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ON RESOLUTIONS FOR PAIRS OF SPACES

by

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1. Introduction

Let (X, A) be a pair of topological spaces, $A\subseteq X$, and let $(\boldsymbol{X}, \boldsymbol{A})=((X_\lambda, A_\lambda), p_{\lambda\lambda'}, \Lambda)$ be an inverse system of pairs of spaces and maps of pairs indexed by a directed set Λ. By a morphism $\boldsymbol{p}:(X, A)\rightarrow(\boldsymbol{X}, \boldsymbol{A})$ of pro-Top^2 we mean a collection of maps of pairs $p_\lambda:(X, A)\rightarrow(X_\lambda, A_\lambda)$ such that

$$p_{\lambda\lambda'}p_{\lambda'}=p_\lambda,\ \lambda\leq\lambda'.$$

A resolution of (X, A) is a morphism $\boldsymbol{p}=(p_\lambda):(X, A)\rightarrow(\boldsymbol{X}, \boldsymbol{A})$ of pro-Top^2, which satisfies the following two conditions.

(R1) Let (P, Q) be an ANR-pair, i.e., a pair of ANR's for metric spaces such that Q is a closed subset of P. Let $\mathcal{V}$ be an open covering of P and let $f:(X, A)\rightarrow (P, Q)$ be a map of pairs. Then there there exists a $\lambda\in\Lambda$ and a map of pairs $g:(X_\lambda, A_\lambda)\rightarrow(P, Q)$ such that gp_λ and f are $\mathcal{V}$-near maps.

(R2) Let (P, Q) be an ANR-pair and let $\mathcal{V}$ be an open covering of P. Then these exists an open covering $\mathcal{V}'$ of P such that whenever $\lambda\in\Lambda$ and $g, g':(X_\lambda, A_\lambda)\rightarrow (P, Q)$ are maps such that the maps gp_λ and $g'p_\lambda$ are $\mathcal{V}'$-near, then there exists a $\lambda'\geq\lambda$ such that the maps $gp_{\lambda\lambda'}$ and $g'p_{\lambda\lambda'}$ are $\mathcal{V}$-near.

If all (X_λ, A_λ) are ANR-pairs (polyhedral pairs), we speak of an ANR-resolution (polyhedral resolution) of the pair (X, A).

If we leave out A, A_λ and Q, the above definition reduces to the definition of a resolution $\boldsymbol{p}:X\rightarrow\boldsymbol{X}=(X_\lambda, p_{\lambda\lambda'}, \Lambda)$ (ANR-resolution or polyhedral resolution, resp.) of a single space X.

The notion of resolution of a space was introduced in 1981 by the author [4] (also see [5] and [6]). Resolutions for pairs were first considered in [6].

Resolutions can be viewed as special inverse limits. In fact, these notions coincide for compact spaces [6]. In the non-compact case resolutions appear to be the appropriate substitutes for inverse limits, the latter notion being only of little value for non-compact spaces.

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The notion of resolution is basic to the recent development ([1], [3]) of strong shape theory and Steenrod-Sitnikov homology [2] for arbitrary spaces. In order to extend these theories also to the case of pairs of spaces, we need various results on resolutions of pairs of spaces, not previously considered in [6]. This primarily motivates the choice of the topics of this paper.

The main results in the paper are Theorems 2 and 6. The first theorem gives a useful internal characterization of resolutions of pairs and the second theorem establishes the existence of ANR-resolutions for pairs. The analogous result for polyhedral resolutions was proved in [6]. However, the method of proof used in [6] could not be used here, because generally, closed subsets of an ANR fail to have a basis of neighborhoods all of whose members are closed ANR's.

2. Characterizing resolutions of spaces

Let $\boldsymbol{p}: X\to \boldsymbol{X}$ be a morphism of pro-Top. We will consider the following properties of $\boldsymbol{p}$.

(B1) For every $\lambda\in\Lambda$ and every open neighborhood U of $\overline{p_\lambda(X)}$ in X_λ there exists a $\lambda'\geq\lambda$ such that $p_{\lambda\lambda'}(X_{\lambda'})\subseteq U$.

(B2) For every normal covering $\mathcal{U}$ of X there is a $\lambda\in\Lambda$ and a normal covering $\mathcal{V}$ of X_λ such that $(p_\lambda)^{-1}(\mathcal{V})$ refines $\mathcal{U}$.

It was proved in [4] that a morphism $\boldsymbol{p}: X\to\boldsymbol{X}$, which has properties (B1) and (B2) is a resolution. Conversely, if all X_λ are normal spaces and $\boldsymbol{p}$ is a resolution, then $\boldsymbol{p}$ has properties (B1) and (B2) (for alternate proofs see [6], I, §6, Theorems 3, 4 and 5).

Recently, T. Watanabe [7] has introduced the following property (B1)* (he denotes it by (B4))

(B1)* For every $\lambda\in\Lambda$ and every normal covering $\mathcal{U}$ of X_λ there exists a $\lambda'\geq\lambda$ such that

$$p_{\lambda\lambda'}(X_{\lambda'})\subseteq \mathrm{St}\,(p_\lambda(X),\mathcal{U}). \tag{1}$$

Modifying the proofs given in [4], Watanabe has obtained the following characterization theorem.

THEOREM 1. (Watanabe). A morphism $\boldsymbol{p}: X\to\boldsymbol{X}$ of pro-Top is a resolution if and only if $\boldsymbol{p}$ has properties (B1)* and (B2).

The value of Watanabe's theorem is that it holds without any restrictions to the spaces X_λ, and condition (B1)* is more natural than (B1). However, Watanabe has shown that in the case of normal spaces X_λ the properties (B1) and (B1)* are equivalent.

For the sake of completeness we give here an alternate and somewhat simpler proof of Watanabe's theorem based on the corresponding proofs in [6].

Proof. Let us assume that $\boldsymbol{p}$ is a resolution. We will first show that $\boldsymbol{p}$ has property (B1)*.

Let $\lambda\in\Lambda$ and let $\mathcal{U}$ be a normal covering of X_λ. By definition, this means that there exists a metric space M, an open covering $\mathcal{V}$ of M and a map $g: X_\lambda \to M$ such that $(g^{-1})(\mathcal{V})$ refines $\mathcal{U}$. Clearly,

(2) $$g^{-1}(\mathrm{St}\,(gp_\lambda(X), \mathcal{V})) \subseteq \mathrm{St}\,(p_\lambda(X), \mathcal{U}),$$

(3) $$\overline{gp_\lambda(X)} \subseteq \mathrm{St}\,(gp_\lambda(X), \mathcal{V}).$$

Let $h: M \to I = [0, 1]$ be a map such that

(4) $$h|\overline{gp_\lambda(X)} = 0$$

(5) $$h|M \backslash \mathrm{St}\,(gp_\lambda(X), \mathcal{V}) = 1.$$

We now put $f = hg: X_\lambda \to I, f' = 0: X_\lambda \to I$. By (4), $fp_\lambda = f'p_\lambda = 0$. Therefore, by (R2), there is a $\lambda' \geq \lambda$ such that $fp_{\lambda\lambda'}$ and $f'p_{\lambda\lambda'} = 0$ are $\mathcal{W}$-near, where $\mathcal{W}$ is the covering of I, which consists of the open sets $[0, 1)$ and $(0, 1]$. Consequently, $fp_{\lambda\lambda'}(X_{\lambda'}) \subseteq [0, 1)$, and thus, by (5),

(6) $$gp_{\lambda\lambda'}(X_{\lambda'}) \subseteq \mathrm{St}\,(gp_\lambda(X), \mathcal{V}).$$

Now (2) yields the desired relation (1).

In order to show that $\boldsymbol{p}$ also has property (B2) we need this simple Lemma.

LEMMA 1. Let $\mathcal{U}$ be a normal covering of a space X. Then there exists an ANR P, an open covering $\mathcal{W}$ of P and a map $h: X \to P$ such that $h^{-1}(\mathcal{W})$ refines P.

Proof of Lemma 1. By definition there exists a metric space M, an open covering $\mathcal{V}$ of M and a map $f: X \to M$ such that $f^{-1}(\mathcal{V})$ refines $\mathcal{U}$. By the Wojdislawski-Kuratowski embedding theorem ([6], I, §3.1, Theorem 2), one can assume that M is a closed subset of a convex set P of a normed vector space. For every $V \in \mathcal{V}$ there exists an open set W_V of P such that $V = W_V \cap M$. Therefore, $\mathcal{W} = (W_V,\ V \in \mathcal{V}) \cup \{P \backslash M\}$ is an open covering of M. If we take for h the composition of f with the inclusion $M \to P$, then $h^{-1}(\mathcal{V}) = f^{-1}(\mathcal{V}) \cup \{\bigcirc\}$ refines $\mathcal{U}$. Moreover, P is an AR by the Dugundji extension theorem ([6], I, §3.1. Theorem 3).

Proof of property (B2). Let $\mathcal{U}$ be a normal covering of X. We choose $P, \mathcal{W}$ and h as in Lemma 1. Let $\mathcal{W}'$ be a star-refinement of $\mathcal{W}$. By (R1), there is a $\lambda \in \Lambda$ and a map $f: X_\lambda \to P$ such that the maps fp_λ and h are $\mathcal{W}'$-near. Let us put $\mathcal{V}=f^{-1}(\mathcal{W}')$. We chaim that $p_\lambda^{-1}(\mathcal{V})$ refines $\mathcal{U}$. Indeed, let $V=f^{-1}(W')$, $W' \in \mathcal{W}'$. Let $W \in \mathcal{W}$ be such that

$$\mathrm{St}(W', \mathcal{W}') \subseteq W.$$

It suffices to show that

$$(p_\lambda)^{-1}(V) \subseteq h^{-1}(W).$$

If $x \in (p_\lambda)^{-1}(V)$, then there is a $W_1' \in \mathcal{W}'$ such that

$$fp_\lambda(x) h(x) \in W_1'.$$

Since $fp_\lambda(x) \in W'$, we conclude that $W' \cap W_1' \neq \emptyset$ and therefore

$$h(x) \in W_1' \subseteq \mathrm{St}(W', \mathcal{W}') \subseteq W.$$

Consequently, $x \in h^{-1}(W)$.

Let us now assume that $\boldsymbol{p}: X \to \boldsymbol{X}$ has properties (B1)* and (B2). We will first verify property (R1). Let P be an ANR, $\mathcal{V}$ an open covering of P and $f: X \to P$ a map. One can assume that P is a closed subset of a convex set K in a normed vector space. Let G be an open neighborhood of P in K, which admits a retraction $r: G \to P$. Let $\mathcal{V}'=r^{-1}(\mathcal{V})$ and let $\mathcal{V}''$ be an open covering of G, which refines $\mathcal{V}'$ and all of its members are convex. Then $\mathcal{U}=f^{-1}(\mathcal{V}'')$ is a normal covering of X. By (B2) there is a $\mu \in \Lambda$ and a normal covering $\mathcal{U}'$ of X_μ such that $(p_\mu)^{-1}(\mathcal{U}')$ refines $\mathcal{U}$. Let $\mathcal{U}''$ be a locally finite normal covering of X_μ, which is a star-refinement of $\mathcal{U}'$. One can assume that $\mathcal{U}''=k^{-1}(\mathcal{K})$, where $k: X \to M$ is a mapping into a metric space M and $\mathcal{K}$ is a locally finite open covering of M. Then $\mathcal{W}=\{W \in \mathcal{U}'': W \cap p_\mu(X) \neq \emptyset\}$ is a normal locally finite open covering of $N=\mathrm{St}(p_\mu(X), \mathcal{U}')$. Let $(\varphi_W, W \in \mathcal{W})$ be a partition of unity on N subordinated to the cover $\mathcal{W}$. For every $W \in \mathcal{W}$ we choose a point $y_W \in f((p_\mu)^{-1}(W))$ and we then define a map $h: N \to K$ by the formula

$$h(z)=\sum_{W \in \mathcal{W}} \varphi_W(z) y_W, \quad z \in N. \tag{7}$$

We will now show that h is actually a map into G and the maps hp_μ and f are $\mathcal{V}''$-near.

Let $z \in N$ and let $W_0, \cdots, W_n$ be all the members of $W \in \mathcal{W}$ for which $\varphi_W(z) \neq 0$. Then

$$z \in W_0 \cap \cdots W_n \subseteq \mathrm{St}(W_0, \mathcal{U}'') \subseteq U' \tag{8}$$

for some $U' \in \mathcal{U}'$. Let $U \in \mathcal{U}$ be such that

(9) $$(p_\mu)^{-1}(U') \subseteq U.$$

Then

(10) $$y_{w_i} \in f((p_\mu)^{-1}(W_i)) \subseteq f((p_\mu)^{-1}(U')) \subseteq f(U),\ i=0, \cdots, n.$$

Since $f(U)$ is contained in some $V'' \in \mathcal{V}''$ and V'' is convex, it follows that also $h(z) \in V'' \subseteq G$.

Now let $x \in X$ and let $z = p_\mu(x)$. Since $p_\mu(x) = z \in W_0$, we see, by (8) and (9), that

(11) $$f(x) \in f((p_\mu)^{-1}(W_0)) \subseteq f(U) \subseteq V''.$$

Since, by (10), also $hp_\mu(x) = h(z) \in V''$, we see that indeed, f and hp_μ are $\mathcal{V}''$-near maps. Therefore, the maps rhp_μ and $f = rf$ are $\mathcal{V}$-near maps.

We now apply property (B1)* and find a $\lambda \geq \mu$ such that $p_{\mu\lambda}(X_\lambda) \subseteq N$. Clearly, the map $g = rhp_{\mu\lambda}: X_\lambda \to P$ has the desired property that the maps gp_λ and f are $\mathcal{V}$-near.

We will now verify property (R2). Let P be an ANR and let $\mathcal{V}$ be an open covering of P. Let $\mathcal{V}'$ be a star-refinement of $\mathcal{V}$. We will show that for any $\lambda \in \Lambda$ and any maps $f_1, f_2: X_\lambda \to P$ such that $f_1 p_\lambda$ and $f_2 p_\lambda$ are $\mathcal{V}'$-near, there exists a $\lambda' \geq \lambda$ such that $f_1 p_{\lambda\lambda'}$ and $f_2 p_{\lambda\lambda'}$ are $\mathcal{V}$-near maps.

Let $U_i = (f_i)^{-1}(\mathcal{V}')$, $i=1, 2$. Then U_1, U_2 are normal coverings of X_λ. Let $\mathcal{U}$ be a normal covering of X_λ, which refines both coverings $\mathcal{U}_1$ and $\mathcal{U}_2$. Let $N = \mathrm{St}(p_\lambda(X), \mathcal{U})$. We claim that the maps $f_1|N$ and $f_2|N$ are $\mathcal{V}$-near. Indeed, let $y \in N$. Then there is a member U of $\mathcal{U}$ and an element $x \in X$ such that $y \in U$ and $p_\lambda(x) \in U$. Then there are elements $V_1', V_2' \in \mathcal{V}'$ such that $f_1(U) \subseteq V_1'$, $f_2(U) \subseteq V_2'$. Moreover, by assumption, there is an element $V' \in \mathcal{V}'$ such that $f_1 p_\lambda(x), f_2 p_\lambda(x) \in V'$. Since $p_\lambda(x) \in U$, we see that $V_1' \cap V' \neq \emptyset$ and $V_2' \cap V' \neq \emptyset$. Consequently, there is an element $V \in \mathcal{V}$ such that $V_1' \cup V' \cup V_2' \subseteq \mathrm{St}(V', \mathcal{V}') \subseteq V$. Clearly, $f_1(y), f_2(y) \in V$, i.e., $f_1|N$ and $f_2|N$ are $\mathcal{V}$-near. We now apply (B1)* and conclude that there is a $\lambda' \geq \lambda$ such that $p_{\lambda\lambda'}(X_{\lambda'}) \subseteq N$. Therefore, $f_1 p_{\lambda\lambda'}$ and $f_2 p_{\lambda\lambda'}$ are also $\mathcal{V}$-near maps. This completes the proof of Theorem 1.

3. Characterizing resolutions of pairs

For a morphism $\boldsymbol{p}: (X, A) \to (\boldsymbol{X}, \boldsymbol{A})$ of pro-Top^2 we now introduce a relative version of property (B1)*.

(B1)** For every $\lambda \in \Lambda$ and every normal covering $\mathcal{U}$ of X_λ there exists a $\lambda' \geq \lambda$ such that

(1) $$p_{\lambda\lambda'}(A_{\lambda'}) \subseteq \mathrm{St}(p_\lambda(A), \mathcal{U}).$$

With every morphism $\boldsymbol{p}:(X,A)\to(\boldsymbol{X},\boldsymbol{A})$ of pro-Top² we can associate two morphisms of pro-Top $\boldsymbol{p}_X:X\to\boldsymbol{X}$ and $\boldsymbol{p}_A:A\to\boldsymbol{A}$, which are defined by restricting $p_\lambda:(X,A)\to(X_\lambda,A_\lambda)$ to $p_\lambda:X\to X_\lambda$ and $p_\lambda:A\to A_\lambda$ respectively. The main result of this section is the following theorem.

THEOREM 2. A morphism $\boldsymbol{p}:(X,A)\to(\boldsymbol{X},\boldsymbol{A})$ of pro-Top² is a resolution if and only if $\boldsymbol{p}_X:X\to\boldsymbol{X}$ has properties (B1)*, (B2) and $\boldsymbol{p}$ has property (B1)**.

Proof. Let us first assume that $\boldsymbol{p}_X:X\to\boldsymbol{X}$ has properties (B1)*, (B2), i.e. is a resolution of X, and $\boldsymbol{p}$ has property (B1)**. We will first verify property (R1) for $\boldsymbol{p}$.

Let (P,Q) be an ANR-pair, let $\mathcal{V}$ be an open covering of P and let $f:(X,A)\to(P,Q)$ be a map. We choose for $\mathcal{V}'$ a star-refinement of $\mathcal{V}$. Since (P,Q) is an ANR-pair, it is easy to find an open neighborhood G of Q in P and a map $k:P\to P$ such that $k|G$ is a retraction $G\to Q$ and the maps 1_P and k are $\mathcal{V}'$-near (see [6], I, §6, Lemma 4). Let $\mathcal{V}''$ be an open covering of P, which refines $\mathcal{V}'$ and star-refines the covering $\{G,P\backslash Q\}$. Since $\boldsymbol{p}_X:X\to\boldsymbol{X}$ is a resolution, there exists a $\lambda\in\Lambda$ and a map $g:X_\lambda\to P$ such that the maps gp_λ and f are $\mathcal{V}''$-near. Let $\mathcal{U}=g^{-1}(\mathcal{V}'')$. We claim that

$$g(\mathrm{St}\,(p_\lambda(A),\mathcal{U}))\subseteq G. \tag{2}$$

Indeed, if $y\in\mathrm{St}\,(p_\lambda(A),\mathcal{U})$, then there exist a point $a\in A$ and a member $U\in\mathcal{U}$ such that $y\in U$ and $p_\lambda(a)\in U$. Let V'' be an element of $\mathcal{V}''$ such that $U=g^{-1}(V'')$. Then $g(y),gp_\lambda(a)\in V''$. There is also an element $V_1''\in\mathcal{V}''$ such that $gp_\lambda(a),f(a)\in V_1''$. Therefore, some element of $\{G,P\backslash Q\}$ must contain $\{g(y),f(a)\}\subseteq V''\cup V_1''$. This cannot be P\Q, because $f(a)\in Q$. Consequently, $g(y)\in G$, which establishes (2). We apply (B1)** and obtain an index $\lambda'\geq\lambda$ such that (1) holds. We now define a map of pairs $g':(X_{\lambda'},A_{\lambda'})\to(P,Q)$ by putting

$$g'=kgp_{\lambda\lambda'}. \tag{3}$$

By assumption on k, the maps $g'p_{\lambda'}=kgp_\lambda$ and gp_λ are $\mathcal{V}'$-near. Since also gp_λ and f are $\mathcal{V}'$-near, it follows that the maps $g'p_{\lambda'}$ and f are $\mathcal{V}$-near, which establishes (R1). That property (R2) for p holds is an immediate consequence of the same property for $\boldsymbol{p}_X$.

We will now prove the converse. Let $\boldsymbol{p}:(X,A)\to(\boldsymbol{X},\boldsymbol{A})$ be a resolution of pairs. Then $\boldsymbol{p}_X:X\to\boldsymbol{X}$ is a resolution of X. This is so because one can view maps $f:X\to P$, $P\in$ANR, as maps of pairs $f:(X,A)\to(P,P)$. Therefore, p_X has properties (B1)* and (B2). We will now establish property (B1)**.

Let $\lambda\in\Lambda$ and let $\mathcal{U}$ be a normal covering of X. Then there exists a metric space M, an open covering $\mathcal{W}$ of M and a map $g: X_\lambda \to M$ such that $g^{-1}(\mathcal{W})$ refines $\mathcal{U}$. Clearly,

$$\overline{gp_\lambda(A)} \subseteq \mathrm{St}\,(gp_\lambda(A), \mathcal{W}). \tag{4}$$

Let $k: M \to I=[0,1]$ be a map such that

$$k(\overline{gp_\lambda(A)})=0, \tag{5}$$

$$k(M \setminus \mathrm{St}\,(\overline{gp_\lambda(A)}, \mathcal{W}))=1. \tag{6}$$

Consider the ANR-pair (I, {0}) and let $\mathcal{V}$ be the covering $\{[0,1), (0,1]\}$ of I. Applying (R2) for $\boldsymbol{p}_X$, we associate with $\mathcal{V}$ a covering $\mathcal{V}'$. Since $kgp_\lambda(A)=0$, property (R1) for $\boldsymbol{p}:(X, A) \to (\boldsymbol{X}, \boldsymbol{A})$ implies the existence of an index $\lambda' \in \Lambda$ and of a map $h:(X_{\lambda'}, A_{\lambda'}) \to (\mathrm{I}, \{0\})$ such that the maps kgp_λ and $hp_{\lambda'}$ are $\mathcal{V}'$-near. We now choose an index $\lambda'' \geq \lambda, \lambda'$ and consider the maps

$$f_1 = kgp_{\lambda\lambda''},\ f_2 = hp_{\lambda'\lambda''}: X_{\lambda''} \to I.$$

Note that the maps $f_1 p_{\lambda''} = kgp_\lambda$ and $f_2 p_{\lambda''} = hp_{\lambda'}$ are $\mathcal{V}'$-near. Therefore, there exists an index $\lambda^* \geq \lambda''$ such that the maps $f_1 p_{\lambda''\lambda^*} = kgp_{\lambda\lambda^*}$ and $f_2 p_{\lambda''\lambda^*} = hp_{\lambda'\lambda^*}$ are $\mathcal{V}$-near. We claim that

$$p_{\lambda\lambda^*}(A_{\lambda^*}) \subseteq \mathrm{St}\,(p_\lambda(A), \mathcal{U}). \tag{7}$$

Indeed, for any $x \in A_{\lambda^*}$ we have

$$f_2 p_{\lambda''\lambda^*}(x) = hp_{\lambda'\lambda^*}(x) \in h(A_{\lambda'}) = \{0\}. \tag{8}$$

Since $[0,1)$ is the only element of $\{[0,1), (0,1]\}$, which contains 0, it follows that

$$f_1 p_{\lambda''\lambda^*}(x) = kgp_{\lambda\lambda^*}(x) \in [0,1). \tag{9}$$

We conclude, by (6), that

$$gp_{\lambda\lambda^*}(x) \in \mathrm{St}\,(gp_\lambda(A)), \mathcal{W}). \tag{10}$$

Consequently, there is an element $W \in \mathcal{W}$ and a point $a \in A$ such that $gp_{\lambda\lambda^*}(x), gp_\lambda(a) \in W$. Therefore, $p_{\lambda\lambda^*}(x), p_\lambda(a) \in g^{-1}(W) \subseteq U$ for some $U \in \mathcal{U}$. This yields the desired relation $p_{\lambda\lambda^*}(x) \in \mathrm{St}\,(p_\lambda(A), \mathcal{U})$.

REMARK 1. If $\mathcal{U}$ is a normal covering of X_λ, then $\mathcal{U}|A_\lambda$ is a normal covering of A_λ. Therefore, property (B1)* for $\boldsymbol{p}_A$ implies property (B1)** for $\boldsymbol{p}:(X, A) \to (\boldsymbol{X}, \boldsymbol{A})$.

REMARK 2. We say that a subset $A \subseteq X$ is normally embedded (or P-embedded) in a space X provided every normal covering $\mathcal{V}$ of A admits a normal covering $\mathcal{U}$ of X such that the restriction $\mathcal{U}|A$ refines $\mathcal{V}$. If $A_\lambda \subseteq X_\lambda$ is normally embedded in X_λ for each $\lambda\in\Lambda$, then property (B1)** for $\boldsymbol{p}:(X, A)\to(\boldsymbol{X}, \boldsymbol{A})$ implies property (B1)* for $\boldsymbol{p}_A$.

THEOREM 3. Let $\boldsymbol{p}:(X, A)\to(\boldsymbol{X}, \boldsymbol{A})$ be a resolution such that A_λ is normally embedded in X_λ for each $\lambda\in\Lambda$. The induced morphism $\boldsymbol{p}_A: A\to\boldsymbol{A}$ is a resolution if and only if A is normally embedded in X.

Proof. By Theorem 2, $\boldsymbol{p}$ has property (B1)**. Therefore, by Remark 2, $\boldsymbol{p}_A$ has property (B1)* and it suffices to show that $\boldsymbol{p}_A$ also has property (B2). However, this is an immediate consequence of the fact that $\boldsymbol{p}_X$ has property (B2) and A is normally embedded in X.

Now assume that p_A is a resolution. Let $\mathcal{V}$ be a normal covering of A. By Lemma 1, there is an ANR Q an open covering $\mathcal{W}$ of Q and a map $h: A\to Q$ such that $h^{-1}(\mathcal{W})$ refines $\mathcal{V}$. Let $\mathcal{W}'$ be a star-refinement of $\mathcal{W}$. By (R1) for p_A there is a $\lambda\in\Lambda$ and there is a map $f: A_\lambda\to Q$ such that $fp_\lambda|A$ and h are $\mathcal{W}'$-near maps. Then $f^{-1}(\mathcal{W}')$ is a normal covering of A_λ. Since A_λ is normally embedded in X_λ, there is a normal covering $\mathcal{U}'$ of X_λ such that $\mathcal{U}'|A_\lambda$ refines $f^{-1}(\mathcal{W}')$. We now put $\mathcal{U}=p^{-1}(\mathcal{U}')$. Clearly, $\mathcal{U}$ is a normal convering of X. Moreover, $\mathcal{U}|A$ refines $\mathcal{V}$. Indeed, let $U\in\mathcal{U}$. Then there is an element $U'\in\mathcal{U}'$ and an element $W'\in\mathcal{W}'$ such that $U=p_\lambda^{-1}(U')$, $U'\cap A_\lambda\subseteq f^{-1}(W')$. Let $W\in\mathcal{W}$ and $V\in\mathcal{V}$ be chosen in such a way that $\mathrm{St}(W', \mathcal{W}')\subseteq W$, $h^{-1}(W)\subseteq V$. We claim that $U\cap A\subseteq V$. Indeed, if $a\in U\cap A$, then $p_\lambda(a)\in U'\cap A_\lambda$ and therefore $fp_\lambda(a)\in W'$. Moreover, since $fp_\lambda|A$ and h are $\mathcal{W}'$-near, there is an element $W_1'\in\mathcal{W}'$ such that $fp_\lambda(a), h(a)\in W_1'$. Therefore, $h(a)\in \mathrm{St}(W', \mathcal{W}')\subseteq W$, i.e., $a\in h^{-1}(W)\subseteq V$.

4. Resolutions and direct products

Let $\boldsymbol{p}:(X, A)\to(\boldsymbol{X}, \boldsymbol{A})$ be a morphism of pro-Top². For any space K, we associate with $\boldsymbol{p}$ the system $K\times(\boldsymbol{X}, \boldsymbol{A})=((K\times X_\lambda, K\times A_\lambda), 1\times p_{\lambda\lambda'}, \Lambda)$ and the morphism $1\times\boldsymbol{p}: K\times(X, A)\to K\times(\boldsymbol{X}, \boldsymbol{A})$, given by the maps $1\times p_\lambda:(K\times X, K\times A)\to(K\times X_\lambda, K\times A_\lambda)$. Similarly, we associate with $\boldsymbol{p}: X\to\boldsymbol{X}$ the morphism $1\times\boldsymbol{p}: K\times X\to K\times\boldsymbol{X}$.

THEOREM 4. If $\boldsymbol{p}: X\to\boldsymbol{X}$ is a resolution and K is a compact Hausdorff space then $1\times\boldsymbol{p}: K\times X\to K\times\boldsymbol{X}$ is also a resolution.

In the proof we use the following lemma, proved in [3], II, 1, Lemma 2.

LEMMA 2. Let X be a topological space and K a compact Hausdorff space. Then every normal covering $\mathcal{U}$ of $K\times X$ admits a normal covering $\mathcal{V}$ of X such that each $V\in\mathcal{V}$ admits an open covering $\mathcal{W}_V$ of K such that $\mathcal{W}=(\mathcal{W}_V\times V, V\in\mathcal{V})$

is a covering of $K\times X$ (stacked covering), which refines $\mathcal{U}$.

Proof of Theorem 4. It suffices to verify properties (B1)* and (B2) for $1\times \boldsymbol{p}$.

Verification of (B1)*. Let $\lambda\in\Lambda$ and let $\mathcal{U}$ be a normal covering of $K\times X_\lambda$. Let $\mathcal{W}=(\mathcal{W}_V\times V,\ V\in\mathcal{V})$ be a stacked covering of $K\times X_\lambda$ such that $\mathcal{V}$ is a normal covering of X_λ and $\mathcal{W}$ refines $\mathcal{U}$ (Lemma 2). Clearly,

$$\text{St}\,((1\times p_\lambda)(K\times X),\mathcal{W})=K\times \text{St}\,(p_\lambda(X),\mathcal{V}). \tag{1}$$

Therefore, by property (B1)* for $\boldsymbol{p}$, there is a $\lambda'\geq\lambda$ such that

$$p_{\lambda\lambda'}(X_{\lambda'})\subseteq \text{St}\,(p_\lambda(X),\mathcal{V}). \tag{2}$$

Consequently,

$$(1\times p_{\lambda\lambda'})(K\times X_{\lambda'})\subseteq \text{St}\,((1\times p_\lambda)(K\times X),\mathcal{W})\subseteq \text{St}\,((1\times p_\lambda)(K\times X),\mathcal{U}), \tag{3}$$

Verification of (B2). Let $\mathcal{U}$ be a normal covering of $K\times X$ and let $\mathcal{W}=(\mathcal{W}_V\times V,\ V\in\mathcal{V})$ be a stacked covering of $K\times X$, such that $\mathcal{V}$ is a normal covering of X and $\mathcal{W}$ refines $\mathcal{U}$. By (B2) for for $\boldsymbol{p}$, there is a $\lambda\in\Lambda$ and a normal covering $\mathcal{V}_\lambda$ of X_λ such that $(p_\lambda)^{-1}(\mathcal{V}_\lambda)$ refines $\mathcal{V}$. We now put

$$\mathcal{W}_\lambda=(\mathcal{W}_V\times V_\lambda,\ V_\lambda\in\mathcal{V}_\lambda), \tag{4}$$

where $(p_\lambda)^{-1}(V_\lambda)\subseteq V\in\mathcal{V}$. Clearly, $\mathcal{W}_\lambda$ is a normal covering of $K\times X_\lambda$ and $(1\times p_\lambda)^{-1}(\mathcal{W}_\lambda)$ refines $\mathcal{W}$ and thus also refines $\mathcal{U}$.

The analogous theorem for pairs assumes the following form.

THEOREM 5. Let $\boldsymbol{p}:(X,A)\rightarrow(\boldsymbol{X},\boldsymbol{A})$ be a resolution such that A_λ is normally embedded in X_λ for each $\lambda\in\Lambda$. If K is a compact Hausdorff space, then $1\times\boldsymbol{p}:K\times(X,A)\rightarrow K\times(\boldsymbol{X},\boldsymbol{A})$ is a resolution of pairs.

Proof. By Theorem 2, it suffices to show that $1\times\boldsymbol{p}_X:K\times X\rightarrow K\times\boldsymbol{X}$ has properties (B1)* and (B2) and $1\times\boldsymbol{p}$ has property (B1)**. The first assertion follows from Theorems 2, 1 and 4. Since $\boldsymbol{p}$ has property (B1)** (Theorem 2), Remark 2 implies that $\boldsymbol{p}_A$ has property (B1)*. This implies that also $1\times\boldsymbol{p}_A$ has property (B1)*, because the argument given in the first part of the proof of Theorem 4 applies (since it only uses property (B1)* of $\boldsymbol{p}_A$). We now apply Remark 1 and conclude that $1\times\boldsymbol{p}$ has property (B1)**.

5. ANR-resolutions of pairs

The main purpose of this section is to prove the following theorem.

THEOREM 6. Every pair of topological spaces (X, A) admits an ANR-resolution $\boldsymbol{p}:(X, A)\rightarrow(\boldsymbol{X}, \boldsymbol{A})$ indexed by a cofinite set Λ.

The analogous theorem for single spaces was established in [4]. The proof for pairs, presented here, follows the same general idea.

In the proof we will need the following lemma.

LEMMA 3. Let $f:(X, A)\rightarrow(Y, B)$ be a map of a pair of topological spaces to an ANR-pair. There exists an ANR-pair (Z, C) with density

(1) $$s(Z)\leq \max(s(X), s(A)),$$

(2) $$s(C)\leq s(A)$$

and there exist maps $g:(X, A)\rightarrow(Z, C)$, $h:(Z, C)\rightarrow(Y, B)$ such that $f=hg$.

Recall that $s(X)$ is the least cardinal of subsets dense in X. Therefore, $s(\bar{A})\leq s(A)$ and $s(f(A))\leq s(A)$. Moreover, if $s(A)$ and $s(B)$ are not both finite, then $s(A\cup B)\leq s(A)+s(B)\leq \max(s(A), s(B))$.

Proof. We first consider the case when $f(A)$ is an infinite set. Let $\overline{f(A)}$ denote the closure of $f(A)$ in $f(X)$. By the Kuratowski-Wojdislawski embedding theorem ([6], I, §3.1. Theorem 2) one can assume that $\overline{f(A)}$ is embeded in a normed vector space and is closed in its convex hull L. Since $\overline{f(A)}$ is infinite, one has

(3) $$s(L)=s(\overline{f(A)})\leq s(fA))\leq s(A).$$

Now note that B is closed in Y and therefore $\overline{f(A)}\subseteq B$. Since B is an ANR, the inclusion $i:\overline{f(A)}\rightarrow B$ extends to a map $j:C\rightarrow B$, where C is an open neighbourhood of $\overline{f(A)}$ in L. Since L is an AR, C is an ANR and $s(C)\leq s(L)\leq s(A)$.

We now consider the space W obtained from the topogical sum $f(X)\oplus C$ identifying the two copies of $\overline{f(A)}$. Clearly, W is a metric space with

(4) $$s(W)\leq \max(s(f(X)), s(L))\leq \max(s(X), s(A)).$$

Moreover, since $\overline{f(A)}$ is closed in $f(X)$ and in C, there is a unique map $k: W\rightarrow Y$ such that $k|f(X)$ is the inclusion into Y and $k|C$ is the composition of j with the inclusion $B\rightarrow Y$.

We can now assume that W is embedded in a normed vector space and is closed in its convex hull K. Since $W\supseteq f(A)$, it is infinite and therefore

(5) $$s(K)=s(W)\leq \max(s(X), s(A)).$$

Since Y is an ANR, one can extend $k: W\rightarrow Y$ to a map $h:Z\rightarrow Y$, where Z is an open neighborhood of W in K. Hence, Z is an ANR and

(6) $$s(Z)\leq s(K)\leq \max(s(X), s(A)).$$

Since C is closed in W, we see that C is also closed in Z and therefore (Z, C) is an ANR-pair. Finally, we take for $g: X\to Z$ the composition of $f: X\to f(X)$ with the inclusion $f(X)\to Z$. Clearly, $f=hg$.

In the case when $f(A)$ is finite and $f(X)$ is infinite, the proof is simpler. We immediately consider $f(X)$ as a closed subset of its convex hull W in some normed vector space. Then

(7) $$s(W)=s(f(X))\leq s(X)=\max(s(X), s(A)).$$

We then extend the inclusion $f(X)\to Y$ to a map $h: Z\to Y$, where Z is an open neighborhood of W. Therefore, Z is an ANR and

(8) $$s(Z)\leq s(W)\leq \max(s(X), s(A)).$$

We take for $g: X\to Z$ the composition of $f: X\to f(X)$ with the inclusion $f(X)\to Z$. Moreover, $g(A)=f(A)$ is finite and thus an ANR and a closed subset of Z. We then put $C=g(A)$. Note that $s(C)\leq s(A)$.

Finally, if both $f(A)$ and $f(X)$ are finite, we put $(Z, C)=(f(X), f(A))$, we take for $g: X\to Z$ the map f and for $h: Z\to Y$ the inclusion $f(X)\to Y$. Clearly, (1) and (2) hold and (Z, C) is an ANR-pair. This completes the proof of Lemma 2.

Proof of Theorem 6. We say that two maps $p:(X, A)\to(P, Q), p':(X, A)\to(P', Q')$ are equivalent if there is a homeomorphism $h:(P, Q)\to(P', Q')$ such that $hp=p'$. Let Γ be the set of all equivalence classes of maps of (X, A) into ANR-pairs (P, Q) with density satisfying

(9) $$s(P)\leq \max(s(X), s(A))$$

(10) $$s(Q)\leq s(A).$$

That Γ is indeed a set follows from the fact that the weight $w(P)=s(P)$ and card $(P)\leq 2^{w(P)}$. For every $\gamma\in\Gamma$ let $q_\gamma:(X, A)\to(Y_\gamma, B_\gamma)$ be a map from the class γ. Let Δ be the set of all finite subsets of Γ ordered by inclusion. If $\delta=\{\gamma_1, \cdots, \gamma_n\}\in\Delta$, we put $(Y_\delta, B_\delta)=(Y_{\gamma_1}\times\cdots\times Y_{\gamma_n}, B_{\gamma_1}\times\cdots\times B_{\gamma_n})$. If $\delta\leq\delta'=\{\gamma_1, \cdots, \gamma_n, \cdots, \gamma_m\}\in\Delta$, we define $q_{\delta\delta'}:\ (Y_{\delta'}, B_{\delta'})\to(Y_\delta, B_\delta)$ to be the projection

$$Y_{\gamma_1}\times\cdots\times Y_{\gamma_n}\times\cdots\times Y_{\gamma_m}\to Y_{\gamma_1}\times\cdots\times Y_{\gamma_n}.$$

We also define $q_\delta:(X, A)\to(Y_\delta, B_\delta)$ to be the map

$$q_\delta=q_{\gamma_1}\times\cdots\times q_{\gamma_n}\ X\to Y_{\gamma_1}\times\cdots\times Y_{\gamma_n}.$$

Clearly, (Y_δ, B_δ) is an ANR-pair and

$$q_{\delta\delta'}q_{\delta'\delta''}=q_{\delta\delta''},\quad \delta\leq\delta'\leq\delta'',$$
$$q_{\delta\delta'}q_{\delta'}=q_{\delta},\quad \delta\leq\delta'.$$

Consequently, $(Y, B))=((Y_\delta, B_\delta), q_{\delta\delta'}, \Delta)$ is an inverse system of ANR-pairs and the maps q_δ define a morphism $\boldsymbol{q}:(X, A)\rightarrow(\boldsymbol{Y}, \boldsymbol{B})$ of pro-Top².

As an immediate consequence of Lemma 3, we have the following property (R1)', which is even stronger than (R1).

(R1)' For every ANR-pair (P, Q) and every map $f:(X, A)\rightarrow(P, Q)$ there exist an index $\delta\in\Delta$ and a map $g:(Y_\delta, B_\delta)\rightarrow(P, Q)$ such that $f=gq_\delta$.

In order to obtain property (R2), we will replace $(\boldsymbol{Y}, \boldsymbol{B})$ by a larger system. We let M be the set of all pairs $\mu=(\delta, U)$, where $\delta\in\Delta$ and U is an open neighborhood of the set $q_\delta(X)$ in Y_δ. We order M by putting $\mu\leq\mu'=(\delta', U')$ whenever $\delta\leq\delta'$ and $q_{\delta\delta'}(U')\subseteq U$. For $\mu=(\delta, U)\in M$, we put $(Z_\mu, C_\mu)=(U, U\cap B_\delta)$ and $r_\mu=q_\delta: X\rightarrow U$. For $\mu\leq\mu'$ we put $r_{\mu\mu'}=q_{\delta\delta'}|U':U'\rightarrow U$. Clearly, $(\boldsymbol{Z}, \boldsymbol{C})=((Z_\mu, C_\mu), r_{\mu\mu'}, M)$ is an inverse system of ANR-pairs and $\boldsymbol{r}=(r_\mu):(X, A)\rightarrow(\boldsymbol{Z}, \boldsymbol{C})$ is a morphism of pro-Top². It is also clear that $\boldsymbol{r}$ satisfies condition (R 1)'.

We will now show that $\boldsymbol{r}$ also satisfies the following stronger form of (R2):

(R2)' Let (P, Q) be an ANR-pair and $\mathcal{V}$ be an open covering of P. If $\mu\in M$ and $g, g':(Z_\mu, C_\mu)\rightarrow(P, Q)$ are maps such that the maps gr_μ and $g'r_\mu$ are $\mathcal{V}$-near, then there is a $\mu'\geq\mu$ such that also the maps $gr_{\mu\mu'}$ and $g'r_{\mu\mu'}$ are $\mathcal{V}$-near.

Indeed, let $\mu=(\delta, U)$ and let $g, g':(U, U\cap B_\delta)\rightarrow(P, Q)$ be such that gr_μ and $g'r_\mu$ are $\mathcal{V}$-near for some open covering $\mathcal{V}$ of P. Then also $g|q_\delta(X)$ and $g'|q_\delta(X)$ are $\mathcal{V}$-near. Therefore, any point $z\in q_\delta(X)$ admits a $V\in\mathcal{V}$ such that $g(z), g'(z)\in V$. By continuity, there exists an open neighborhood $U(z)$ of z in U such that for any $z'\in U(z)$ one has $g(z'), g'(z')\in V$. Let U' be the union of all $U(z)$, when z ranges over $q_\delta(X)$. Then U' is an open neighborhood of $q_\delta(X)$ in Y_δ and $U'\subseteq U$. Moreover, the maps $g|U', g'|U'$ are $\mathcal{V}$-near. Therefore, $\mu'=(\delta, U')\in M, \mu\leq\mu'$, and the maps $gr_{\mu\mu'}=g|U'$ and $g'r_{\mu\mu'}=g'|U'$ are $\mathcal{V}$-near.

It now only remains to achieve cofiniteness of the index set Λ, i.e., to achieve that every element of Λ has only a finite number of predecessors. We define Λ as the set of all finite subsets of M ordered by inclusion. We then define an increasing function $\varphi:\Lambda\rightarrow M$ such that $\varphi(\{\mu\})=\mu$. This is obtained by induction on n, where $\lambda=\{\mu_1, \cdots, \mu_n\}$. We then put

$$(X_\lambda, A_\lambda)=(Z_{\varphi(\lambda)}, C_{\varphi(\lambda)}),$$
$$p_{\lambda\lambda'}=r_{\varphi(\lambda)\varphi(\lambda')},\quad p_\lambda=r_{\varphi(\lambda)}.$$

Clearly, $(\boldsymbol{X}, \boldsymbol{A})=((X_\lambda, A_\lambda), p_{\lambda\lambda'}, \Lambda)$ is a cofinite inverse system of ANR-pairs and $\boldsymbol{p}=$

$(p_\lambda):(X, A)\to(\boldsymbol{X}, \boldsymbol{A})$ is a morphism of pro-Top^2, which obviously has property (R1)$'$.

Now assume that (P, Q) is an ANR-pair, $\mathcal{V}$ is an open covering of P and $g, g':(X_\lambda, A_\lambda)\to(P, Q)$ are maps such that $gp_\lambda, g'p_\lambda$ are $\mathcal{V}$-near maps, i.e. $gr_{\varphi(\lambda)}, g'r_{\varphi(\lambda)}$ are $\mathcal{V}$-near. Then there is a $\mu\geq\varphi(\lambda)$ such that also $gr_{\varphi(\lambda)\mu}, g'r_{\varphi(\lambda)\mu}$ are $\mathcal{V}$-near maps. Let $\lambda'=\lambda\cup\{\mu\}$. Then $\lambda\leq\lambda'$ and $\{\mu\}\leq\lambda'$ and thus $\mu=\varphi(\{\mu\})\leq\varphi(\lambda')$ and

$$gp_{\lambda\lambda'}=gr_{\varphi(\lambda)\mu}r_{\varphi(\lambda')},\ g'p_{\lambda\lambda'}=g'r_{\varphi(\lambda)\mu}r_{\mu\varphi(\lambda')}.$$

Consequently, the maps $gp_{\lambda\lambda'}, g'p_{\lambda\lambda'}$ are also $\mathcal{V}$-near. This completes the proof of Theorem 6.

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