

ON COMPLETE SPACE-LIKE SURFACES WITH CONSTANT MEAN CURVATURE IN A LORENTZIAN 3-SPACE FORM

By

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Introduction

By a *Lorentzian* $(n+1)$ -space form $M_1^{n+1}(c)$ we mean a Minkowski space R_1^{n+1} , a de Sitter space $S_1^{n+1}(c)$ or an anti-de Sitter space $H_1^{n+1}(c)$, according as $c > 0$, $c = 0$ or $c < 0$, respectively. That is, a Lorentzian space form $M_1^{n+1}(c)$ is a complete connected $(n+1)$ -dimensional Lorentzian manifold with constant curvature c .

A hypersurface in a Lorentzian manifold is said to be *space-like* if the induced metric on the hypersurface is positive definite. On a space-like hypersurface, the first fundamental form, the second fundamental form and the mean curvature are defined in the same way as those on a hypersurface in a Riemannian manifold [§ 1].

It has been proved by Bernstein and others that the only entire minimal hypersurface in a Euclidean space R^{n+1} is a linear hyperplane for $n \leq 7$, but there are other examples for $n > 7$. So, Calabi proposed to study a Lorentzian analogue, called the Bernstein type problem, in Minkowski space R_1^{n+1} , and this was solved by Cheng and Yau [4] for every n .

More precisely, a space-like hypersurface in a Lorentzian manifold is said to be *maximal*, if the mean curvature is zero. The Bernstein type problem has led to the conclusion that the only entire maximal space-like hypersurface in R_1^{n+1} is a linear hyperplane. In order to prove this, Cheng and Yau [4] established the following result:

- (*) *If an entire space-like hypersurface M in R_1^{n+1} has a constant mean curvature H , then the induced Lorentzian metric on M is a complete Riemannian metric and the length of second fundamental form of M is bounded from above by $n|H|$.*

It follows from this result that if M is maximal, then it is totally geodesic.

Moreover, Nishikawa [11] studied the Bernstein type problem for complete maximal space-like hypersurfaces in other Lorentzian manifolds, and Ishihara

[8] found a similar result for complete maximal space-like submanifolds M^n in a semi-Riemannian space form $M_p^{n+p}(c)$.

On the other hand, in the theory of relativity, certain space-like hypersurfaces with constant mean curvature in arbitrary space-times are also investigated. For instance, Choque-Bruhat, Fischer and Marsden [5] studied the Bernstein type problem in a space-time $M_1^4(c)$ ($c \geq 0$) in connection with the positivity of mass, and proved that a compact maximal space-like hypersurface in $M_1^4(c)$ ($c \geq 0$) must be totally geodesic.

We shall consider, in this paper, complete space-like hypersurfaces with non-zero constant mean curvature in a Lorentzian space form $M_1^{n+1}(c)$. The well-known standard models of these are the totally umbilical space-like hypersurfaces and the following product manifolds:

$$\begin{aligned}
 & H^k(c_1) \times M^{n-k}(c_2) \\
 &= \begin{cases} H^k(c_1) \times S^{n-k}(c_2) & \text{in } S_1^{n+1}(c) & \left[\frac{1}{c_1} + \frac{1}{c_2} = \frac{1}{c}, c_2 > 0 \right], \\ H^k(c_1) \times R^{n-k} & \text{in } R_1^{n+1} & [c = c_2 = 0], \\ H^k(c_1) \times H^{n-k}(c_2) & \text{in } H_1^{n+1}(c) & \left[\frac{1}{c_1} + \frac{1}{c_2} = \frac{1}{c}, c_2 < 0 \right]. \end{cases}
 \end{aligned}$$

where $k=1, \dots, n-1$. $H^1(c_1) \times M^{n-1}(c_2)$ is, in particular, called a *hyperbolic cylinder*.

Goddard [6] conjectured that the only complete space-like hypersurfaces of constant mean curvature in $M_1^4(c)$ ($c \geq 0$) are the above standard models. However, it is proved by Treibergs [16] that many other examples of complete space-like surfaces with constant mean curvature exist in R_1^3 . Thus, conversely, it seems to be interesting to characterize the above standard models among these space-like surfaces.

In this direction, Akutagawa [2], Ramanathan [14] and Cheng and Nakagawa [3] obtained the conditions for a complete space-like hypersurfaces with constant mean curvature in $S_1^{n+1}(c)$ to be totally umbilical.

On the other hand, K. Milnor [10] and Yamada [17] characterized the hyperbolic cylinder $H^1(c_1) \times R^1$ in R_1^3 as the only complete "uniformly" non-umbilical space-like surface with non-zero constant mean curvature. In particular, K. Milnor proved this result by making use of the Cheng-Yau result (*).

The purpose of this paper is to prove a certain extension of the Cheng-Yau result (*) as stated in Theorem 1 [§2]. This theorem means that a complete space-like surface with constant mean curvature in $M_1^3(c)$ is totally umbilical, or the Gaussian curvature is non-positive. Furthermore, by applying

theorem 1, a characterization of the hyperbolic cylinder $H^1(c_1) \times M^1(c_2)$ in $M_1^3(c)$ is obtained in Theorem 2 [§ 3].

THEOREM 1. *Let M be a complete space-like surface with constant mean curvature H in a Lorentzian 3-space form $M_1^3(c)$. Let α be the second fundamental form of M . Then the following hold:*

- (1) *If c is non-positive, then $|\alpha|^2 \leq 4H^2 - 2c$.*
- (2) *If c is positive, then M is totally umbilical or $|\alpha|^2 \leq 4H^2 - 2c$.*

THEOREM 2. *The hyperbolic cylinder is the only complete space-like surface in $M_1^3(c)$ with non-zero constant mean curvature whose principal curvatures λ and μ satisfy $\inf(\lambda - \mu)^2 > 0$.*

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1. Space-like hypersurfaces in a Lorentzian manifold.

Let \tilde{M} be an $(n+1)$ -dimensional Lorentzian manifold and M be a space-like hypersurface in \tilde{M} . Throughout this paper, manifolds are always assumed to be connected and geometric objects are assumed to be smooth, unless otherwise stated. We choose a local field of Lorentzian orthonormal frames $\{E_A\} = \{E_0, E_1, \dots, E_n\}$ defined on a neighborhood of \tilde{M} in such a way that, restricted to M , $\{E_1, \dots, E_n\}$ are space-like and tangent to M and E_0 is time-like and normal to M . Let $\tilde{\nabla}$ (resp. ∇) denote the Levi-Civita connection of \tilde{M} (resp. M).

We use the following convention on the ranges of indices throughout this paper, unless otherwise stated:

$$A, B, \dots = 0, 1, \dots, n; \quad i, j, \dots = 1, \dots, n.$$

With respect to the frame field $\{E_A\}$, let $\{\omega_A\} = \{\omega_0, \omega_i\}$ denote its dual frame field. Then the Lorentzian metric tensor \tilde{g} of \tilde{M} is given by $\tilde{g} = \sum \varepsilon_A \omega_A \otimes \omega_A$, where ε_A is defined by $\varepsilon_0 = -1$ and $\varepsilon_i = 1$. The connection forms on \tilde{M} are denoted by ω_{AB} , that is, ω_{AB} is defined by $\omega_{AB}(E_C) = \tilde{g}(E_A, \tilde{\nabla}_{E_C} E_B)$. The canonical forms ω_A and the connection forms ω_{AB} of the ambient space satisfy the structure equations

$$(1.1) \quad d\omega_A + \sum \varepsilon_B \omega_{AB} \wedge \omega_B = 0, \quad \omega_{AB} + \omega_{BA} = 0,$$

$$(1.2) \quad \begin{aligned} d\omega_{AB} + \sum \varepsilon_C \omega_{AC} \wedge \omega_{CB} &= \tilde{\Omega}_{AB}, \\ \tilde{\Omega}_{AB} &= -\frac{1}{2} \sum \varepsilon_C \varepsilon_D \tilde{R}_{ABCD} \omega_C \wedge \omega_D. \end{aligned}$$

where $\tilde{\Omega}_{AB}$ is called the Riemannian curvature form on \tilde{M} , and \tilde{R}_{ABCD} denotes the component of the Riemannian curvature tensor \tilde{R} on \tilde{M} . That is, \tilde{R}_{ABCD} is defined by

$$\begin{aligned} \tilde{R}_{ABCD} &= \tilde{g}(\tilde{R}(E_A, E_B)E_C, E_D), \\ \tilde{R}(E_A, E_B)E_C &= \tilde{\nabla}_{E_A} \tilde{\nabla}_{E_B} E_C - \tilde{\nabla}_{E_B} \tilde{\nabla}_{E_A} E_C - \tilde{\nabla}_{[E_A, E_B]} E_C. \end{aligned}$$

Restricting these forms to the hypersurface M , we have

$$(1.3) \quad \omega_0 = 0,$$

and the Riemannian metric g of M induced from the Lorentzian metric \tilde{g} on \tilde{M} is given by $g = \sum \omega_j \otimes \omega_j$. Then, with respect to this metric, $\{E_j\}$ becomes a local orthonormal frame field and $\{\omega_j\}$ is a local dual frame field of $\{E_j\}$. Further, ω_{ij} is the connection form on M satisfying $\omega_{ij}(E_k) = g(E_i, \nabla_{E_k} E_j)$. From the structure equations of \tilde{M} it follows that the structure equations for M are given by

$$(1.4) \quad d\omega_i + \sum \omega_{ij} \wedge \omega_j = 0, \quad \omega_{ij} + \omega_{ji} = 0,$$

$$(1.5) \quad \begin{aligned} d\omega_{ij} + \sum \omega_{ik} \wedge \omega_{kj} &= \Omega_{ij}, \\ \Omega_{ij} &= -\frac{1}{2} \sum R_{ijkl} \omega_k \wedge \omega_l, \end{aligned}$$

where Ω_{ij} is the Riemannian curvature form on M and R_{ijkl} is the component of the Riemannian curvature tensor R on M . That is, R_{ijkl} is defined by

$$\begin{aligned} R_{ijkl} &= g(R(E_i, E_j)E_k, E_l), \\ R(E_i, E_j)E_k &= \nabla_{E_i} \nabla_{E_j} E_k - \nabla_{E_j} \nabla_{E_i} E_k - \nabla_{[E_i, E_j]} E_k. \end{aligned}$$

It follows from (1.3) and Cartan's lemma that the exterior derivative of (1.3) gives rise to

$$(1.6) \quad \omega_{0i} = \sum h_{ij} \omega_j, \quad h_{ij} = h_{ji}.$$

On the other hand, the second fundamental form α of M is defined by

$$\tilde{\nabla}_X Y = \nabla_X Y + \alpha(X, Y),$$

where X and Y are local vector fields on M . Then α is the symmetric bilinear form with values in the normal bundle and it can be written as

$$\alpha = \varepsilon_0 \sum h_{ij} \omega_i \otimes \omega_j E_0.$$

It follows from (1.2), (1.5) and (1.6) that the Gauss equation is given by

$$(1.7) \quad R_{ijkl} = \tilde{R}_{ijkl} + \varepsilon_0(h_{il}h_{jk} - h_{ik}h_{jl}).$$

The components of a Ricci tensor S are given by

$$(1.8) \quad S_{ij} = \sum R_{kijk} = \sum \tilde{R}_{kijk} + \varepsilon_0 h h_{ij} - \varepsilon_0 h_{ij}^2,$$

where $h = \text{trace } h = \sum h_{kk}$ is n times the mean curvature function H of M and $h_{ij}^2 = \sum h_{ik}h_{kj}$.

Now, the components h_{ijk} of the covariant derivative $\nabla\alpha$ of the second fundamental form α of M are given by

$$\sum h_{ijk}\omega_k = dh_{ij} - \sum (h_{kj}\omega_{ki} + h_{ik}\omega_{kj}).$$

Then, by substituting dh_{ij} in this definition into the exterior derivative of (1.6), we obtain the Codazzi equation

$$(1.9) \quad h_{ijk} - h_{ikj} = \tilde{R}_{0ijk}.$$

Similarly, the components h_{ijkl} of the second covariant derivative $\nabla^2\alpha$ of α can be defined by

$$\sum h_{ijkl}\omega_l = dh_{ijk} - \sum (h_{lj}k\omega_{li} + h_{il}k\omega_{lj} + h_{ij}l\omega_{lk}),$$

and the simple calculation gives rise to the Ricci formula

$$(1.10) \quad h_{ijkl} - h_{ijlk} = \sum (h_{mj}R_{milk} + h_{im}R_{mjlk}).$$

In particular, let the ambient space \tilde{M} be a Lorentzian space from $M_1^{n+1}(c)$ of constant curvature c . In this case, the Riemannian curvature \tilde{R} of \tilde{M} is given by

$$\tilde{R}_{ABCD} = c\varepsilon_A\varepsilon_B(\delta_{AD}\delta_{BC} - \delta_{AC}\delta_{BD}).$$

Then the Gauss equation and the Codazzi equation are given by

$$(1.11) \quad R_{ijkl} = c(\delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl}) + \varepsilon_0(h_{il}h_{jk} - h_{ik}h_{jl}),$$

$$(1.12) \quad h_{ijk} = h_{ikj}.$$

The Ricci curvature is given by

$$(1.13) \quad S_{ij} = c(n-1)\delta_{ij} + \varepsilon_0 h h_{ij} - \varepsilon_0 h_{ij}^2.$$

By means of (1.9) and (1.10), the Laplacian $\Delta h_{ij} = \sum h_{ijkk}$ of the function h_{ij} is given by

$$\Delta h_{ij} = (h)_{ij} + c(nh_{ij} - h\delta_{ij}) - \varepsilon_0 h_2 h_{ij} + \varepsilon_0 h h_{ij}^2,$$

where $(h)_{ij} = \nabla_{E_j}\nabla_{E_i}h$ and h_2 is a function on M defined by $h_2 = |\alpha|^2 = \sum h_{ij}^2 = \sum h_{ij}h_{ij}$. Then the Laplacian Δh_2 of the function h_2 is given by

$$\Delta h_2 = 2\sum(h)_{ij}h_{ij} + 2c(nh_2 - h^2) - 2\varepsilon_0(h_2)^2 + 2\varepsilon_0 h h_3 + 2|\nabla\alpha|^2,$$

where $h_3 = \sum h_{ij}h_{jk}h_{ki}$ and $|\nabla\alpha|^2 = \sum h_{ijk}h_{ijk}$.

Now, let the mean curvature H of M be constant. Then, since $(h)_{ij} = 0$, the Laplacian of h_2 is given by

$$(1.14) \quad \Delta h_2 = 2c(nh_2 - h^2) - 2\varepsilon_0(h_2)^2 + 2\varepsilon_0 h h_3 + 2|\nabla\alpha|^2.$$

These formulas are obtained by Cheng and Yau [4].

2. Proof of Theorem 1.

Let M be a space-like surface with constant mean curvature H in a Lorentzian 3-space form $M_1^3(c)$, and let λ and μ be the principal curvatures of M . We can choose a local field of Lorentzian orthonormal frames $\{E_0, E_1, E_2\}$ on $M_1^3(c)$ in such a way that, restricted to M , $\{E_1, E_2\}$ are tangent to M and

$$(2.1) \quad h_{11} = \lambda, \quad h_{12} = h_{21} = 0, \quad h_{22} = \mu.$$

In this case, the Gaussian curvature $G = R_{1221}$ of M is given by

$$(2.2) \quad G = c - \lambda\mu,$$

and the constant mean curvature H is represented as

$$(2.3) \quad H = \frac{h}{2}, \quad h = \lambda + \mu.$$

The function $h_2 = |\alpha|^2$ is given by

$$(2.4) \quad h_2 = \lambda^2 + \mu^2 = 2G + h^2 - 2c \quad (\geq 0).$$

It follows from (1.14) that the Laplacian of h_2 is calculated as

$$(2.5) \quad \Delta h_2 = 2G(\lambda - \mu)^2 + 2|\nabla\alpha|^2 \geq 2G(\lambda - \mu)^2.$$

In this section, we prove Theorem 1 which gives the estimate of the function h_2 on a complete space-like surface with constant mean curvature in $M_1^3(c)$. For this purpose, the following generalized maximum principle due to Omori [12] and Yau [18] is needed for the estimate of the Laplacian of the function of class C^2 .

THEOREM (Omori and Yau). *Let N be a complete Riemannian manifold whose Ricci curvature is bounded from below and let F be a function of class C^2 on N . If F is bounded from below, then for any $\varepsilon > 0$ there exists a point q such that*

$$(2.6) \quad |\nabla F(q)| < \varepsilon, \quad \Delta F(q) > -\varepsilon, \quad F(q) < \inf F + \varepsilon.$$

In fact, since M is a complete space-like surface with constant mean curvature H , it follows from (1.13) that the Ricci curvature tensor $S_{ij} = \sum_k R_{kij}$ is given by

$$(2.7) \quad \begin{aligned} S_{11} = S_{22} = G = c - \lambda\mu = c - \lambda(h - \lambda) = c - h\lambda + \lambda^2 = c + (\lambda - H)^2 - H^2 \geq c - H^2, \\ S_{12} = S_{21} = 0. \end{aligned}$$

implying that the Ricci curvature is bounded from below by constant $c - H^2$. Accordingly, we can apply this theorem to prove Theorem 1.

PROOF OF THEOREM 1. Given any positive number a , we define a smooth function F on M by $(h_2 + a)^{-1/2}$, which is positive and is also bounded from above by positive constant $a^{-1/2}$. So we can apply the generalized maximum principle due to Omori and Yau to F .

First, we compute the gradient and the Laplacian of F :

$$\begin{aligned} \nabla F &= -\frac{1}{2}(h_2 + a)^{-3/2} \nabla h_2 = -\frac{1}{2} F^3 \nabla h_2, \\ \Delta F &= -\frac{3}{2} F^2 \nabla F \nabla h_2 - \frac{1}{2} F^3 \Delta h_2 = 3F^{-1} |\nabla F|^2 - \frac{1}{2} F^3 \Delta h_2. \end{aligned}$$

Consequently, the following inequality

$$(2.8) \quad F^4 G(\lambda - \mu)^2 \leq 3 |\nabla F|^2 - F \Delta F$$

is obtained by (2.5).

For a convergent sequence $\{\epsilon_m\}$ such that $\epsilon_m > 0$ and $\epsilon_m \rightarrow 0$ ($m \rightarrow \infty$), by the theorem due to Omori and Yau, there is a point sequence $\{q_m\}$ such that F satisfies (2.6) at each q_m for ϵ_m :

$$(2.6') \quad |\nabla F(q_m)| < \epsilon_m, \quad \Delta F(q_m) > -\epsilon_m, \quad \inf F \leq F(q_m) < \inf F + \epsilon_m.$$

Then the sequence $\{F(q_m)\}$ converges to $\inf F$, which implies by the definition of F that $h_2(q_m) \rightarrow \sup h_2$ ($m \rightarrow \infty$). We shall prove that h_2 is bounded.

Suppose $\sup h_2 = +\infty$. Since $h_2 = \lambda^2 + \mu^2 = 2(\lambda - H)^2 + 2H^2$, the sequence $\{\lambda(q_m)\}$ then diverges to positive infinity, by taking a subsequence if necessary. Moreover, we have

$$\frac{\mu(q_m)}{\lambda(q_m)} \longrightarrow -1 \quad (m \rightarrow \infty),$$

for $\mu/\lambda+1=(\mu+\lambda)/\lambda=h/\lambda$. On the other hand, from the inequality (2.8), we get the relation

$$(2.9) \quad F(q_m)^4 G(q_m) \{\lambda(q_m) - \mu(q_m)\}^2 < 3\varepsilon_m^2 + \varepsilon_m F(q_m),$$

in which the right hand side converges to 0, because the function F is bounded. Hence the left hand side of (2.9) converges to a non-positive number. But, since the left hand side is

$$\frac{\{c - \lambda(q_m)\mu(q_m)\} \{\lambda(q_m) - \mu(q_m)\}^2}{\{\lambda(q_m)^2 + \mu(q_m)^2 + a\}^2} = \frac{\left\{ \frac{c}{\lambda(q_m)^2} - \frac{\mu(q_m)}{\lambda(q_m)} \right\} \left\{ 1 - \frac{\mu(q_m)}{\lambda(q_m)} \right\}^2}{\left\{ 1 + \frac{\mu(q_m)^2}{\lambda(q_m)^2} + \frac{a^2}{\lambda(q_m)^2} \right\}},$$

it should converge to 1 as $m \rightarrow \infty$. This is a contradiction. Accordingly, h_2 is bounded.

This implies that the sequence $\{G(q_m)\}$ converges to $\sup G$ which is bounded. So we have

$$\{\lambda(q_m) - \mu(q_m)\}^2 \longrightarrow \sup (\lambda - \mu)^2 < \infty \quad (m \rightarrow \infty),$$

since $(\lambda - \mu)^2 = (\lambda + \mu)^2 - 4\lambda\mu = h^2 + 4G - 4c$. Then it follows from (2.9) that we have

$$(2.10) \quad \sup G \cdot \sup (\lambda - \mu)^2 \leq 0.$$

Hence, if $\sup (\lambda - \mu)^2$ is positive, then G is non-positive. On the other hand, when $\sup (\lambda - \mu)^2 = 0$, $\lambda - \mu$ is identically zero. In consequence, under the assumption of Theorem 1, M is either totally umbilical or $G \leq 0$. Note that, when c is non-positive, if M is totally umbilical then G is non-positive, for $G = c - \lambda\mu = c - \lambda^2$. Finally, it follows from (2.4) that the condition $G \leq 0$ is equivalent to $h_2 = |\alpha|^2 \leq 4H^2 - 2c$. q. e. d.

REMARK 1. Cheng and Nakagawa [3] extend the Cheng-Yau result and give an estimate of $|\alpha|$ for a complete space-like hypersurface with constant mean curvature in $M_1^{n+1}(c)$, $c \leq 0$. In the case $c \leq 0$, Theorem 1 is equivalent to of their result, but the method of proof is different from theirs.

REMARK 2. In the case $c > 0$, a totally umbilical surface $S^2(c_2)$ in $S_1^3(c)$ has positive Gaussian curvature c_2 , and the other surfaces in $S_1^3(c)$ have non-positive curvature. On the other hand, Akutagawa [2] gave the condition for a complete space-like hypersurface M^n in $S_1^{n+1}(c)$ to be totally umbilical. In the case $n=2$, Akutagawa's theorem can also be proved by Theorem 1:

COROLLARY (Akutagawa). *Let M be a complete space-like surface with constant mean curvature H in $S_1^3(c)$. Suppose $c \geq H^2$, then M is totally umbilical.*

PROOF. Since the Gaussian curvature G is given by $G=c-\lambda\mu=(\lambda-H)^2+c-H^2$, G is non-negative by the assumption $c \geq H^2$. Then, it follows from Theorem 1 that G is positive constant or identically zero. Hence, M is a totally umbilical surface in $S_1^3(c)$. q. e. d.

3. Proof of Theorem 2.

In this section, we prove Theorem 2 which characterizes a hyperbolic cylinder in a Lorentzian space form $M_1^3(c)$.

First, it is to be remarked that hyperbolic cylinders are the only flat space-like surfaces with non-zero constant mean curvature in $M_1^3(c)$. This fact is proved by the use of a theorem due to Abe, Koike and Yamaguchi [1]. Hence we have only to prove that the Gaussian curvature of a “uniformly” non-umbilical space-like surface with constant mean curvature in $M_1^3(c)$ is identically zero. On the other hand, Theorem 1 asserts that if a space-like surface with constant mean curvature in $M_1^3(c)$ is not totally umbilical, then the Gaussian curvature is non-positive. Accordingly, Theorem 2 will follow immediately from the following lemma.

LEMMA. *Let M be a complete space-like surface with constant mean curvature H in $M_1^3(c)$. If the principal curvatures λ and μ of M satisfy*

$$\inf (\lambda - \mu)^2 > 0,$$

then the Gaussian curvature G of M is non-negative.

In order to prove this lemma, the generalized maximum principle due to Omori [12] and Yau [18] is used here again. So, we are going to compute the Laplacian of the Gaussian curvature G of M .

Now, since the mean curvature $H=h/2$ and c are constant, the relation $\Delta h_2=2\Delta G$ is obtained from (2.4). Then it follows from (2.5) that the Laplacian ΔG is given by

$$(3.1) \quad \Delta G = G(\lambda - \mu)^2 + |\nabla \alpha|^2,$$

where $|\nabla \alpha|^2 = (h_{111})^2 + 3(h_{221})^2 + (h_{222})^2 + 3(h_{112})^2$. Since the principal curvatures λ and μ are mutually different everywhere by the assumption $\inf (\lambda - \mu)^2 > 0$, it is known that they are both smooth functions on M (see Szabó [15], for example).

Recalling the definition of the components h_{ijk} , the derivatives of λ and μ are given by

$$\begin{aligned}d\lambda &= dh_{11} = h_{111}\omega_1 + h_{112}\omega_2, \\d\mu &= dh_{22} = h_{221}\omega_1 + h_{222}\omega_2.\end{aligned}$$

Since h is constant, the derivative $dh = d\lambda + d\mu$ is identically zero, and hence the following relations are obtained;

$$h_{111} + h_{221} = 0 \quad \text{and} \quad h_{112} + h_{222} = 0.$$

Also, from (2.2), the derivative of G is given by

$$\nabla G = dG = -(d\lambda)\mu - \lambda(d\mu) = (\lambda - \mu)d\lambda.$$

Hence we have

$$|\nabla \alpha|^2 = 4\{(h_{111})^2 + (h_{112})^2\} = 4|d\lambda|^2 = \frac{4}{(\lambda - \mu)^2} |\nabla G|^2,$$

which combined with (3.1), implies that the Laplacian ΔG is given by

$$(3.2) \quad \Delta G = G(\lambda - \mu)^2 + \frac{4}{(\lambda - \mu)^2} |\nabla G|^2.$$

PROOF OF LEMMA. It follows from (2.7) that we can apply the generalized maximum principle due to Omori and Yau to a smooth function F bounded from below. Here, we define F to be $\exp[aG]$ for any given positive number a . Note that F is a smooth function bounded from below by a positive constant $F_0 = \exp[a(c - H^2)]$, because of (2.7).

The gradient and the Laplacian of F are then given by

$$\begin{aligned}\nabla F &= a \exp[aG] \nabla G = aF \nabla G, \\ \Delta F &= a \nabla F \nabla G + aF \Delta G = a^2 F |\nabla G|^2 + aF \Delta G.\end{aligned}$$

Further, it follows from (3.2) that the Laplacian ΔF is given by

$$(3.3) \quad \Delta F = aFG(\lambda - \mu)^2 + \left\{ 2a^2 - \left(a^2 - \frac{4a}{(\lambda - \mu)^2} \right) \right\} F |\nabla G|^2.$$

We put $k = \inf(\lambda - \mu)^2$, which is positive by the assumption of the lemma. Let a be greater than $4/k$. Then

$$a^2 - \frac{4a}{(\lambda - \mu)^2} \geq a^2 - \frac{4a}{k} = a \left(a - \frac{4}{k} \right) > 0.$$

Accordingly, from (3.3), the Laplacian ΔF is evaluated by

$$\Delta F \leq aFG(\lambda - \mu)^2 + 2a^2 F |\nabla G|^2,$$

which implies, since $\nabla F = aF\nabla G$ and $F > 0$, the following inequality;

$$(3.4) \quad aF^2G(\lambda - \mu)^2 \geq F\Delta F - 2|\nabla F|^2.$$

For a convergent sequence $\{\varepsilon_m\}$ such that $\varepsilon_m > 0$ and $\varepsilon_m \rightarrow 0$ ($m \rightarrow \infty$), the theorem due to Omori and Yau implies that there is a point sequence $\{q_m\}$ such that F satisfies (2.6'). Then the sequence $\{F(q_m)\}$ converges to $\inf F$, which satisfies $\inf F \geq F_0 > 0$. So the definition of F implies that $G(q_m) \rightarrow \inf G$ ($m \rightarrow \infty$), where $\inf G$ is bounded.

Moreover, by taking subsequences if necessary, $\lambda(q_m)$ and $\mu(q_m)$ tends to some numbers;

$$\lambda(q_m) \longrightarrow \lambda_1, \quad \mu(q_m) \longrightarrow \mu_1 = h - \lambda_1 \quad (m \rightarrow \infty).$$

This is proved in the following way. Suppose $\{\lambda(q_m)\}$ is not bounded. Then we can regard $\{\lambda(q_m)\}$ and $\{\mu(q_m)\}$ as sequences which diverge to positive infinity and negative infinity, respectively. It follows from (2.2), that $G(q_m)$ must diverge to positive infinity. This contradicts the fact that $G(q_m)$ converges to its infimum. Thus $\{\lambda(q_m)\}$ is bounded and hence it contains a subsequence converging to some finite number.

On the other hand, from the inequality (3.4), the following relation is obtained;

$$aF(q_m)^2G(q_m)\{\lambda(q_m) - \mu(q_m)\}^2 > -\varepsilon_m\{F(q_m) + 2\varepsilon_m\},$$

in which the right hand side converges to 0 as m tends to ∞ , since the function F is bounded. Accordingly, we get

$$(3.5) \quad a(\inf F)^2(\inf G)(\lambda_1 - \mu_1)^2 \geq 0.$$

Since $a > 0$, $\inf F > 0$ and $(\lambda_1 - \mu_1)^2 \geq k > 0$, the inequality (3.5) now implies that $\inf G$ is non-negative. Hence the Gaussian curvature G is non-negative everywhere. q. e. d.

As mentioned above, Theorem 2 is proved by this lemma and Theorem 1 immediately.

REMARK 1. Recently, various kinds of surfaces of revolution with constant mean curvature in Minkowski space R_1^3 are constructed by Hano and Nomizu [7] and Ishihara and Hara [9], which shows that the condition $\inf(\lambda - \mu)^2 > 0$ in this theorem cannot be omitted in the case $c = 0$.

REMARK 2. The examples given by Akutagawa [2], each of which is a space-like rotation surface in $S_1^3(c)$, are complete space-like surfaces with con-

stant mean curvature and negative Gaussian curvature. They are not totally umbilical and satisfy $\inf(\lambda - \mu)^2 = 0$. This shows that there are many surfaces with constant mean curvature in $S_1^3(c)$ such that $G \leq 0$ which are different from the hyperbolic cylinders.

Finally, it is to be noted that the fact that all the above examples of complete space-like surfaces in R_1^3 and $S_1^3(c)$ have negative Gaussian curvature leads us to the following conjecture: Let M be a complete space-like surface with constant mean curvature in $M_1^3(c)$. If there is a point p in M at which the Gaussian curvature is zero, the Gaussian curvature is identically zero on M .

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Added in proof.

1. There are also many complete space-like surfaces with constant mean curvature in $H_1^3(c)$, which are not totally umbilical and satisfy $\inf(\lambda - \mu)^2 = 0$. These examples are constructed by the method similar to Akutagawa's one.

2. Recently, Ki, Kim and Nakagawa [19] gave an estimate of $|\alpha|$ for a complete space-like hypersurface with constant mean curvature in $M_1^{n+1}(c)$ for any c and $n \geq 2$. If $n=2$, their result are equivalent to Theorem 1.

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