TSUKUBA J. MATH.
Vol. 15 No. 1 (1991), 193-203

GLOBAL HYPOELLIPTICITY AND CONTINUED FRACTIONS

By

Masafumi YOSHINO

§1. Introduction.

In this paper, we are concerned with differential operators which are not hypoellipitic, that are globally hypoelliptic. Let $\boldsymbol{T}^{d}=\boldsymbol{R}^{d}/2\pi\boldsymbol{Z}^{d}$ $(d\geqq 2)$ be a d-dimensional torus. We denote by $\mathcal{D}'(\boldsymbol{T}^{d})$ and $C^{\infty}(\boldsymbol{T}^{d})$ the sets of distributions and smooth functions on $\boldsymbol{T}^{d}$, respectively. For a pseudodifferential operator P on $\boldsymbol{T}^{d}$, P is said to be hypoelliptic in $\boldsymbol{T}^{d}$ if, for any domain $\Omega\subset\boldsymbol{T}^{d}$ and $u\in\mathcal{D}'(\Omega)$, $Pu\in C^{\infty}(\Omega)$ implies $u\in C^{\infty}(\Omega)$. We say that P is globally hypoelliptic in $\boldsymbol{T}^{d}$, if every distribution u such that $Pu\in C^{\infty}(\boldsymbol{T}^{d})$ is smooth on $\boldsymbol{T}^{d}$. Clearly, hypoelliptic operators are globally hypoelliptic.

We know that there is much difference between these notions (cf. [2]). One trend in the study is, as to second order equations, characterizing global hypoellipticity by properties of certain vector fields determined by equations, which would be the same way as to the famous work of Hörmander (cf. [2]). But this way cannot explain all globally hypoelliptic operators because we have Greenfield's example; $Q=(\partial/\partial x_{1}-\tau\partial/\partial x_{2})^{2}$ with $\tau>0$, irrational. Q is globally hypoelliptic in $\boldsymbol{T}^{2}$ if and only if τ satisfies a Siegel condition (cf. [1]).

In [4] we investigated operators for which the global hypoellipticity is equivalent to a Siegel condition. Indeed, let us consider the operators

$$P=\sum_{|\alpha|\leq m}a_{\alpha}D^{\alpha}+\sum_{|\beta|\leq m-s}b_{\beta}(x)D^{\beta}\qquad\text{on } \boldsymbol{T}^{d}$$

where $m\geqq 1$, $s\geqq 0$ are integers, $a_{\alpha}\in\boldsymbol{C}$, $b_{\beta}\in C^{\infty}(\boldsymbol{T}^{d})$, and $D^{\alpha}=(-i\partial/\partial x_{1})^{\alpha_{1}}\cdots(-i\partial/\partial x_{d})^{\alpha_{d}}$. We define the cone Γ_{p} as the convex hull of the closure of the set $\{t\gamma;\ t\geqq 0,\ b_{\beta,\gamma}\neq 0$ for some β in $P\}\cup\{0\}$, where $b_{\beta}(x)=\sum b_{\beta,\gamma}e^{i\gamma x}$. Then the class of operators for which the global hypoellipticity is equivalent to a Siegel condition is, roughly speaking, characterized by the properness of Γ_{p}, i.e., Γ_{p} contains no ray. (For the detail we refer [4]).

If we drop the properness of Γ_{p}, a Siegel condition is no longer adequete to describe the global hypoellipticity. (cf. [4]). Indeed, the operater $Q=-(\partial/\partial x_{1})^{2}+1$ on $\boldsymbol{T}^{2}$ satisfies a Siegel condition, and $\Gamma_{Q}=\{0\}$. Q is globally

Received November 27, 1989, Revised April 23, 1990.

hypoelliptic on T^2. On the other hand, the Mathieu operator $M=-(\partial/\partial x_1)^2+2\sin x_1$ on T^2 does not satisfy a Siegel condition, but it is globally hypoelliptic. (cf. [4]). We note that though the operators Q and M give examples of globally hypoelliptic operators which do not satisfy the Hörmander condition in [2], they are quite different, because $\boldsymbol{Q}$ is controled by a Siegel condition, and $\boldsymbol{M}$ by a different principle. (cf. Theorem 2.1 which follows.)

We shall study global hypoellipticity of Mathieu-type pseudodifferential operators from the viewpoints of eigenvalue problems. We express necessary and sufficient conditions for global hypoellipticity in terms of Hill's infinite determinants and continued fractions, which make clear why a Mathieu operator is globally hypoelliptic in T^2. Our theorem is applied to the distribution of eigenvalues for pseudodifferential Mathieu-type operators.

Finally, the author would like to give sincere thanks to the referee for giving the author useful suggestions in preparing this paper.

§2. Notations and results.

Let $p(\eta_1)$, $\eta\in \boldsymbol{R}$ be a smooth function on $\boldsymbol{R}$, and let us consider the pseudodifferential equation.

(2.1) $$Pu\equiv(p(D_1)+2\cos x_1)u=f(x) \quad \text{on } T^d,\ D_1=-i\partial/\partial x_1 .$$

We say that P is of Mathieu-type if $p(\eta_1)$ satisfies the following conditions.

(2.2) $$p(t)=p(-t) \quad \text{for all } t\in \boldsymbol{R}.$$

(2.3) $$\text{There exist } \varepsilon>0 \text{ such that } |p(t)t^{1+\varepsilon}| \longrightarrow \infty \quad (t\to\infty).$$

For $z\in \boldsymbol{C}$ we define an infinite matrix H by

(2.4) $$H\equiv(H_{n,m})_{-\infty}^{\infty}=((z-p(n))^{-1}g_{n-m}+\delta_{n,m})_{-\infty}^{\infty},$$

that is, the (n, m)-component is given by $g_{n-m}/(z-p(n))+\delta_{n,m}$, where

$$g_{n-m}=\begin{cases} 1 & (\text{if } n-m=\pm 1) \\ 0 & (\text{if otherwise}) \end{cases}, \qquad \delta_{n,m}=\begin{cases} 1 & (\text{if } n=m) \\ 0 & (\text{if otherwise}) \end{cases}.$$

We denote by $(H_{n,m})_{-l}^{l}$ the l-th section of H. We define the infinite determinant, det H by

$$\lim_{l\to\infty}\det(H_{n,m})_{-l}^{l}$$

if the limit exists. Hill's determinant exists under the condition (2.3) (cf. Proposition 3.2 which follows). We set

(2.5) $$D(z)=\det H .$$

We define a meromorphic function $T(z)$ by continued fraction

(2.6) $$T(z)=\mathop{\mathbf{K}}_{n=1}^{\infty}\left(\frac{-1}{z-p(n)}\right)=\cfrac{-1}{z-p(1)+\cfrac{-1}{z-p(2)+\cfrac{-1}{\ddots}}},$$

or equivalently, by

(2.7) $$T(z)=\lim_{n\to\infty} A_n/B_n$$

where $\{A_n\}$ and $\{B_n\}$ are solutions of the difference equation

(2.8) $$y_n=(z-p(n))y_{n-1}-y_{n-2}$$

for $n=1, 2, \cdots$, with the initial conditions, $A_{-1}=1$, $A_0=0$; $B_{-1}=0$, $B_0=1$, respectively (cf. [3]). Then we have

THEOREM 2.1. *Suppose that P satisfies* (2.2) *and* (2.3). *Then the followings are equivalent.*

(a) *$P-\lambda$ is globally hypoelliptic in $\boldsymbol{T}^d$.*

(b) *$P-\lambda$: $C^\infty(\boldsymbol{T})\to C^\infty(\boldsymbol{T})$ is injective.*

(c) *λ is not a pole of $T(z)$ and $2T(\lambda)\neq p(0)-\lambda$.*

(d) *$D(\lambda)\neq 0$.*

REMARKS 2.2. (i) Theorem 2.1 is valid if we replace $C^\infty(\boldsymbol{T})$ in (b) with any one of $C^\omega(\boldsymbol{T})$, $C^\infty(\boldsymbol{T}^d)$ and $C^\omega(\boldsymbol{T}^d)$, where $C^\omega(\boldsymbol{T})$ denotes the space of analytic functions on the torus $\boldsymbol{T}$ and so on. Theorem 2.1 is still valid if we replace (a) with the following:

(a)′ $P-\lambda$ is globally analytically hypoelliptic in $\boldsymbol{T}^d$.

Here $P-\lambda$ is said to be globally analytically hypoelliptic if $u\in\mathcal{D}'(\boldsymbol{T}^d)$ and $Pu\in C^\omega(\boldsymbol{T}^d)$ implies $u\in C^\omega(\boldsymbol{T}^d)$. All these results follows by modifications of the proof of Theorem 2.1.

(ii) If we replace (2.4) with

(2.3)′ $$|p(t)| \longrightarrow \infty \qquad \text{as } |t|\to\infty\,.$$

then $T(z)$ is still well-defined, but Hill's determinant does not have a sense in general. Nevertheless, the equivalence of (a), (b) and (c) in Theorem 2.1 is valid.

COROLLARY 2.3. *The Mathieu operator $-(\partial/\partial x_1)^2+2\cos x_1$ is globally hypoelliptic in $\boldsymbol{T}^d$.*

REMARK 2.4. Theorem 2.3 can be extended to more general pseudodifferential operators. In fact, we have: Suppose (2.2) and (2.3) are satisfied. Moreover, suppose that $p(t)$ is real for all integers $t\in \boldsymbol{Z}$. Then, there exists a discrete set $E\subset \boldsymbol{R}$ such that $P-\lambda$ is globally hypoelliptic if and only if λ is not in E. The proof of this fact is given in § 3.

We can easily see, from the proof that in the case of a Mathieu operator, the set E coincides with the characteristic values of a Mathieu operator. Hence it does not contain 0. Hence, Corollary 2.3 is a special case of the above result.

COROLLARY 2.5. (*Asymptotic distribution of eigenvalues*). *Suppose that P satisfies* (2.2) *and* (2.3). *Then, for any δ, $0<\delta<\varepsilon$, there exists $k\geqq 0$ such that the eigenvalues of P are contained in the set*

$$F=\{z\in \boldsymbol{C};\ |z-p(n)|\leqq 2 \text{ for some } n\in \boldsymbol{N},\ 0\leqq n\leqq k \text{ or }$$
$$|z-p(n)|\leqq n^{-\delta} \text{ for some } n\in \boldsymbol{N},\ n\geqq k\}.$$

§ 3. Proof of Theorems.

3.1. Preliminary lemmas.

Let $\{A_n^*\}_{n=-1}^{\infty}$ and $\{B_n^*\}_{n=-1}^{\infty}$ be defined by an equivalent difference equation to (2.8),

$$y_n^*=y_{n-1}^*-(z-p(n))^{-1}(z-p(n-1))^{-1}y_{n-2}^*, \qquad n=1, 2, \cdots \tag{3.1}$$

with the initial conditions

$$A_{-1}^*=1, \quad A_0^*=0; \quad B_{-1}^*=0, \quad B_0^*=1. \tag{3.2}$$

Then we have

LEMMA 3.1. *The limits*

$$A^*\equiv A^*(z)=\lim_{n\to\infty} A_n^*, \qquad B^*\equiv B^*(z)=\lim_{n\to\infty} B_n^*$$

exist as meromorphic functions on $\boldsymbol{C}$. *A^* and B^* have poles at $z=p(n)$ ($n=1, 2, \cdots$) of degree* 1, *and A^* has a pole of degree* 1 *at $z=p(0)$. Moreover, we have the identity*

$$B^*(z)T(z)=(z-p(0))A^*(z). \tag{3.3}$$

REMARK 3.2. We say that (2.8) and (3.1) are equivalent, because two continued fractions defined by (2.7) and by A^*/B^* are identical except for the factor $z-p(0)$. For the detail we refer [3].

PROOF OF LEMMA 3.1. We set $b_n=z-p(n)$. Then, by adding (3.1) from $n=1$ to $n=k$ we have

$$y_k^*=y_0^*-\sum_{n=1}^{k} y_{n-2}^*/b_n b_{n-1}\,. \tag{3.4}$$

If $\{y_k^*\}$ is bounded by some constant independent of k, then (2.3) implies the existence of $\lim_{k\to\infty} y_k^*$. Let c_n be so defined that $|y_k^*|\leqq c_n$ $(k=1, \cdots, n)$. Then, by (3.1) and (2.3) we have

$$|y_n^*|\leqq c_{n-1}(1+K(1+n)^{-2-2\varepsilon})$$

for some $K>0$ independent of n. This implies that $c_n\leqq c_{n-1}(1+K(1+n)^{-2-2\varepsilon})$. By iterating this we see that $\{c_n\}$ is bounded in n.

By (2.3), $A^*(z)$ and $B^*(z)$ are meromorphic functions of z. By the recurrence relation (3.1) $A^*(z)$ and $B^*(z)$ have poles at $z=p(n)$ $(n=1, 2, \cdots)$ of degree 1, and $A^*(z)$ has a pole of degree 1 at $z=p(0)$. This implies that A^*b_0/B^* does not have a pole at $z=p(n)$, $n=0, 1, \cdots$. If $b_n=z-p(n)\neq 0$ for $n=0, 1, 2, \cdots$, then we set

$$y_n=y_n^* b_n b_{n-1}\cdots b_0 b_{-1},\ b_{-1}=1\,, \qquad n=0, 1, 2, \cdots\,. \tag{3.5}$$

By (2.8), y_n^* satisfies (3.1). Since $\{A_n\}$ and $\{B_n\}$ correspond to $\{A_n^*\}$ and $\{B_n^*/b_0\}$, respectively, we get, from (2.7) and (3.5)

$$T(z)=\lim_{n\to\infty} A_n/B_n=\lim_{n\to\infty}(A_n^* b_0)/B_n^*=(A^*b_0)/B^*\,. \tag{3.6}$$

This proves (3.3). □

PROPOSITION 3.3. (*Continued fraction representation of Hill's determinant*). *Let $D(z)$ be given by* (2.5), *and let A^* and B^* be given by Lemma* 3.1. *Then we have*

$$D(z)=B^*(2A^*+B^*)\,. \tag{3.7}$$

PROOF. We set $s_n=(z-p(n))^{-1}$ and

$$D_{k,l}(z)=\det(H_{n,m})_{n,m=-k}^{l}=\det\begin{bmatrix} 1 & s_{-k} & & & 0 \\ s_{-k+1} & 1 & s_{-k+1} & & \\ & \ddots & \ddots & \ddots & \\ 0 & & & 1 & s_{l-1} \\ & & & s_l & 1 \end{bmatrix}. \tag{3.8}$$

Then the limit $D_{\infty,l}(z)=\lim_{k\to\infty} D_{k,l}(z)$ exists. Indeed, by (2.2), if we expand $D_{k,l}(z)$

with respect to k-th column or l-th column, we have recurrence formulas similar to (3.1);

(3.9) $$D_{k,l}=D_{k-1,l}-s_k s_{k-1}D_{k-2,l}\,,\qquad k=1,2,\cdots,$$

(3.10) $$D_{k,l}=D_{k,l-1}-s_l s_{l-1}D_{k,l-2}\,,\qquad l=1,2,\cdots.$$

Hence, by (3.9) and the argument of Lemma 3.1 the limit $D_{\infty,l}$ exists.

Next, we shall show that

(3.11) $$D(z)=\lim_{l\to\infty}D_{\infty,l}(z)\,.$$

Indeed, by letting $k\to\infty$ in (3.10) we see that $\{D_{\infty,l}\}$ satisfies (3.1) with n replaced by l. Hence the limit in (3.11) exists by the arguments of Lemma 3.1

In order to prove (3.11), it suffices to prove that

(3.12) $$\lim_{l\to\infty}(D_{l,l}(z)-D_{\infty,l}(z))=0\,.$$

By summing up (3.9) from $k=l+1$ to infinity, we have

(3.13) $$D_{l,l}-D_{\infty,l}=\sum_{i=l-1}^{\infty}D_{i,l}s_{i-1}s_i\,.$$

On the other hand, it follows from (3.1), (3.2) and (3.9) that

(3.14) $$D_{i,l}=D_{-1,l}A_i^*+D_{0,l}B_i^*\,,\qquad i=-1,0,1,\cdots,$$

where $D_{-1,l}=(D_{0,l}-D_{1,l})/(s_1s_0)$. By (3.10) and the arguments of Lemma 3.1 the sequences $\{D_{-1,l}\}$ and $\{D_{0,l}\}$ are bounded in l. Because $A_i^*\to A^*$, $B_i^*\to B^*$ $(i\to\infty)$ it follows from (3.14) that $\{D_{i,l}\}$ is uniformly bounded in i and l. Hence, by (2.3) and (3.13) we have (3.12).

We shall prove (3.7). By letting $i\to\infty$ in (3.14) we have

(3.15) $$D_{\infty,l}=D_{-1,l}A^*+D_{0,l}B^*\,.$$

Ih view of (3.10) we express $D_{0,l}$ in terms of A_l^* and B_l^*. By (3.8) we have

$$D_{0,1}=\det\begin{bmatrix}1 & s_0\\ s_1 & 1\end{bmatrix}=1-s_0s_1\,.$$

$$D_{0,2}=\det\begin{bmatrix}1 & s_0 & 0\\ s_1 & 1 & s_1\\ 0 & s_2 & 1\end{bmatrix}=1-s_0s_1-s_1s_2\,.$$

On the other hand, it follows from (3.1), (3.2) and $s_n=(z-p(n))^{-1}$ that $A_1^*=A_2^*=-s_0s_1$, $B_1^*=1$, $B_2^*=1-s_1s_2$. Therefore we have

(3.16) $$D_{0,l}=A_l^*+B_l^*\,.$$

Similarly, we have

$$D_{-1,l}=B_l^*. \tag{3.17}$$

By (3.15), (3.16), (3.17) and (3.11) we have

$$D(z)=\lim_{l\to\infty}(D_{-1,l}A^*+D_{0,l}B^*)=B^*A^*+(A^*+B^*)B^*=(2A^*+B^*)B^*. \quad \square$$

For $\nu\geqq 1$ we set

$$T_\nu(z)=\mathop{\mathbf{K}}_{n=\nu}^{\infty}\left(\frac{-1}{b_n}\right), \qquad b_n=z-p(n). \tag{3.18}$$

Then we have

LEMMA 3.4. *Suppose* (2.3). *Then there exist* $N\geqq 0$ *and* $\varepsilon>0$ *such that for any* $\nu\geqq N$, $T_\nu(z)$ *in* (3.18) *converges, and satisfies* $0<|T_\nu(z)|<1-\varepsilon$ *for all* $\nu\geqq N$. *Moreover if* $T(z)\neq\infty$, *then there exists* $K>0$ *such that*

$$|A_n-T(z)B_n|\leqq K|B_n|^{-1} \qquad \textit{for } n=1, 2, \cdots. \tag{3.19}$$

PROOF. By (2.3), there exists $N_1\geqq 0$ and $0<\eta<1$ such that $|z-p(n)|>2+\eta$ for all $n\geqq N_1$. Hence, in order to show the convergence of $T_\nu(z)$ we may assume that $\nu=1$ and $|b_n|>2+\eta$ for $n=1, 2, \cdots$. Since $B_{-1}=0$, $B_0=1$, it follows from (2.8) that $|B_1|>2|B_0|>1+\eta$. Similarly, we have $|B_n|>(1+\eta)|B_{n-1}|$, $n=1, 2, \cdots$.

On the other hand, for $m>n\geqq N$

$$\frac{A_n}{B_n}-\frac{A_m}{B_m}=\sum_{k=n+1}^{m}\left(\frac{A_k}{B_k}-\frac{A_{k-1}}{B_{k-1}}\right)=\sum_{k=n+1}^{m}\frac{A_kB_{k-1}-A_{k-1}B_k}{B_kB_{k-1}}. \tag{3.20}$$

By the determinant formula, $A_kB_{k-1}-A_{k-1}B_k=-1$ which follows from (2.8), we see that the limit $T_1(z)\equiv T(z)$ in (2.7) exists. It is also clear that $|T_\nu(z)|$ is uniformly bounded in $\nu\geqq N_1$. By (2.3), $|b_{\nu-1}|$ tends to infinity as $\nu\to\infty$. It follows that $|T_\nu(z)|<1-\varepsilon$ for $\nu\geqq N$ if N is sufficiently large and ε is small.

By letting $m\to\infty$ in (3.20), we have

$$|B_n|^2\left|\frac{A_n}{B_n}-T(\lambda)\right|\leqq\sum_{k=n+1}^{\infty}\frac{|B_n|^2}{|B_k||B_{k-1}|}.$$

Since $|B_{n+1}|/|B_n|<1+\eta$ for sufficiently large n, this proves (3.19) for large n. Hence, by taking K sufficiently large, we have (3.19). $\square$

3.2. Proof of Theorem 2.1.

PROOF OF THEOREM 2.1. We shall prove that (b) implies (c). Suppose that (c) is not satisfied. We first assume that λ is a pole of $T(z)$. We substitute the Fourier expansion of u, $u=\sum_{\eta}u_{\eta}e^{i\eta x}$ into the equation $(P-\lambda)u=0$. Then we have the recurrence relation (2.8) for $y_n=u_{(n+1,\eta')}$, $\eta'\in \boldsymbol{Z}^{d-1}$ and $z=\lambda$ for $n=0, \pm1, \pm2, \cdots$. If we define B_n $(n=-1, 0, 1, \cdots)$ as in (2.7), it follows from (2.2) that the sequence y_n, $y_n=B_n$ $(n\geqq-1)$, B_{-n-2} $(n<-1)$ solves (2.8) for $n=0, \pm1, \pm2, \cdots$.

In order to show that $\{B_n\}$ is rapidly decreasing as $n\rightarrow\infty$ we prove

$$B_m/B_{m-1}=-T_{m+1}(\lambda), \qquad m\geqq 1. \tag{3.21}$$

By (2.6) and $T(\lambda)=T_1(\lambda)=\infty$, we have that $b_1=-T_2(\lambda)$. Since $B_0=1$ and $B_1=\lambda-p(1)=b_1$ by definition, this implies (3.21) for $m=1$.

Suppose that (3.21) holds for $m\leqq k$. By (2.8) we have that $B_{k+1}=b_{k+1}B_k-B_{k-1}$. If $B_k\neq 0$, then (2.6) yields

$$B_{k+1}/B_k=b_{k+1}-B_{k-1}/B_k=b_{k+1}-T_{k+1}(\lambda)^{-1}=-T_{k+2}(\lambda)^{-1}.$$

Hence we have (3.21) for $m=k+1$. If $B_k=0$, then $B_{k+1}B_{k-1}\neq 0$, to yield $B_{k+1}/B_k=\infty$ and $B_k/B_{k-1}=0$. By the induction hypothesis, the latter equation is equivalent to $b_{k+1}+T_{k+2}(\lambda)=\infty$. Hence both sides of (3.21) for $m=k+1$ are equal to infinity. This proves (2.21).

By Lemma 3.4 the right-hand side of (3.21) is smaller than $1-\varepsilon$ if m is sufficiently large. Hence $\{B_n\}$ is rapidly decreasing. This contradicts to (b).

In the case, $2T(\lambda)=p(0)-\lambda$, we can easily see that $y_n=A_n-TB_n$ $(n>-1)$; $=A_{-n-2}-TB_{-n-2}$ $(n<-1)$ is a solution of (2.8) for $n=0, \pm1, \pm2, \cdots$. On the other hand, by the same arguments as in the proof of (3.21), we can prove (3.18) with B_m replaced by $A_m-T(\lambda)B_m$. Indeed, for $m=1$ (3.21) follows from the identity, $1/T_1=-b_1-T_2$. For general m, (3.21) is proved by (2.8). Hence, we have that $\{y_n\}$ is rapidly decreasing as $n\rightarrow\infty$, a contradiction to (b).

We prove that (c) implies (d). Suppose that $D(\lambda)=0$. Then, it follows from Proposition 3.3 that $B^*(2A^*+B^*)=0$. If $B^*\neq 0$, then $2A^*+B^*=0$. By (3.3), this implies $2T(\lambda)+\lambda-p(0)=0$, which is impossible.

If $B^*=0$, then we have $A^*\neq 0$. Indeed, by (2.3), (3.1) has an exponentially growing solution. Since $\{A_n^*\}$ and $\{B_n^*\}$ are linearly independent, it follows that $A^*\neq 0$. It follows from Lemma 3.1 that $(z-p(0))A^*\neq 0$. By (3.3) this implies $T(\lambda)=\infty$. This contradicts (c).

We shall prove that (d) implies (b). Suppose that there exists a $u=\sum_{\eta}u_{\eta}e^{i\eta x}\in C^{\infty}(\boldsymbol{T})$ such that $Pu=\lambda u$. By substitution, we have (2.8) with $z=\lambda$, $y_n=u_{(n+1,\eta')}$, $\eta'\in\boldsymbol{Z}^{d-1}$ for $n=0, \pm1, \pm2, \cdots$. By (2.4), this implies that the infinite equation $HU=0$, $U=(u_{(n,\eta')})_{-\infty}^{\infty}$ has a nontrivial solution.

Let $k, l\geqq 1$. Because $D_{k,l}(\lambda)$ given by (3.8) tends to $D(\lambda)\neq 0$, and because

$\Pi(1+|\lambda-p(n)|^{-1})$ converges, by (2.3), all components of the cofactor matrix, $\mathrm{co}\,(H'_{n,m})^{l}_{n,m=-k}\equiv\Delta_{k,l}$ are uniformly bounded in k and l. Applying $\Delta_{k,l}$ to the finite section of $HU=0$,

$$(H'_{n,m})^{l}_{-k}(u_{(n,\eta')})^{l}_{-k}=\begin{bmatrix} u_{(-k-1,\eta')}(\lambda-p(k))^{-1} \\ 0 \\ \vdots \\ 0 \\ u_{(-l-1,\eta')}(\eta-p(l))^{-1} \end{bmatrix}\equiv g\,,$$

we have that

$$D_{k,l}(\lambda)(u_{(n,\eta')})^{l}_{-k}=\Delta_{k,l}g\,. \tag{3.22}$$

The right-hand side tends to zero as $k, l\to\infty$, while the left-hand side converges to $D(\lambda)U$. Hence it follows that $D(\lambda)=0$, a contradiction to (d).

Since (a) trivially follows from (b) by the condition $d\geqq 2$, we shall prove that (c) implies (a). Let $u=\sum_{\eta}u_{\eta}e^{i\eta x}\in\mathcal{D}'(\boldsymbol{T}^{d})$ and $(P-\lambda)u\equiv f(x)=\sum_{\eta}f_{\eta}e^{i\eta x}\in C^{\infty}(\boldsymbol{T}^{d})$. By substitution, we have the recurrence relation

$$y_{n}-b_{n}y_{n-1}+y_{n-2}=f_{n}\,,\qquad n=0,\ \pm 1,\ \pm 2,\ \cdots, \tag{3.23}$$

where $y_{n}=u_{(n+1,\eta')}$, $f_{n}=f_{(n,\eta')}$, $b_{n}=\lambda-p(n)$.
We first solve (3.23) for $n\geqq -1$ with the initial conditions $y_{-1}=\alpha$, $y_{0}=\beta$ where α and β will be determined later. By (2.8) the solution is given by

$$y_{n}=\alpha A_{n}+\beta B_{n}+\varphi_{n}\,, \tag{3.24}$$

where

$$\varphi_{n}=-A_{n}\sum_{\nu=1}^{n-1}f_{\nu}B_{\nu-1}+B_{n}\sum_{\nu=1}^{n-1}f_{\nu}A_{\nu-1}+f_{n},\ \varphi_{-1}=\varphi_{0}=0\,.$$

We rewrite this as

$$y_{n}=\alpha A_{n}+(\beta+\gamma_{1})B_{n}+\psi_{n}+f_{n}=(\alpha T(\lambda)+\beta+\gamma_{1})B_{n}+\alpha(A_{n}-T(\lambda)B_{n})+\psi_{n}+f_{n}$$

where $\gamma_{1}=\sum_{\nu=1}^{\infty}f_{\nu}(A_{\nu-1}-T(\lambda)B_{\nu-1})$, $\psi_{0}=\psi_{1}=0$ and, for $n\geqq 3$,

$$\psi_{n}=-(A_{n}-T(\lambda)B_{n})\sum_{\nu=1}^{n-1}f_{\nu}B_{\nu-1}-B_{n}\sum_{\nu=n}^{\infty}f_{\nu}(A_{\nu-1}-T(\lambda)B_{\nu-1})\,. \tag{3.25}$$

Because $\{f_{\nu}\}$ is rapidly decreasing as $\nu\to\infty$, it follows from (3.25) and (3.19) that ψ_{n} in (3.25) is rapidly decreasing. Since y_{n} is of polynomial growth as $n\to\infty$, by assumption, we have

$$\alpha T(\lambda)+\beta+\gamma_{1}=0\,. \tag{3.26}$$

Next we shall solve (3.23) for $n\leqq -1$. We note that $y_{-2}=(\lambda-p(0))y_{-1}-y_{0}+f_{0}=\alpha(\lambda-p(0))-\beta+f_{0}$. We replace n in (3.23) with $-n$, and we set $v_{n}=y_{-n-2}$, to obtain $v_{n}=b_{n}v_{n-1}-v_{n-2}+f_{-n}$, $n=1, 2, \cdots$; $v_{-1}=\alpha$, $v_{0}=(\lambda-p(0))\alpha-\beta+f_{0}$. We make the same arguments as above. Then we have

$$\alpha T(\lambda)+(\lambda-p(0))\alpha+f_0-\beta+\gamma_2=0,\quad \gamma_2=\sum_{\nu=1}^{\infty} f_{-\nu}(A_{\nu-1}-TB_{\nu-1}). \tag{3.27}$$

By (3.26), (3.27) and $2T(\lambda)\neq p(0)-\lambda$ we can determine α and β uniquely as rapidly decreasing functions of η as $|\eta|\to\infty$. Hence u is smooth. This proves (a). □

3.3. Proof of Corollaries.

PROOF OF COROLLARY 2.3. We have $p(\eta_1)=\eta_1^2$. By Lemma 3.4 $T_\nu(0)$ in (3.18) satisfies that $0<|T_\nu(0)|<1$ if ν is sufficiently large. Since $|p(\eta_1)|\geqq 1$ we see that $0<|T_n(0)|<1$ for all n. Recalling that $T(0)=T_1(0)$, we have the corollary from (c) of Theorem 2.1.

PROOF OF COROLLARY 2.5. In view of (c) of Theorem 2.1 we show that $g(z)\equiv 2T(z)+z-p(0)$ has no zeros or poles if $z\notin F$. If z satisfies $|z-p(n)|>2$ for $n=0, 1, 2, \cdots$, then the argument of Corollary 2.3 implies that $0<|T(z)|<1$. Since $|z-p(0)|>2$, $g(z)$ has no zeros.

Next we consider the case $n^{-\delta}<|z-p(n)|\leqq 2$ for some n. By (2.3), there exist $c_0>0$, $k\geqq 0$ such that, if $m\geqq k$ and $m\neq n$, then $|z-p(m)|>c_0 m^{\varepsilon}$. It follows that $|T_{n+2}(z)|<1$. Hence $|T_{n+1}(z)|<(c_0(n+1)^{\varepsilon}-1)^{-1}$, to yield

$$|T_n(z)|<1/(n^{-\delta}-(c_0(n+1)^{\varepsilon}-1)^{-1}). \tag{3.28}$$

If we take k sufficiently large, then, for any $n\geqq k$ the right-hand side of (3.28) is smaller than $2n^{\delta}$. Since $|z-p(n-1)|>c_0 n^{\varepsilon}$ and $\varepsilon>\delta$, it follows that $|T_{n-1}(z)|<1$. This proves that $T(z)$ has no poles or zeros. □

PROOF OF REMARK 2.4. We give a rough sketch of Remark 2.4. We define E as the set of all poles and zeros of a meromorphic function $g(z)=2T(z)+z-p(0)$. E is discrete, by definition. It follows from Theorem 4.61 of [3] that the poles of $T(z)$ are real.

On the other hand, by (9.48) of [3] we have representations

$$T(z)=-\int_{-\infty}^{\infty}(t+z)^{-1}d\psi(t) \qquad (\mathrm{Im}\, z>0 \text{ or } \mathrm{Im}\, z>0) \tag{3.29}$$

for some real-valued, bounded, monotone non-decreasing function $\psi(t)$ with infinitely many points of increase on $-\infty\leqq t\leqq\infty$. Hence the condition $g(z)=0$ implies

$$2\,\mathrm{Im}\, z\int_{-\infty}^{\infty}|t+z|^{-2}d\psi(t)+\mathrm{Im}\, z=0. \tag{3.30}$$

(3.30) has a solution only if $\operatorname{Im} z=0$. Hence we have $E\subset \boldsymbol{R}$. The result follows from Theorem 2.1. □

References

[1] Greenfield, S. and Wallach, N., Global hypoellipticity and Liouville numbers, Proc. Amer. Math. Soc. **31** (1972), 112-114.

[2] Taira, K., Le principe du maximum et l'hypoellipticité globale, Seminaire Bony-Sjöstrand-Meyer 1984-1985, $n\circ 1$.

[3] Jones, W. B. and Thron, W. J., Continued fractions, Analytic theory and applications, Addison-Wesley Publishing Company, 1980.

[4] Yoshino, M., A class of globally hypoelliptic operators on the torus. Math. Z. **201** (1989), 1-11.

Faculty of Economics
Chuo University
Higashinakano, Hachioji
192-03 Japan