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ROOTS OF KAC-MOODY LIE ALGEBRAS

By

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0. Introduction.

The notion of Kac-Moody Lie algebras has recently been introduced and studied as a natural generalization of a finite dimensional split semisimple Lie algebra with successful applications to Macdonald type identities (cf. Lepowsky [3]). Such a Lie algebra has a root system, which is a natural analogue of the usual root systems in the sense of Bourbaki [1]. In this paper, we will give a characterization of positive root systems of Kac-Moody Lie algebras as a subset of a lattice. (See Proposition 1 and Theorem 1 below.)

Let A be a generalized Cartan matrix and L the Kac-Moody Lie algebra associated with A, and let Δ (resp. Δ_+) be the root system (resp. the positive root system) of L (for the definition, see § 1). In § 2, we will consider the special positive root system $P(A)$ associated with L. This system $P(A)$ satisfies the conditions (X1), (X2), (Y1), (Y2) and (Y3), which are specified in § 2. Conversely we will show that any set satisfying these conditions coincides with the system $P(A)$ arising from some Kac-Moody Lie algebra. In particular, Δ_+ is uniquely determined by (Y1), (Y2) and (Y3) when A is given.

On the other hand, there are two kinds of roots in Δ, called real roots and imaginary roots respectively. In § 3, we will present a characterization of imaginary roots. In § 4, we will give a way to produce the roots of L inductively from simple roots.

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1. Kac-Moody Lie algebras.

In this section, we will review the notion of a Kac-Moody Lie algebra and its root system (cf. Kac [2], Lepowsky [3], Moody [4]). Let l be a positive integer, and set $I=\{1, \cdots, l\}$. Let $A=(a_{ij})$ be an $l\times l$ *generalized Cartan matrix* —that is $a_{ij}\in Z$ for all $i, j\in I$, $a_{ij}=2$ for all $i\in I$, $a_{ij}\leqq 0$ for distinct $i, j\in I$,

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$a_{ij}=0$ whenever $a_{ji}=0$ for each $i, j\in I$. For any generalized Cartan matrix $A=(a_{ij})$ and for any field K of characteristic zero, L denotes the Lie algebra over K generated by $3l$ generators $e_1, \cdots, e_l, h_1, \cdots, h_l, f_1, \cdots, f_l$ with the defining relations $[h_i, h_j]=0$, $[e_i, f_j]=\delta_{ij}h_i$, $[h_i, e_j]=a_{ij}e_j$, $[h_i, f_j]=-a_{ij}f_j$ for all $i, j\in I$, and $(\mathrm{ad}\, e_i)^{-a_{ij}+1}e_j=0$, $(\mathrm{ad}\, f_i)^{-a_{ij}+1}f_j=0$ for distinct $i, j\in I$. We call the algebra L the (standard) *Kac-Moody Lie algebra* over K associated with A.

Let $\Gamma=\sum_{i\in I}Z\alpha_i$ be the free Z-module with free generators $\alpha_1, \cdots, \alpha_l$. We give the structure of a Γ-graded Lie algebra to L by defining $\deg(e_i)=\alpha_i$, $\deg(h_i)=0$ and $\deg(f_i)=-\alpha_i$ for all $i\in I$. For any $\alpha\in\Gamma$, let L_α be the subspace of L consisting of all elements with degree α. Set $H=L_0$, which equals $Kh_1\oplus\cdots\oplus Kh_l$. We call a nonzero element $\alpha\in\Gamma$ a *root* of L if $L_\alpha\neq 0$. Let Δ be the set of all roots of L, called the root system of L, so $L=H\oplus\sum_{\alpha\in\Delta}L_\alpha$. Then Δ is contained in $\Gamma_+\cup(-\Gamma_+)$, where $\Gamma_+=\{\alpha=\sum_{i\in I}k_i\alpha_i\in\Gamma\mid k_i\geqq 0, \alpha\neq 0\}$. Set $\Delta_+=\Delta\cap\Gamma_+$ and $\Delta_-=\Delta\cap(-\Gamma_+)$. We call Δ_+ the positive root system of L. We note $\Delta_-=-\Delta_+$. For each $\alpha=\sum_{i\in I}k_i\alpha_i\in\Delta_+$ (resp. Δ_-), L_α is the subspace of L spanned by the elements

$$[e_{i_1}, [e_{i_2}, \cdots, [e_{i_{r-1}}, e_{i_r}]\cdots]]$$
$$(\text{resp. } [f_{i_1}, [f_{i_2}, \cdots, [f_{i_{r-1}}, f_{i_r}]\cdots]]),$$

where e_j (resp. f_j) occurs $|k_j|$ times. In particular, $L_{\alpha_i}=Ke_i$ and $L_{-\alpha_i}=Kf_i$. Set $\Pi=\{\alpha_1, \cdots, \alpha_l\}$, called simple roots. For each $i\in I$, let U_i be the subalgebra of L generated by e_i, h_i, f_i, which is isomorphic to $sl(2, K)$.

LEMMA 1. (cf. Lepowsky [3, Proposition 1.4]). *The subspace $\sum_{\alpha\in\Delta_+-\{\alpha_i\}}L_\alpha$ is a direct sum of finite dimensional irreducible U_i-modules for each $i\in I$.*

LEMMA 2. *Let $\alpha\in\Delta_+-\Pi$. Then there is $\alpha_i\in\Pi$ such that $\alpha-\alpha_i\in\Delta_+$.*

PROOF. Since $L_\alpha\neq 0$, there is a nonzero generator $[e_{i_1}, [e_{i_2}, \cdots [e_{i_{r-1}}, e_{i_r}]\cdots]]$ in L_α. Then $[e_{i_2}, \cdots [e_{i_{r-1}}, e_{i_r}]\cdots]$ is a nonzero element of $L_{\alpha-\alpha_{i_1}}$. Therefore such $\alpha-\alpha_{i_1}$ is in Δ_+. q. e. d.

2. Abstract positive root systems.

Let $\Gamma=\sum_{i\in I}Z\alpha_i$ be the free Z-module with free generators $\Pi=\{\alpha_1, \cdots, \alpha_l\}$, and let $\Gamma_+=\{\alpha=\sum_{i\in I}k_i\alpha_i\in\Gamma\mid k_i\geqq 0, \alpha\neq 0\}$ as in § 1. Set $\Gamma^*=\mathrm{Hom}_Z(\Gamma, Z)$, the dual of Γ. A pair $\Phi=(X, Y)$ consisting of a subset X of Γ^* and a subset Y of Γ is called an *abstract positive root system* (in the lattice Γ) if the following axioms (X1), (X2), (Y1), (Y2) and (Y3) are satisfied.

(X1) X consists of l distinct elements $\phi_1, \cdots, \phi_l$, labeled by I.

(X2) $\phi_i(\alpha_i)=2$ for all $i\in I$.

(Y1) $\Pi\subseteqq Y\subseteqq \Delta_+$.

(Y2) For each $i\in I$, if $\alpha\in Y-\{\alpha_i\}$, then there are nonnegative integers $p=p(i, \alpha)$ and $q=q(i, \alpha)$ satisfying

(*) $p-q=\phi_i(\alpha)$

and

(**) $\alpha+k\alpha_i\in Y$ if and only if $-p\leqq k\leqq q$, where $k\in \boldsymbol{Z}$.

(Y3) If $\alpha\in Y-\Pi$, then there exists $\alpha_i\in\Pi$ for which $\alpha-\alpha_i\in Y$.

Let L be the (standard) Kac-Moody Lie algebra associated with a generalized Cartan matrix A, and let Δ_+ be the positive root system of L (see § 1). Let $\psi_1, \cdots, \psi_l$ be elements of Γ^* defined by $\psi_i(\alpha_j)=a_{ij}$ for all $j\in I$. Set $\Psi=\{\psi_1, \cdots, \psi_l\}$ and $P(A)=(\Psi, \Delta_+)$. We call $P(A)$ the *special positive root system* of A or of L. For each $\alpha=\sum_{i\in I}k_i\alpha_i\in\Gamma$, let $\mathrm{ht}(\alpha)=k_1+\cdots+k_l$, the height of α.

PROPOSITION 1. *Let A be a generalized Cartan matrix. Then the special positive root system $P(A)$ is an abstract positive root system.*

PROOF. By Lemma 1 and Lemma 2, we see that (Y2) and (Y3) hold. The other conditions are easily verified. q. e. d.

For each abstract positive root system Φ, set $C_\Phi=(c_{ij})$, where $c_{ij}=\phi_i(\alpha_j)$ for all $i, j\in I$.

TTEOREM 1. *Let Φ be an abstract pasitive root system. Then*: (1) C_Φ *is a generalized Cartan matrix,*

(2) $\Phi=P(C_\Phi)$.

PROOF. (1) By the definition, $\phi_i(\alpha_j)\in\boldsymbol{Z}$. The axiom (X2) says $\phi_i(\alpha_i)=2$. For distinct $i, j\in I$, we see $\alpha_j-\alpha_i\notin Y$ by (Y1), so $p(i, \alpha_j)=0$ and $\phi_i(\alpha_j)=-q(i, \alpha_j)\leqq 0$ by (Y2). Furthermore, for distinct $i, j\in I$,the condition $\phi_i(\alpha_j)=0$ means $\alpha_i+\alpha_j\notin Y$. Therefore $\phi_i(\alpha_j)=0$ if and only if $\phi_j(\alpha_i)=0$.

(2) This follows from Proposition 2 below.

PROPOSITION 2. *Let $\Phi=(X, Y)$ and $\Phi'=(X', Y')$ be abstract positive root systems in the lattices Γ and Γ' respectively. Suppose $C_\Phi=C_{\Phi'}$. Then there is an isomorphism $\lambda:\Gamma\rightarrow\Gamma'$ such that $\lambda(\Phi)=\Phi'$.*

PROOF. Let $l=\mathrm{rank}\,\Gamma=\mathrm{rank}\,\Gamma'$. Since $C_\Phi=C_{\Phi'}$, we have $\phi_i(\alpha_j)=\phi'_i(\alpha'_j)$ for all $i, j\in I$, where $\phi_i\in X$, $\phi'_i\in X'$, $\alpha_j\in\Pi$, and $\alpha'_j\in\Pi'$. Let λ be the isomorphism of Γ to Γ' defined by $\lambda(\alpha_i)=\alpha'_i$. Then $\phi'_i=\phi_i\cdot\lambda^{-1}$. Put $Y_n=\{\alpha\in Y\,|\,\mathrm{ht}(\alpha)\leqq n\}$

and $Y'_n=\{\alpha\in Y'\,|\,\mathrm{ht}(\alpha)\leqq n\}$, where $n\in\boldsymbol{Z}_{>0}$. If $n=1$, then $\lambda(Y_1)=Y'_1=\Pi'$ by (Y1). Assume $n>1$ and $\lambda(Y_{n-1})=Y'_{n-1}$. Let $\alpha\in Y_n$. By (Y3), there exists $\alpha_i\in\Pi$ for which $\alpha-\alpha_i\in Y_{n-1}$. Since $\lambda(Y_{n-1})=Y'_{n-1}$, we have $\lambda(\alpha)\in Y'_n$ by (Y2). Hence $\lambda(Y_n)\subseteqq Y'_n$. Similarly $\lambda^{-1}(Y'_n)\subseteqq Y_n$. Therefore $\lambda(Y_n)=Y'_n$ for all $n\in\boldsymbol{Z}_{>0}$, which implies $\lambda(Y)=Y'$. q. e. d.

Moreover we will prove the following two results without recourse to the theory of Lie algebras.

PROPOSITION 3. *Let $\Phi=(X, Y)$ be an abstract positive root system. Then $\boldsymbol{Z}\alpha_i\cap Y=\{\alpha_i\}$ for any $\alpha_i\in\Pi$.*

PROOF. Let $\alpha_i\in\Pi$. Clearly $m\alpha_i\notin Y$ if $m\leqq 0$. Suppose $m\alpha_i\in Y$ for some $m\in\boldsymbol{Z}_{>1}$. By (Y2), we see $\alpha_i, 2\alpha_i, \cdots, m\alpha_i\in Y$. Thus $p(i, m\alpha_i)=m-1$ by (Y1) and (Y2). Then $m-1\geqq p(i, m\alpha_i)-q(i, m\alpha_i)=\phi_i(m\alpha_i)=2m$ by (Y2), so $m\leqq -1$, which is a contradiction. q. e. d.

For each $i\in I$, let w_i be an involutive endomorphism of Γ defined by $w_i(\alpha)=\alpha-\phi_i(\alpha)\alpha_i$ for all $\alpha\in\Gamma$.

PROPOSITION 4. *Let $\Phi=(X, Y)$ be an abstract positive root system. Then $Y-\{\alpha_i\}$ is w_i-stable for any $i\in I$.*

PROOF. Let $\beta\in Y-\{\alpha_i\}$. Then $w_i(\beta)=\beta-\phi_i(\beta)\alpha_i=\beta+(q-p)\alpha_i$. By (Y2), we see $w_i(\beta)\in Y$ since $-p\leqq q-p\leqq q$. Suppose $w_i(\beta)=\alpha_i$, then $\beta=(\phi_i(\beta)+1)\alpha_i\in\boldsymbol{Z}\alpha_i\cap Y=\{\alpha_i\}$, which is a contradiction. Therefore $w_i(\beta)\in Y-\{\alpha_i\}$. q. e. d.

3. Imaginary roots.

Let W be the subgroup of $GL(\Gamma)$ generated by w_i for all $i\in I$. The group W is called the Weyl group. Set $\Delta^{\mathrm{re}}=W(\Pi)$, real roots, and set $\Delta^{\mathrm{im}}=\Delta-\Delta^{\mathrm{re}}$, imaginary roots. Put $\Delta_+^{\mathrm{im}}=\Delta_+\cap\Delta^{\mathrm{im}}$. For each $\alpha=\sum_{i\in I}k_i\alpha_i\in\Gamma$, let S_α be the diagram, called the support of α, with vertices v_i for all i satisfying $k_i\neq 0$ such that v_i and v_j are joined whenever $\phi_i(\alpha_j)\neq 0$ for distinct i, j. Let J be the set consisting of all elements α of Γ_+ such that the support S_α is connected, and let $J^0=\bigcap_{w\in W}(wJ)$. We call a generalized Cartan matrix *indecomposable* if the corresponding Dynkin diagram is connected (cf. Kac [2], Lepowsky [3], Moody [4]).

THEOREM 2. *Let A be an indecomposable generalized Cartan matrix. Then* $\Delta_{+}^{\mathrm{im}}=J^{0}$.

PROOF. We note that the support S_{α} of a root $\alpha\in\Delta$ is connected. Therefore $\Delta_{+}^{\mathrm{im}}\subseteqq J$, and $\Delta_{+}^{\mathrm{im}}\subseteqq J^{0}$ since Δ_{+}^{im} is W-invariant. Conversely let $\alpha\in J^{0}$. Then choose an element $w\alpha$ in the orbit $W(\alpha)$ of minimal height. Then $w\alpha\in J$ and in fact in the fundamental set M, where $M=\{\alpha\in J\mid\phi_{i}(\alpha)\leqq 0 \text{ for all } i\in I\}$. We know $M\subseteqq\Delta_{+}^{\mathrm{im}}$ (see Kac [2], or Remark (3) below). Hence $w\alpha\in\Delta_{+}^{\mathrm{im}}$ and $\alpha\in\Delta_{+}^{\mathrm{im}}$.

q. e. d.

4. Successive computation of roots.

Let A be a generalized Cartan matrix. For each $\alpha\in\Gamma_{+}$, it is difficult to determine whether α is in Δ_{+} or not, since in general the Weyl group is infinite. Here we will give an actual method of constructing the positive roots inductively from Π. Let $P(A)=(X, Y)$ be the special positive root system of A. Let $Y_{n}=\{\alpha\in Y\mid \mathrm{ht}(\alpha)\leqq n\}$ for each $n\in\boldsymbol{Z}_{>0}$.

PROPOSITION 5. *Suppose $n\geqq 1$. Let $\phi_{i}\in X$ and $\alpha_{i}\in\Pi$ for each $i\in I$, and let $\alpha\in Y_{n}$.*

(1) *If $\phi_{i}(\alpha)<0$, then $\alpha+\alpha_{i}\in Y_{n+1}$.*

(2) *If $\phi_{i}(\alpha)\geqq 0$, then $\alpha+\alpha_{i}\in Y_{n+1}$ if and only if $\alpha-(\phi_{i}(\alpha)+1)\alpha_{i}\in Y_{n}$.*

Therefore we can construct the set of roots recursively. (That an inductive construction is possible is already known—cf. Moody [4, Proposition 1].)

REMARKS.

(1) We can prove Propositions 2 and 3 without the condition (Y3).

(2) Let

$$A=\begin{pmatrix} 2 & -2 & 0 & 0 & 0 \\ -2 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -2 \\ 0 & 0 & 0 & -2 & 2 \end{pmatrix}$$

and $S=\bigcap_{w\in W}(w\Gamma_{+})$. Set $Y=\Delta_{+}\cup S$. Then we see that Y satisfies the conditions (Y1) and (Y2). Take a minimal height element α in $Y-\Delta_{+}$. This is possible since $\delta=\alpha_{1}+\alpha_{2}+\alpha_{4}+\alpha_{5}\in Y-\Delta_{+}$. For such an element α, there is no element $\alpha_{i}\in\Pi$ such that $\alpha-\alpha_{i}\in Y$. (This example appears in a letter from Mr. M. Kaneda to Prof. N. Iwahori.)

(3) (Kac [2]). Let Δ_+^{im}, W and M be the positive imaginary roots, the Weyl group and the fundamental set respectively. Then $\Delta_+^{im}=W(M)$.

(4) In Moody and Yokonuma [5], a geometric axiomatic foundation for real root systems of Kac-Moody Lie algebras has been developed.

References

[1] Bourbaki, N., Groupes et algèbres de Lie, Chap. 4-6, Hermann, Paris, 1968.
[2] Kac, V.G., Infinite root systems, representations of graphs and invariant theory, Invent. Math., **56** (1980), 57-92.
[3] Lepowsky, J., Lectures on Kac-Moody Lie algebras, Paris Univ., 1978.
[4] Moody, R.V., Root systems of hyperbolic type, Advance Math., **33** (1979), 144-160.
[5] Moody, R.V. and Yokonuma, T., Root systems and Cartan matrices, to appear.

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