exists a representation of U which induces the representation $(\rho_{-\xi}, W_{-\xi})$ and we denote it by $(\tilde{\rho}_{-\xi}, W_{-\xi})$. Let (ν, V) be a holomorphic representation of G. We denote by $((\nu|_U)\otimes\tilde{\rho}_{-\xi}, V\otimes W_{-\xi})$ the tensor product of the representations $(\nu|_U, V)$ and $(\tilde{\rho}_{-\xi}, W_{-\xi})$ of U. We also denote by E_S the holomorphic vector bundle over M associated to the principal bundle $G\longrightarrow M$ by a representation of U on S. For a holomorphic vector bundle E over M, we denote by ΩE the sheaf of germs of local holomorphic sections of E. We shall consider the cohomology groups $H^j(M, \Omega E_{V\otimes W_{-\xi}})$.

Let W be the Weyl group of $\mathfrak{g}$ and $\Delta_1{}^+$ the set of all positive roots of Δ_1. We define a subset W^1 of W by

$$W^1=\{\sigma\in W\,;\ \sigma^{-1}(\Delta_1{}^+)\subset\Delta^+\}\,.$$

Let δ be the half of sum of all positive roots of $\mathfrak{g}$.

THEOREM 1. *Let $\xi\in D_1$ and (ν, V) be a holomorphic representation of G. If $\xi+\delta$ is not regular, then*

$$H^j(M, \Omega E_{V\otimes W_{-\xi}})=(0) \quad \text{for all} \quad j=0,1,\cdots\,.$$

If $\xi+\delta$ is regular, $\xi+\delta$ is expressed uniquely as $\xi+\delta=\sigma(\lambda+\delta)$, where $\lambda\in D$ and *$\sigma\in W^1$, and*

$$H^j(M, \Omega E_{V\otimes W_{-\xi}})=(0) \quad \text{for all} \quad j\neq n(\sigma)\,,$$

$$H^{n(\sigma)}(M, \Omega E_{V\otimes W_{-\xi}})=V\otimes V_{-\lambda} \quad \text{(as G-module)},$$

where $n(\sigma)$ is the index of σ and $(\nu_{-\lambda}, V_{-\lambda})$ is the irreducible G-module with the lowest weight $-\lambda$.

If (ν, V) is the trivial representation of G, the theorem coincides with the generalized Borel-Weil theorem of Bott [1].

We prepare some lemmas to prove this theorem. Let (f, S) be a representation of $\mathfrak{u}$ and let $H^j(\mathfrak{n}^+, S)$ be the j-th cohomology group formed with respect to the representation $f|_{\mathfrak{n}^+}$ of $\mathfrak{n}^+$ on S. We may regard $H^j(\mathfrak{n}^+, S)$ as $\mathfrak{g}_1$-module in a canonical way. We denote by $H^j(\mathfrak{n}^+, S)^0$ the subspace of $H^j(\mathfrak{n}^+, S)$ annihilated by all $X\in\mathfrak{g}_1$. We may easily get the following lemma from theorems of Bott [1].

LEMMA 1. *Let $\lambda\in D$. Then*

$$\textit{the multiplicity of } \nu^\lambda \textit{ in } H^j(M, \Omega E_{V\otimes W_{-\xi}})$$
$$=\dim H^j(\mathfrak{n}^+, \mathrm{Hom}(V^\lambda, V\otimes W_{-\xi}))^0 \quad \text{for } j=0, 1, \cdots,$$

where (ν^λ, V^λ) is an irreducible representation of $\mathfrak{g}$ with the highest weight λ.

Since the representation $(\rho_{-\xi}|_{\mathfrak{n}^+}, W_{-\xi})$ is trivial, we have

$$H^j(\mathfrak{n}^+, \mathrm{Hom}(V^\lambda, V\otimes W_{-\xi}))$$
$$=H^j(\mathfrak{n}^+, V_{-\lambda}\otimes V\otimes W_{-\xi}))$$
$$=H^j(\mathfrak{n}^+, V_{-\lambda}\otimes V)\otimes W_{-\xi}.$$

From Schur's lemma we have

$$\dim H^j(\mathfrak{n}^+, \mathrm{Hom}(V^\lambda, V\otimes W_{-\xi}))^0$$
$$=\text{the multiplicity of } \nu^\xi{}_1 \text{ in } H^j(\mathfrak{n}^+, V_{-\lambda}\otimes V),$$

where $\nu^\xi{}_1$ is an irreducible representation of $\mathfrak{g}_1$ with the highest weight ξ.

LEMMA 2. *Let $\lambda\in D$. Then*

$$\textit{the multiplicity of } \nu^\lambda \textit{ in } H^j(M, \Omega E_{V\otimes W_{-\xi}})$$
$$=\textit{the multiplicity of } \nu^\xi{}_1 \textit{ in } H^j(\mathfrak{n}^+, V_{-\lambda}\otimes V).$$

Now we recall Kostant's result of Lie algebra cohomology.

THEOREM OF KOSTANT ([3]). *Let $\lambda\in D$. Then $\mathfrak{g}_1$-module $H^j(\mathfrak{n}^+, V^\lambda)$ is decomposed into direct sums:*

$$H^j(\mathfrak{n}^+, V^\lambda)=\sum_{\sigma\in W^1(j)}\oplus W^{\sigma(\lambda+\delta)-\delta},$$

where $W^1(j)=\{\sigma\in W^1;\ n(\sigma)=j\}$ *and* (ν''_1, W'') *is the irreducible representation of* $\mathfrak{g}_1$ *with the highest weight* μ.

PROOF OF THEOREM 1. Assume that the multiplicity of ν^{ξ}_1 in $H^j(\mathfrak{n}^+, V^{\gamma})$, $\gamma\in D$, is not 0. By the above theorem there exists an element $\sigma\in W^1(j)$ so that $\xi+\delta=\sigma(\gamma+\delta)$. Since $\gamma+\delta$ is regular, $\xi+\delta$ is also regular. Therefore by Lemma 2 we see that if $\xi+\delta$ is not regular then $H^j(M, \Omega E_{V\otimes W_{-\xi}})=(0)$ for any j.

Assume that $\xi+\delta$ is regular. Then $\xi+\delta$ is expressed uniquely as $\xi+\delta=\sigma(\lambda+\delta)$, where $\lambda\in D$ and $\sigma\in W^1$ (Kostant [3]). If $j\neq n(\sigma)$, we see immedietly that $H^j(M, \Omega E_{V\otimes W_{-\xi}})=(0)$ by Lemma 2 and Theorem of Kostant.

Let G_u be a maximal compact subgroup of G. Denote by χ_{ϕ} the charactor of a representation ϕ of G. Then by Theorem of Kostant we get the following:

$$\begin{aligned}&\text{the multiplicity of } \nu^{\xi}_1 \text{ in } H^{n(\sigma)}(\mathfrak{n}^+, V_{-\gamma}\otimes V)\\ &=\text{the multiplicity of } \nu^{\lambda} \text{ in } V_{-\gamma}\otimes V\\ &=\int_{G_u}\bar{\chi}_{\nu^{\gamma}}\cdot\chi_{\nu}\cdot\bar{\chi}_{\nu^{\lambda}}dg\\ &=\text{the multiplicity of } \nu^{\gamma} \text{ in } V\otimes V_{-\lambda},\end{aligned}$$

where dg is the normalized Haar measure on G_u. Therefore by Lemma 2, we get

$$H^{n(\sigma)}(M, \Omega E_{V\otimes W_{-\xi}})=V\otimes V_{-\lambda}\quad\text{(as G-module)}.\qquad Q.E.D.$$

§2. Proof of the main theorem.

We retain the same notations and assumptions introduced in §1. Let Λ be an integral form such that $(\Lambda, \alpha_i)=0$ for $\alpha_i\in\Pi_1$ and $(\Lambda, \alpha_j)>0$ for $\alpha_j\notin\Pi_1$. We denote by $(\tilde{\nu}^{\Lambda}, V^{\Lambda})$ the irreducible representation of G with highest weight Λ. Let $P(V^{\Lambda})$ be the complex projective space consisting of all 1-dimensional subspace of V^{Λ}. Since the dimension of the weight space (v) in V^{Λ} correspanding to the highest weight Λ is equal to 1, (v) is an element of $P(V^{\Lambda})$. Moreover G acts canonically on $P(V^{\Lambda})$ via the representation $(\tilde{\nu}^{\Lambda}, V^{\Lambda})$, and it is known that U coincides with the isotropy subgroup of G at (v). Therefore we get a G-equivariant imbedding f^{Λ}: $M=G/U\longrightarrow P(V^{\Lambda})$. Since $\tilde{\nu}^{\Lambda}$ is an irreducible representation, f^{Λ} is a full imbedding. Conversely every full Kähler imbedding of a Kähler C-space M in $P_n(\boldsymbol{C})$ is obtained in this way (Nakagawa and Takagi [4]).

THEOREM 2. *Let* $M=G/U$ *be a Kähler C-space fully imbedded in* $P_n(\boldsymbol{C})$. *Then the nullity* $n(M)$ *of* M *in* $P_n(\boldsymbol{C})$ *is given by*

$$n(M)=\dim_{\boldsymbol{R}}\mathfrak{a}(P_n(\boldsymbol{C}))-\dim_{\boldsymbol{R}}(M),$$

where $\mathfrak{a}(P_n(\boldsymbol{C}))$ (*resp.* $\mathfrak{a}(M)$) *is the vector space of all analytic vector fields on* $P_n(\boldsymbol{C})$ (*resp.* M).

PROOF. Assume that the imbedding of M in $P_n(C)$ is induced by the irreducible representation $(\rho^\Lambda, V^\Lambda)$, $\Lambda \in D$ and $\dim V^\Lambda = n+1$, of G. Denote by $(h, (v))$ the representation of U on (v) induced by ρ^Λ and denote by $(h^*, (v)^*)$ the contragredient representation of $(h, (v))$. Then we get the following exact sequence of U-modules:

$$0 \longrightarrow (v)\otimes(v)^* \longrightarrow V\otimes(v)^* \longrightarrow V\otimes(v)^*/(v)\otimes(v)^* \longrightarrow 0 .$$

It is easy to see that $E_{V\otimes(v)^*/(v)\otimes(v)^*} = T(P_n(\boldsymbol{C}))|_M$. Therefore we get the following exact sequence of holomorphic vector bundles over M:

$$0 \longrightarrow 1 \longrightarrow E_{V\otimes(v)^*} \longrightarrow T(P_n(\boldsymbol{C}))|_M \longrightarrow 0 ,$$

where 1 is the trivial line bundle over M. Since M is a Kähler C-space, $H^1(M, \Omega 1) = (0)$. Therefore we get the following exact esquence of cohomology groups:

$$0 \longrightarrow H^0(M, \Omega 1) \longrightarrow H^0(M, \Omega E_{V\otimes(v)^*}) \longrightarrow H^0(M, \Omega(T(P_n(\boldsymbol{C})|_M)) \longrightarrow 0 .$$

Since the lowest weight of $(h^*, (v)^*)$ is $-\Lambda$, it follows, by Theorem 1, that $H^0(M, \Omega E_{V\otimes(v)^*}) = V \otimes V_{-\Lambda}$ as G-modules. It is obvious that $\dim H^0(M, \Omega 1) = 1$. Therefore we get

$$\dim H^0(M, \Omega(T(P_n(\boldsymbol{C}))|_M)) = (n+1)^2 - 1 .$$

Since $\dim_{\boldsymbol{R}} \mathfrak{a}(P_n(\boldsymbol{C}) = 2\{(n+1)^2 - 1\}$, we get

$$(1) \qquad \dim_{\boldsymbol{R}} H^0(M, \Omega(T(P_n(\boldsymbol{C}))|_M)) = \dim_{\boldsymbol{R}} \mathfrak{a}(P_n(\boldsymbol{C})) .$$

The exact sequence of holomorphic vector bundles over M:

$$0 \longrightarrow T(M) \longrightarrow T(P_n(\boldsymbol{C}))|_M \longrightarrow N(M) \longrightarrow 0$$

and $H^1(M, \Omega T(M)) = (0)$ (Bott [1]) induce the following exact sequence of cohomology groups:

$$(2) \quad 0 \longrightarrow H^0(M, \Omega T(M)) \longrightarrow H^0(M, \Omega(T(P_n(\boldsymbol{C})|_M)) \longrightarrow H^0(M, \Omega N(M)) \longrightarrow 0 .$$

Recall that the nullity $n(M)$ of M is given by

$$(3) \qquad n(M) = \dim_{\boldsymbol{R}} H^0(M, \Omega N(M))$$

(Kimura [2]). From (1), (2), (3) and $\dim_{\boldsymbol{R}} H^0(M, \Omega T(M)) = \dim_{\boldsymbol{R}} \mathfrak{a}(M)$, we get

$$n(M) = \dim_{\boldsymbol{R}} \mathfrak{a}(P_n(\boldsymbol{C})) - \dim_{\boldsymbol{R}} \mathfrak{a}(M)$$

Q.E.D.

From the above theorem and Lemma 3.4 in Kimura [2] we have the following result.

COROLLARY. *Let M be a Kähler C-space holomorphically and isometrically imbedded in $P_N(\boldsymbol{C})$. Then*

$$n(M)=n_a(M).$$

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