VERY AMPLE INVERTIBLE SHEAVES OF NEW TYPE ON ABELIAN VARIETIES

By

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Introduction.

In 1919 Comessatti [1] proved the following theorem, which we learned by Lange's paper [2].

THEOREM (Comessatti). *Let* Jac(C) *denote the Jacobian variety of a smooth projective curve C of genus 2. If an ample divisor D on* Jac(C) *satisfies* $(D^2)=2$ *and* $(C \cdot D)=n$ *for* $n \geq 3$, *then the divisor* $C+D$ *is very ample.*

The aim of the present paper is to generalize this theorem. Our result is

THEOREM. *Let A be an abelian variety defined over an algebraically closed field of any characteristic. Let L and M be ample invertible sheaves on A with* $h^0(A, L)=h^0(A, M)=1$. *Let D and E be positive divisors such that* $L=\mathcal{O}_A(D)$ *and* $M=\mathcal{O}_A(E)$. *Assume that any component of D is not algebraically equivalent to a component of E. Then* $L \otimes M$ *is very ample.*

We prove the theorem in §1. In §2 we show that the Commessatti's theorem is a special case of ours. In the last §3 we discuss projective embeddings of abelian varieties with real multiplication.

At first I set up unnecessary assumption in the theorem. I could find the above theorem as a result of the referee's pertinent suggestion. Here I thank the referee for his kind advice.

1. Proof of theorem.

We shall use the following notation. For details we refer to [4]. Let A be an abelian variety of dimension g defined over an algebraically closed field k of arbitrary characteristic and let $\hat{A}=\mathrm{Pic}^0(A)$ denote its dual variety. The translation $x \to x+a$ by a point a of A is denoted by T_a. We denote by P the

Poincaré invertible sheaf on the product $A \times \hat{A}$ and by P_α the restriction $P|_{A \times (\alpha)}$. For an invertible sheaf L on A, the homomorphism $a \to T_a{}^*(L) \otimes L^{-1}$ of A to $\hat{A}$ is denoted by φ_L and its kernel by $K(L)$. When L is ample, we have $P_{\varphi_L(a)} \cong T_a{}^*(L) \otimes L^{-1}$. The Riemann-Roch theorem asserts $\deg \varphi_L = \chi(L)^2$ and $\chi(L) = (L^g)/g!$ where $\chi(L)$ is the Euler-Poincaré characteristic of L and (L^g) is the g-fold self-intersection number of L. If L is ample and $h^0(A, L) = 1$, then φ_L is an isomorphism and $(L^g) = g!$.

Now we shall prove the theorem. Let

$$\Phi = \Phi_{|L \otimes M|} : A \longrightarrow P(\Gamma(A, L \otimes M))$$

be the rational map associated with the complete linear system $|L \otimes M|$. What we should do is to establish the following statements:

(1.1)   Given $a, b \in A$ with $a \neq b$, there is a divisor $F \in |L \otimes M|$ such that $a \in$ Supp(F) and $b \notin$ Supp(F).

(1.2)   Given any tangent t to A at a, there is a dividor $F \in |L \otimes M|$ such that $a \in$ Supp(F) and t is not tangential to F.

In the following we shall use the same letter for a divisor and its support. Let

$$D = \sum_{i=1}^{r} D_i \quad \text{and} \quad E = \sum_{j=1}^{s} E_j$$

be decompositions into irreducible components and $\mathcal{O}_A(D_i) = L_i$, $\mathcal{O}_A(E_j) = M_j$. Since $h^0(A, L) = 1$, it follows that L_i and $L_{i'}$ are not algebraically equivalent for $i \neq i'$. We denote by A_i the quotient of A by the connected component $K(L_i)^0$ of $K(L)$ containing the origin 0. Then there is an ample invertible sheaf $\bar{L}_i$ on A_i such that $h^0(A_i, \bar{L}_i) = 1$ and $\pi^*(\bar{L}_i) \cong L_i$, where π is the canonical surjection. Moreover we have

$$A \cong A_1 \times \cdots \times A_r \quad \text{and} \quad L \cong p_1{}^*(\bar{L}_1) \otimes \cdots \otimes p_r{}^*(\bar{L}_r),$$

where $p_i : A_1 \times \cdots \times A_r \to A_i$ is the i-th projection; cf. [7], Lem. 1.6. The same results hold for M: there is an ample invertible sheaf $\bar{M}_j$ on $B_j = A/K(M_j)^0$ such that $h^0(B_j, \bar{M}_j) = 1$ and we have

$$A \cong B_1 \times \cdots \times B_s \quad \text{and} \quad M \cong p_1{}^*(\bar{M}_1) \otimes \cdots \otimes p_s{}^*(\bar{M}_s).$$

Now we shall prove (1.1). Let $\psi = -\varphi_M{}^{-1} \circ \varphi_L$, then we have

$$T_{\psi(a)}{}^*(M) \cong M \otimes P_{\varphi_M(\psi(a))} \cong M \otimes P_{-\varphi_L(a)} \cong M \otimes L \otimes T_a{}^*(L)^{-1}.$$

Hence we have

(1.3) $T_a{}^*(D) + T_{\psi(a)}{}^*(M) \in |L \otimes M|$ for all $a \in A$.

Let a and b be points in A. Suppose that, for any $F \in |L \otimes M|$, $a \in F$ implies $b \in F$. For every i, if $p \in T_a{}^*(D_i)$ then $a \in T_p{}^*(D_i) \subset T_p{}^*(D) + T_{\psi(p)}(E)$. This last divisor is a member in $|L \otimes M|$ by (1.3); hence it contains b. If $b \in T_p{}^*(D)$, then $p \in T_b{}^*(D)$. If $b \in T_{\psi(p)}{}^*(E)$, then $\psi(p) \in T_b{}^*(E)$, i.e., $p \in \psi^*(T_b{}^*(E))$. Thus we have

$$T_a{}^*(D_i) \subset T_b{}^*(D) \cup \psi^*(T_b{}^*(E)).$$

Since D_i is irreducible, we have

(1.4) $T_a{}^*(D_i) = T_b{}^*(D_{i'})$ for some i'

or

(1.5) $T_a{}^*(D_i) = \psi^*(T_b{}^*(E_j))$ for some j.

Suppose (1.5) holds. Since $T_b \circ \psi = \psi \circ T_{\psi^{-1}(b)}$, we have

$$T_a{}^*(D_i) = \psi^*(T_b{}^*(E_j)) = T_{\psi^{-1}(b)}{}^*(\psi^*(E_j)).$$

This implies that $\varphi_L(D_i)$ is algebraically equivalent to $\varphi_M(E_j)$. Therefore $K(\varphi_L(L_i))^0 = K(\varphi_M(M_j))^0$ and there are ample inveritible sheaves $(\bar{L}_i)\hat{}$ and $(\bar{M}_j)\hat{}$ such that $h^0(X, (\bar{L}_i)\hat{}) = h^0(X, (\bar{M}_j)\hat{}) = 1$ and $\pi^*((\bar{L}_i)\hat{}) \cong \varphi_L(L_i)$, $\pi^*((\bar{M}_j)\hat{}) \cong \varphi_M(M_j)$, where $X = \hat{A}/K(\varphi_L(L_i))^0$ and $\pi : \hat{A} \to X$ is the canonical surjection. Then $(\bar{L}_i)\hat{}$ and $(\bar{M}_j)\hat{}$ are algebraically equivalent. Moreover X is isomorphic to both of the dual abelian varieties of A_i and B_j; hence $A_i \cong B_j$, and $(\bar{L}_i)\hat{} \cong \varphi_{L_i}(\bar{L}_i)$, $(\bar{M}_j)\hat{} \cong \varphi_{\bar{M}_j}(\bar{M}_j)$. We identify A_i with $\hat{A}_i$ via the canonical isomorphism induced by the Poincaré invertible sheaf P; cf. [4] §13. Then $\varphi_{L_i}^{-1} = \varphi_{(L_i)\hat{}}$ and $\varphi_{\bar{M}_j}^{-1} = \varphi_{(\bar{M}_i)\hat{}}$. Since $\varphi_{(L_i)\hat{}} = \varphi_{(\bar{M}_j)\hat{}}$, $\varphi_{L_i} = \varphi_{\bar{M}_j}$; hence $\bar{L}_i$ is algebraically equivalent to $\bar{M}_j$. It follows that L_i is algebraically equivalent to M_j. This contradicts to the assumption. Thus we see that (1.5) does not occur.

If (1.4) holds, then D_i is algebraically equivalent to $D_{i'}$; hence $i = i'$ and $T^*_{a-b}(D_i) = D_i$. Therefore $T^*_{a-b}(D) = (D)$ and $a - b \in K(L) = \{0\}$, so we have $a = b$. This completes the proof of (1.1).

Now we shall show (1.2). We shall prove this only for $a = 0$, since the general case follows by applying the result to translates of L and M. Suppose (1.2) is not true (with $a = 0$). Then there is a non-zero tangent vector to the origin such that, for any member $F \in |L \otimes M|$ containing 0, $\langle t, df \rangle = 0$ where f is a local equation of F. If $p \in D$ then $0 \in T_p{}^*(D) + T_{\psi(p)}{}^*(E)$. This is a member of $|L \otimes M|$, so t is tangent to it. $0 \in T_{\psi(p)}{}^*(E)$ means $p \in \psi^*(E)$. Since any component D_i does not equal to a component of $\psi^*(E)$ (cf. the proof of (1.1)), t is tangent to $T_p{}^*(D)$ at 0 for general $p \in D$. V be the invariant vector

field defined by t. Then V_p is tangent to D for all i and general $p \in D_i$. It follows that V is tangent to D. This is equivalent to the property:

(1.6) For any open subset $U \subset A$ and any local equation f of D_L on U,

$$V(f) = h \cdot f \quad \text{for some} \quad h \in \mathcal{O}_A(U).$$

Let $\Lambda = \mathrm{Spec}\, k[\varepsilon]/(\varepsilon^2)$. We regard t as a Λ-valued point of A. Then the translation T_t on $A \times \Lambda$ induced by t is given by $(a, s) \to (a + t(s), s)$. Let L_Λ denote the pull-back of L via the projection $A \times \Lambda \to A$. Then we have $T_t^* L_\Lambda \cong L_\Lambda$ by (1.6). This means that t is a Λ-valued point of $K(L) = \{0\}$. Therefore t must be 0. This is a contradiction. Thus we have proved the theorem.

2. Proof of Comessatti's theorem.

In this section we shall show that Comessatti's theorem is a special case of the theorem proved in the previous section.

LEMMA. *Let L_0 and L_1 be ample invertible sheaves on a g-dimensional abelian variety A with $h^0(A, L_0) = h^0(A, L_1) = 1$. Then the following statements are equivalent:*

(2.1)   *L_0 is algebraically equivalent to L_1.*

(2.2)   *$(L_0{}^i \cdot L_1{}^{g-i}) = g!$ for $i = 0, 1, \cdots, g$.*

PROOF. Let $P(n) = P_{L_0, L_0 \otimes L_1^{-1}}(n) = \chi(L_0{}^n \otimes L_1^{-1})$. Then we have

$$(2.3) \qquad P(n) = \frac{1}{g!} \left\{ \sum_{i=0}^{g} (-1)^i \binom{g}{i} (L_0{}^{g-i} \cdot L_1{}^i)(n+1)^{g-i} \right\}.$$

(2.1) is equivalent to $K(L_0 \otimes L_1^{-1}) = A$, and it is also equivalent to $P(n) = n^g$; cf. [5] App. By (2.3), it is equivalent to (2.2).   Q. E. D.

COROLLARY. *Let L_0 and L_1 be ample invertible sheaves on abelian surface A with $h^0(A, L_0) = h^0(A, L_1) = 1$. Then we have the following:*

(2.4)   *$(L_0 \cdot L_1) \geqq 2$;*

(2.5)   *$(L_0 \cdot L_1) = 2$ if and only if L_0 is algebraically equivalent to L_1.*

PROOF. (2.4) Since $(L_0{}^2) > 0$, $(L_0 \cdot L_1)^2 \geqq (L_0{}^2)(L_1{}^2) = 4$; hence $(L_0 \cdot L_1) \geqq 2$. (2.5) follows the lemma.   Q. E. D.

THEOREM (Comessatti). *Let $L \cong \mathcal{O}_A(C)$ and M be ample invertible sheaves on an abelian surface A with $h^0(A, L) = h^0(A, M) = 1$, where C is an irreducible curve*

on A. If $(L \cdot M) \geqq 3$, then $L \otimes M$ is very ample.

PROOF. Combining our theorem and (2.5), we get the resul.   Q. E. D.

3. Application.

Let K be a totally real algebraic number field of degree g and $\mathfrak{o}_K$ the ring of integers of K. Let $\{\sigma_1, \sigma_2, \cdots, \sigma_g\}$ be the set of embeddings of K into the field R of real numbers. Let $\Phi : K \to M_g(C)$ denote the representation of K over the field of complex numbers defined by

$$\Phi(a) = \begin{pmatrix} \sigma_1(a) & & 0 \\ & \ddots & \\ 0 & & \sigma_g(a) \end{pmatrix} \quad (a \in K).$$

Then there are a simple abelian variety A over C of dimension g, an ample invertible sheaf L on A with $h^0(A, L) = 1$ and a ring homomorphism $\theta : K \to \mathrm{End}_Q(A)$ such that

(3.1) $\theta(\mathfrak{o}_K) \subset \mathrm{End}(A)$;

(3.2) $r_a \circ \theta$ is equivalent to Φ where r_a is the analytic representation of $\mathrm{End}_Q(A)$ with respect to some basis for the universal covering space of A,

(3.3) $\rho \cdot \theta = \theta$ where $\rho : \mathrm{End}_Q(A) \to \mathrm{End}_Q(A)$ is the Rosati involution defined by L, i. e., $\rho(f) = \varphi_L^{-1} \cdot \hat{f} \cdot \varphi_L$.

For detais we refer to [8].

We regard $\mathfrak{o}_K$ as a subring of $\mathrm{End}(A)$ via θ. Let $\varepsilon \in K$ be a unit of infinite order. Then we have

PROPOSITION. (1) *$L \otimes \varepsilon^*(L)$ is very ample.*

(2) $h^0(L \otimes \varepsilon^*(L)) = \sum_{i=0}^{g} s_i(\sigma_1(\varepsilon^2), \cdots, \sigma_g(\varepsilon^2))$

where s_i is the i-th fundamental symmetric polynomial and $s_0 = 1$.

PROOF. (1) There is a positive divisor D on A such that $L \cong \mathcal{O}_A(D)$. Then D is irreducible. Otherwise A is isomorphic to a product $B \times C$ of abelian varieties of smaller dimension; cf. [7], Lem. 1.6. This contradicts to the fact that A is simple. If L is algebraically equivalent to $\varepsilon^*(L)$, then ε is an automorphism of the polarized abelian variety (A, L). Therefore the order of ε is finite; cf. [4] § 20 The. 5. This is a contradiction. By the theorem we see that $L \otimes \varepsilon^*(L)$ is very ample.

(2) By the Riemann-Roch theorem, we have

(3.4) $h^0(A, L\otimes\varepsilon^*(L))=\chi(L\otimes\varepsilon^*(L))$

$$=\frac{1}{g!}\left\{\sum_{i=0}^{g}\binom{g}{i}(L^{g-i}\cdot\varepsilon^*(L)^i)\right\}$$

and

(3.5) $\chi(L^n\otimes\varepsilon^*(L)^{-1})=\frac{1}{g!}\left\{\sum_{i=0}^{g}(-1)^i\binom{g}{i}(L^{g-i}\cdot\varepsilon^*(L)^i)n^{g-i}\right\}.$

On the other hand (3.5) is equal to the characteristic polynomial $P(n)$ of the endomorphism; cf. [2] Lem. 2.3:

$$\varphi_L^{-1}\cdot\varphi_{\varepsilon*L}=\varphi_L^{-1}\cdot\hat{\varepsilon}\cdot\varphi_L\cdot\varepsilon=\varphi_L^{-1}\cdot\varphi_L\cdot\varepsilon\cdot\varepsilon=\varepsilon^2$$

Here we used (3.3). By (3.2), we have

$$P(n)=\prod_{i=1}^{g}(n-\sigma_i(\varepsilon^2)).$$

Comparing (3.4) and (3.5), we get (2).   Q. E. D.

EXAMPLE (Lange [2]). Let $K=Q(\sqrt{5})$ and $\varepsilon=1+\sqrt{5}/2$. Let a triplet (A, L, θ) be as above. Then $L\otimes\varepsilon^*L$ is very ample and

$$h^0(A, L\otimes\varepsilon^*L)=1+tr(\varepsilon^2)+Nm(\varepsilon^2)=5.$$

EXAMPLE. Let $K=Q(\varepsilon)$, where ε is a roof of $X^3-2X^2-X+1=0$. Then K is totally real and ε is a unit of infinite order. Let a triplet (A, L, θ) be as above. Then $L\otimes\varepsilon^*L$ is very ample and

$$h^0(A, L\otimes\varepsilon^*L)=1+tr(\varepsilon^2)+\{\sigma_2(\varepsilon^2)\sigma_3(\varepsilon^2)+\sigma_3(\varepsilon^2)\sigma_1(\varepsilon^2)$$

$$+\sigma_1(\varepsilon^2)\sigma_2(\varepsilon^2)\}+Nm(\varepsilon^2)$$

$$=1+6+5+1=13.$$

In conclusion we raise a qustion:

What is the smallest dimension $d(g)+1$ of the space of the global sections of very ample invertible sheaves on abelian varieties of dimension g?

It is well-known that $d(2)=4$. Is $d(3)$ equal to 12?

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