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ON THE LENGTH OF PROOFS IN FORMAL SYSTEMS

By

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§0. Introduction.

This paper is concerned with an aspect of lengths of proofs in formal systems. For a first order system $\boldsymbol{T}$, by $\vdash_{\boldsymbol{T}}^{k} A$, we will mean that A is provable in $\boldsymbol{T}$ with at most k applications of rules of inference. Let $\boldsymbol{PA}^{*}$ be a system for Peano arithmetic with only one function symbol S for successor and two predicate symbols which represent addition and multiplication respectively.

In [2] R. Parikh proved:

(1) For any given formula A and natural number k, it is decidable whether $\vdash_{\boldsymbol{PA}^{*}}^{k} A$ holds or not.

(2) $\vdash_{\boldsymbol{PA}^{*}} \forall xA(x)$ iff there is a k such that $(\forall n) \vdash_{\boldsymbol{PA}^{*}}^{k} A(\bar{n})$.

In this paper we shall prove an analogue of (2) for systems which have a finite number of function symbols and a finite number of axiom schemata, and are complete with respect to formulas [in Presburger arithmetic i.e. formulas which have only $S, +, =$ other than logical symbols.

Let $\boldsymbol{T}$ be any one of such systems. By $\boldsymbol{T}_{k}$ we mean the subsystem of $\boldsymbol{T}$ which has only axioms containing at most k occurrences of bound variables and critical explicit terms (these will be defined in §1)˙ Now our claim is:

$\vdash_{\boldsymbol{T}} \forall xA(x)$ iff there is a k such that $(\forall n) \vdash_{\boldsymbol{T}_{k}}^{k} A(\bar{n})$.

This implies Parikh's result (2), for it is easy to see that $(\forall n) \vdash_{\boldsymbol{PA}^{*}}^{k} A(\bar{n})$ iff there exists r such that $(\forall n) \vdash_{\boldsymbol{PA}_{r}^{*}}^{k} A(\bar{n})$.

T. Yukami has proved an analogous result as (2) for a system of natural numbers with two function symbols for successor and addition, with one predicate symbol which represents multiplication.

This system has as its axioms not only usual ones, but also all valid equations $t=u$. Since his system does not fall under a system with finitely many axiom schemata, we cannot treat his system by the method in this paper.

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§ 1. System $G(\varepsilon_0, \cdots \varepsilon_r)$.

We consider first order systems with finitely many axiom schemata in Hilbert style or in Gentzen style. We shall give a proof for only systems in Gentzen style, but it is easy to see that for the other systems mentioned a similar argument works.

In the following $\mathcal{L}$ is a first order language with constant symbols $0, \cdots$, finitely many function symbols $S, \cdots$, and predicate symbols $=, P_0, P_1, P_2, \cdots$ (countable). $\mathcal{L}^*$ is the language obtained from $\mathcal{L}$ by adding n-ary predicate variables for $n \geqq 0$; $\sigma_0, \sigma_1(a), \sigma_2(a, b), \cdots$. Formulas in $\mathcal{L}^*$ ($\mathcal{L}^*$-formulas) are formed in the usual way. We use $A, B, C, \cdots$ for formulas in $\mathcal{L}$, and $\varepsilon, \cdots$ for $\mathcal{L}^*$-formulas unless otherwise stated. Semi-terms are defined similarly as terms, but admitting bound variables in it.

DEFINITION. Substitution $\mathcal{S}$ is an assignment of formulas in $\mathcal{L}$ to certain predicate variables. $\mathcal{S}$ will induce a map, also called $\mathcal{S}$, from certain $\mathcal{L}^*$-formulas to formulas in $\mathcal{L}$ defined uniquely by;

(1) If ε is an atomic formula in $\mathcal{L}$, then $\mathcal{S}(\varepsilon)=\varepsilon$.

(2) If ε is an atomic $\sigma(t_1, \cdots, t_n)$ with predicate variable σ, and $\mathcal{S}(\sigma(a_1, \cdots, a_n))=A$, then $\mathcal{S}(\varepsilon)=A\binom{a_1\cdots a_n}{t_1\cdots t_n}$. Where $A\binom{a_1\cdots a_n}{t_1\cdots t_n}$ is the formula obtained from A by substituting t_i for a_i.

(3) $\mathcal{S}(\varepsilon_1 \wedge \varepsilon_2)=\mathcal{S}(\varepsilon_1) \wedge \mathcal{S}(\varepsilon_2)$

(4) $\mathcal{S}(\forall x \varepsilon)=\forall x \mathcal{S}(\varepsilon)$, etc.

Of course, if it is necessary, we replace bound variables.

DEFINITION. Measure δ_1 for terms and formulas

(1) $\delta_1(0)=\delta_1(a)=0$, where a is a free variable.

(2) $\delta_1(x)=1$, where x is a bound variable.

(3) $\delta_1(f(t_1 \cdots t_n))=\sum_{i-1}^{n} \delta_1(t_i)$, where t_i are terms and f is an n-ary function symbol.

(4) $\delta_1(P(t_1 \cdots t_n))=\max\{\delta_1(t_i)\}$, where t_i are terms and P is an n-ary predicate symbol.

(5) $\delta_1(A \wedge B)=\delta_1(A \vee B)=\delta_1(A \supset B)=\max\{\delta_1(A), \delta_1(B)\}$

(6) $\delta_1(QxA)=\delta_1(\neg A)=\delta_1(A)$, where Q is $\forall$ or $\exists$.

DEFINITION.

(1) Formula A (in $\mathcal{L}$) is an instance of a $\mathcal{L}^*$-formula ε if there is a substitution $\mathcal{S}$ such that $\mathcal{S}(\varepsilon)=A$.

(2) For each symbol occurrence of S in a $\mathcal{L}^*$-formula ε, we call these occurrences explicit occurrences of S in ε. If A is an instance of this ε by $\mathcal{S}$, we also call

occurrences of S in A corresponding to those in ε explicit occurrences with regard to ε and $\mathcal{S}$.

(Example)

$\sigma(0)\wedge\forall x(\sigma(x)\supset\sigma(S(x)))\supset\forall x\sigma(x)$ has one explicit occurrence of S in it. And if $\mathcal{S}(\sigma)=A(a)\wedge B(a)$, then all occurrences of S denoted in $(A(0)\wedge B(0))\wedge\forall x((A(x)\wedge B(x))\supset(A(Sx)\wedge B(Sx)))\supset\forall x(A(x)\wedge B(x))$ are explicit occurrences of S.

(3) For an $\mathcal{L}^*$-formula ε, we call a term occurrence t in ε a critical (explicit) occurrence if:

(i) t is a maximal semi-term occurrence in ε and

(ii) the outermost function symbol of t is S.

If A is an instance of ε by $\mathcal{S}$, then we also call term occurrences in A correspoding to those in ε critical occurrences with regard to ε and $\mathcal{S}$. (We often call semiterms simply by the word "terms".)

(Example)

(i) In the above example, all occurrences of Sx are critical in $(A(0)\wedge B(0))\wedge\forall x((A(x)\wedge B(x))\supset(A(Sx)\wedge B(Sx))\supset\forall x(A(x)\wedge B(x))$.

(ii) Let ε be $\forall x(\sigma_1(St)\wedge\sigma_2(f(x))\supset\sigma_1(t))$ and

$$\mathcal{S}(\sigma_1)=A(SSa)\wedge B(a)\,,\quad \mathcal{S}(\sigma_2)=C(b)\,.$$

Then in $\forall x((A(SSSt)\wedge B(St)\wedge C(f(x))\supset(A(SSt)\wedge B(t)))$ occurrences of St in $A(SSSt)$ and $B(St)$ are critical and if t is of the form Su, then occurrences of t in $A(SSt)$ and $B(t)$ are also critical.

Let $\varepsilon_0, \cdots, \varepsilon_r$ be $\mathcal{L}^*$-formulas. $\boldsymbol{G}(\varepsilon_0, \cdots, \varepsilon_r)$ is the system obtained by adding following inference rules to $\boldsymbol{LK}$.

(EQ-rules)

$$\frac{t=t,\ \Gamma\rightarrow\Delta}{\Gamma\rightarrow\Delta}$$

$$\frac{t_1=u_1\wedge\cdots\wedge t_n=u_n\wedge P(t_1\cdots t_n)\supset P(t_1\cdots t_n),\ \Gamma\rightarrow\Delta}{\Gamma\rightarrow\Delta}$$

$$\frac{t_1=u_1\wedge\cdots t_n=u_n\supset f(t_1\cdots t_n)=f(u_1\cdots u_n),\ \Gamma\rightarrow\Delta}{\Gamma\rightarrow\Delta}$$

where t, t_i, v_i are terms, P is an n-ary predicate symbol and f is an n-ary function symbol.

(Critical rules)

$$\frac{A,\ \Gamma\rightarrow\Delta}{\Gamma\rightarrow\Delta}$$ where A is an instance of some ε_i $(0\leqq i\leqq r)$ by some substitution $\mathcal{S}$.

In this schema we define

$$\delta(A)=\delta_1(A)+\text{the number of critical term occurrences in } A.$$

In the following we only consider systems $\boldsymbol{G}(\varepsilon_0, \cdots, \varepsilon_r)$ which have $\forall x(\neg(Sx=0))$, $\forall x\forall y(Sx=Sy\supset x=y)$ in $\varepsilon_0, \cdots, \varepsilon_r$. Henceforth we only write $\boldsymbol{G}$ for $\boldsymbol{G}(\varepsilon_0, \cdots, \varepsilon_r)$.

DEFINITION. System $\boldsymbol{G}_k$ is the sub-system of $\boldsymbol{G}$ in which critical rules are admitted under the condition $\delta(A)\leqq k$.

DEFINITION. Suppose there exists a formula $A(a, b, c)$ with only free variable a, b, c such that (1), (2) and (3) are provable in $\boldsymbol{G}$.

(1) $\forall x\forall y\exists! zA(x, y, z)$

(2) $\forall xA(x, 0, x)$

(3) $\forall x\forall y\forall z(A(x, y, z)\supset A(x, Sy, Sz))$

Then we can extend $\boldsymbol{G}$ to $\boldsymbol{G}^*$ by introducing new function symbol $+$, and new axiom $\forall x\forall yA(x, y, x+y)$. $\boldsymbol{G}^*$ is a conservative extension of $\boldsymbol{G}$. If $\vdash_{\boldsymbol{G}^*} B$ for all valid formulas B in Presburger arithmetic, then we say that $\boldsymbol{G}$ is complete with respect to Presburger arithmetic (complete w. r. t. $\boldsymbol{PAR}$ for short).

Now our purpose is to give a proof of the following theorem.

THEOREM. *Let* $\boldsymbol{G}$ *be complete w. r. t.* $\boldsymbol{PAR}$, *then the following* (1) *and* (2) *are equivalent.*

(1) $\vdash_{\boldsymbol{G}} \Gamma(a)\to\Delta(a)$

(2) *For some number* k, $\overset{k}{\vdash}_{\boldsymbol{G}_k} \Gamma(\bar{n})\to\Delta(\bar{n})$ *for all natural number* n. (*Where* $\bar{n}$ *is a term* $\underbrace{S(S(\cdots(S}_{n\text{-times}}(0))\cdots)$.)

§2. Proof of the theorem.

That (1) implies (2) is trivial. So we only prove that (2) implies (1).

Let $\mathfrak{P}_n$ be proof figures of $\Gamma(\bar{n})\to\Delta(\bar{n})$ in $\boldsymbol{G}_k$ with at most k inference rules. We can assume without loss of generality that

(1) $\mathfrak{P}'_n$s are all cut free, and

(2) In all basic sequents $A\to A$ in $\mathfrak{P}_n$, A is an atomic formula. This can be done by the same way as in Parikh [2; proof of lemma B and theorem 2] or in Yukami [3; §1]. We only change the number k by a suitable k'.

DEFINITION. For each term occurrence in $\mathfrak{P}_n$ we define,
(1) If an atomic formula $P(t_1 \cdots t_n)$ is a subformula occurrence of a formula occurrence in $\mathfrak{P}_n$, then each occurrenc t_i is a normal occurrence.
(2) If $f(t_1 \cdots t_n)$ is a normal occurrence and f is not S, then each t_i is a normal occurrence.
(3) If $S(S(\cdots(S(t)\cdots)$ is a normal occurrence and t is not of the form $S(u)$, then t is a normal occurrence.

NOTATION. For terms of the form $\underbrace{S(S(\cdots(S}_{i\text{-times}}(t)\cdots)$, we write $S^i(t)$ for short.

Now we mark each $\mathfrak{P}_n$ with $\sharp$ from end-sequent $\Gamma(\bar{n})\to\Delta(\bar{n})$ up to basic sequents as follows.
(1) For each term occurrence t in end-sequent, we mark it according to its structure. For each minimal normal occurrences,

$$S^i(0) \Rightarrow \sharp S^i \sharp(0)$$

$$S^i(a) \Rightarrow \sharp S^i \sharp(a)$$

$$S^i(x) \Rightarrow \sharp S^i \sharp(x)$$

If $i=0$ or its outermost symbol is not S, then we don't mark it.

For t, we write $\tilde{t}$ for its marked occurrence. Then inductively, $\widetilde{f(t_1 \cdots t_n)}$ is $f(\tilde{t}_1 \cdots \tilde{t}_n)$ and $\widetilde{S^i(t)}$ is $\sharp S^i \sharp(\tilde{t})$, where $S^i(t)$ is normal and t is not of the form $S(u)$.

Finally we add $\sharp$ to enclose those occurrences of S^n of $\bar{n}$ which are substituted for a in $\Gamma(a)\to\Delta(a)$.
(2) For rules of ***LK***

We assume that lower sequents of inference rules have already been marked.

$$\text{(2.1)} \qquad \frac{\Gamma\to\Delta}{\Pi\to\Lambda} \quad \text{or} \quad \frac{\Gamma_1\to\Delta_1 \quad \Gamma_2\to\Delta_2}{\Pi\to\Lambda}$$

is an inference of ***LK*** other than $\forall$- or $\exists$-rules.

In this case we can naturally transfer the marks of the lower sequent to the upper sequent. (Note that no cut rules appear in $\mathfrak{P}_n$.)

$$\text{(2.2)} \qquad \frac{\Gamma\to\Delta,\ A(t)}{\Gamma\to\Delta,\ QxA(x)} \quad \text{where } Q \text{ is } \forall \text{ or } \exists.$$

Let the marked sequent corresponding to the lower sequent to $\tilde{\Gamma}\to\tilde{\Delta},\ \widetilde{QxA(x)}$. Then for the upper sequent, we take $\tilde{\Gamma}\to\tilde{\Delta},\ \tilde{A}(\tilde{t})$ where $\tilde{A}(\tilde{t})$ is the result of substituting $\tilde{t}$ for x in $\widetilde{A(x)}$ and $\tilde{t}$ is a marked occurrence of t which is marked as in (1).

(2.3) The dual inferences of (2.2) are treated similarly as in (2.2).

(3) For the rules not of $\boldsymbol{LK}$

(3.1) EQ-rules

$$\frac{t=t,\ \Gamma\to\Delta}{\Gamma\to\Delta} \quad \text{where} \quad \tilde{\Gamma}\to\tilde{\Delta}$$

is for the lower sequent. In this case we take for the upper sequent, $\tilde{t}=\tilde{t}$, $\tilde{\Gamma}\to\tilde{\Delta}$.

$$\frac{t_1=u_1\wedge\cdots\wedge t_n=u_n\wedge P(t_1\cdots t_n)\supset P(u_1\cdots u_n),\ \Gamma\to\Delta}{\Gamma\to\Delta}$$

In this case we take for the upper sequent,

$$\tilde{t}_1=\tilde{u}_1\wedge\cdots\wedge\tilde{t}_n=\tilde{u}_n\wedge P(\tilde{t}_1\cdots\tilde{t}_n)\supset P(\tilde{u}_1\cdots\tilde{u}_n),\ \tilde{\Gamma}\to\tilde{\Delta}\,.$$

$$\frac{t_1=u_1\wedge\cdots t_n=u_n\supset f(t_1\cdots t_n)=f(u_1\cdots u_n),\ \Gamma\to\Delta}{\Gamma\to\Delta}$$

In this case we take for the upper sequent,

$$\tilde{t}_1=\tilde{u}_1\wedge\cdots\wedge\tilde{t}_n=\tilde{u}_n\supset f(\tilde{t}_1\cdots\tilde{t}_n)=f(\tilde{u}_1\cdots\tilde{u}_n),\ \tilde{\Gamma}\to\tilde{\Delta}$$

if f is not S and $\tilde{t}=\tilde{u}\supset\sharp S\sharp(\tilde{t})=\sharp S\sharp(\tilde{u})$, $\tilde{\Gamma}\to\tilde{\Delta}$ if f is S.

(3.2) Critical rules

$$\frac{A,\ \Gamma\to\Delta}{\Gamma\to\Delta}$$

We have $\tilde{\Gamma}\to\tilde{\Delta}$ for the lower sequent. Now we make $\tilde{A}$ as in (1), and further we enclose the explicit occurrences of S and critical term occurrences as follows. Let $\sharp S^i\sharp(\tilde{t})$ be an occurrence which is marked in $\tilde{A}$, and $S^i(t)$ be $\underbrace{S(\cdots(S}_{i_1\text{-times}}(\underbrace{S\cdots S}_{i_2\text{-times}}(t)\cdots)$, where i_2 S's are consecutive explicit occurrences of S or $\underbrace{S(\cdots(S}_{i_2\text{-times}}(t)\cdots)$ is a critical term occurrence. Now we add $\sharp$ to $\sharp S^i\sharp(\tilde{t})$ and make $\sharp S^{i_1}\sharp S^{i_2}\sharp(\tilde{t})$. In this way we make $\tilde{\tilde{A}}$, and take for the upper sequent $\tilde{\tilde{A}}$, $\tilde{\Gamma}\to\tilde{\Delta}$.

NOTATION. We write $\tilde{\mathfrak{P}}_n$ for the maked proof figure.

DEFINITION. In $\tilde{\mathfrak{P}}_n$ we call occurrences of consecutive S's enclosed by $\sharp$ blocks. And,

(1) The blocks produced at the stage of the end-sequent, we call $\sharp S^n\sharp$ in $\sharp S^n\sharp(0)$, where $S^n(0)$ is the occurrence of $\bar{n}$ in $\Gamma(\bar{n})\to\Delta(\bar{n})$ substituted for a in $\Gamma(a)\to\Delta(a)$, designated blocks. All the other blocks in the end-sequent are called invariant blocks.

(2) In (3.1) for EQ-rules for the function symbol S, outermost blocks $\sharp S\sharp$'s in

$\# S\#(\tilde{t})$ and $\# S\#(\tilde{u})$ are also invariant.

(3) In (3.2) additional blocks $\# S^{i_2}\#$ in $\# S^{i_1}\# S^{i_3}\#(\tilde{t})$ are also invariant.

(4) Designated blocks (d-blocks) and invariant blocks (i-blocks) are transfered from a lower sequent to upper sequents at each stage.

(5) The blocks which are neither designated nor invariant are called neutral blocks (n-blocks).

Now we transform each $\tilde{\mathfrak{P}}_n$ to a proof figure $\mathfrak{P}_n^*$ in an extended system $\boldsymbol{G}^*$ with a function symbol $+$ and construct a set of equations.

Let φ be defined as follows;

$$\varphi(0)=0$$

$$\varphi(a)=a$$

$$\varphi(x)=x$$

$$\varphi(\# S^i\#(t))=\varphi(t)+\tau_i$$

$$\varphi(f(t_1 \cdots t_n))=f(\varphi(t_1) \cdots \varphi(t_n))$$

according to its structure. In the above

$$\tau_i=\begin{cases} a_0 & \text{if } \# S^i\# \text{ is a } d\text{-block} \\ \bar{\imath} & \text{if } \# S^i\# \text{ is an } i\text{-block} \\ b_i & \text{if } \# S^i\# \text{ is an } n\text{-block} \end{cases}$$

where $\{a_0, b_1, b_2, \cdots, b_j, \cdots\}$ is a set of new free variables.

REMARK. Reflections on the marking procedures tell us that each normal occurrence has at most three blocks in its outermost part.

We write $\varphi(A)$ for the formula which is obtained from a marked formula A by replacing each term occurrence t in A by $\varphi(t)$, and $\varphi(\Gamma\to\Delta)$ for the sequent which is obtained from a marked sequent $\Gamma\to\Delta$ by replacing each formula A in $\Gamma\to\Delta$ by $\varphi(A)$.

Let $P(t_1 \cdots t_n)\to P(t_1 \cdots t_n)$ be a basic sequent in $\tilde{\mathfrak{P}}_n$. Observe that two occurrences of t_i in this sequent may have different marks, so we distinguish these two occurrences by denoting $P(t_1^1 \cdots t_n^1)\to P(t_1^2 \cdots t_n^2)$.

We construct a finite set of equations for each pair (t_j^1, t_j^2) as follows. Let u_j^1 and u_j^2 are two corresponding normal subterm occurrences in t_j^1 and t_j^2, then we construct $\Omega(u_j^1, u_j^2)$ such that $\Omega(u_j^1, u_j^2)=\Omega(v_j^1, v_j^2)\cup E(u_j^1, u_j^2)$, where $E(u_j^1, u_j^2)$ is

$$\begin{cases} \phi & \text{If } u_j^1 \text{ is } \boxed{1}v_j^1,\ u_j^2 \text{ is } \boxed{2}v_j^2,\ v_j^1 \text{ and } v_j^2 \text{ are normal, } \boxed{1} \text{ and } \boxed{2} \text{ consist of the same}^{(*)} \text{ blocks.} \\ & (*) \text{ with regard to also their kinds (}d\text{-block, }i\text{-block or }n\text{-block)} \\ h(a_0, \bar{b})=g(a_0, \bar{b}) & \text{Otherwise. Where } h \text{ and } g \text{ are corresponding terms of } \boxed{1} \text{ and } \boxed{2}. \text{ For instance, if } \boxed{1} \text{ is } \#S^{i_1}\#S^{i_2}\# \text{ and } \boxed{2} \text{ is } \#S^{j_1}\#S^{j_2}\#S^{j_3}\#, \text{ then } h(a_0, \bar{b}) \text{ is } \tau_{i_2}+\tau_{i_1} \text{ and } g(a_0, \bar{b}) \text{ is } \tau_{j_3}+\tau_{j_2}+\tau_{j_1}. \end{cases}$$

$$\Omega_{P(t_1^1\cdots t_n^1)\to P(t_1^2\cdots t_n^2)}=\bigcup_{1\leq i\leq n}\Omega(t_i^1, t_i^2)$$

$$\Omega_n=\cup\Omega_{P(t_1^1\cdots t_n^1)\to P(t_1^2\cdots t_n^2)} \quad \text{where } P(t_1^1\cdots t_n^1)\to P(t_1^2\cdots t_n^2)$$

ranges over all basic sequents in $\tilde{\mathfrak{P}}_n$.

By $\varphi(\tilde{\mathfrak{P}}_n)$ we denote the figure obtained from $\tilde{\mathfrak{P}}_n$ by replacing each sequent $\Gamma\to\Delta$ in $\tilde{\mathfrak{P}}_n$ by $\varphi(\Gamma\to\Delta)$. Although inferences in $\varphi(\tilde{\mathfrak{P}}_n)$ are correct derived rules in $\boldsymbol{G}^*$, top sequents in $\varphi(\tilde{\mathfrak{P}}_n)$ are not basic sequents. But for each such sequent

$$P(\varphi(t_1^1)\cdots\varphi(t_n^1))\to P(\varphi(t_1^2)\cdots\varphi(t_n^2)),$$

we can construct a proof figure in $\boldsymbol{G}^*$ of

$$\Omega_n,\ P(\varphi(t_1^1)\cdots\varphi(t_n^1))\to P(\varphi(t_1^2)\cdots\varphi(t_n^2)).$$

From these figures and $\varphi(\tilde{\mathfrak{P}}_n)$, we obtain a proof figure of the seqent

$$\Omega_n,\ \varphi(\Gamma(\bar{n}))\to\varphi(\Delta(\bar{n})).$$

It is easy to see that this sequent is equivalent in $\boldsymbol{G}^*$ to Ω_n, $\Gamma(a_0)\to\Delta(a_0)$. So we get the proof figure of Ω_n, $\Gamma(a_0)\to\Delta(a_0)$.

Ω_n is a set of equations $h(a_0, \bar{b})=g(a_0, \bar{b})$, and h, g are of the form $\alpha_1+\cdots+\alpha_j$ $(1\leqq j\leqq 3)$ where α_j is one of the followings:

(i) free variables $a_0, b_1, \cdots$

(ii) numerals (bounded depending only on $\Gamma(a)\to\Delta(a)$ and schemata $\varepsilon_0, \cdots, \varepsilon_r$)

Now we define

$$C_n(a_0)=\exists\bar{x}\Big[\bigwedge_{h=g\in\Omega_n}\{h(a_0, \bar{x})=g(a_0, \bar{x})\}\Big].$$

(In the above we write $\bar{b}$ for some finite sequence $b_{j_1}, \cdots, b_{j_m}$ which are elements of $\{b_1, b_2, \cdots\}$ and $\exists\bar{x}$ for $\exists x_{j_1}\exists x_{j_2}\cdots\exists x_{j_m}$.) Then we get the proof figure $\tilde{\mathfrak{P}}_n^*$ of $C_n(a_0)$, $\Gamma(a_0)\to\Delta(a_0)$. (Note that free variables $b_1, b_2, \cdots$ do not appear in $\Gamma(a_0)\to\Delta(a_0)$.)

We claim that the number of equations in Ω_n is bounded by some number uniformly in n. If it is the case, then $\{C_n(a_0)\}$ can be divided into finite classes by their logical equivalence in $\boldsymbol{G}^*$.

Let $C_{r_1}(a_0), \cdots, C_{r_s}(a_0)$ be their representatives, then

$$\vdash_{\overline{G^*}} C_{r_j}(a_0),\ \Gamma(a_0) \rightarrow \Delta(a_0) \quad \text{for all } 1 \leqq j \leqq s\,.$$

Now $\bigvee_{1 \leqq j \leqq s} C_{r_j}(a_0)$ is a valid formula in Presburger arithmetic.

In fact for each n, $C_n(\bar{n})$ is valid (suitable numbers $m_1, \cdots$ can be read off from $\mathfrak{P}_n$ such that $h(\bar{n}, \bar{m}_1, \cdots) = g(\bar{n}, \bar{m}_1, \cdots)$ is true for all $h = g$ in Ω_n). $C_n(a_0)$ is equivalent to $C_{r_j}(a_0)$ for some r_j. So $C_{r_j}(\bar{n})$ is valid. From these and that $\boldsymbol{G}$ is complete w. r. t. $\boldsymbol{PAR}$ and $\boldsymbol{G}^*$ is conservative over $\boldsymbol{G}$, we get $\vdash_{\overline{G}} \Gamma(a_0) \rightarrow \Delta(a_0)$.

Now we show the following claim.

Claim: The number of equations in Ω_n is bounded uniformly in n.

Let us ignore term occurrences in $\mathfrak{P}_n$ and look at the logical structure of each sequent and kinds of inference rules in $\mathfrak{P}_n$. Let's call this a skeleton of $\mathfrak{P}_n$. Since lengths of $\mathfrak{P}_n$'s are bounded by k, $\mathfrak{P}_n$'s are cut free and basic sequents in $\mathfrak{P}_n$'s are all atomic, all the skeletons arising from $\mathfrak{P}_n$'s are finite.

We call $\boxed{}$ of blocked normal term occurrence $\boxed{}t$, a building (blg.) of this occurrence (t is normal and not of the form Su). A blg. is said to be regular if it consists of only one n-block. Now if $E(u_j^1, u_j^2)$ becomes non-empty, then at least one of $\boxed{1}$ or $\boxed{2}$ is not regular. (u_j^1 and u_j^2 are two corresponding normal occurrences in a basic sequent and u_j^1 is $\boxed{1}v_j^1$ and u_j^2 is $\boxed{2}v_j^2$, where v_j^1 and v_j^2 are normal occurrences not of the form Sw.) Observe that all non-regular blg.'s are produced at one of the following stages.

(i) End-sequent $\Gamma(\bar{n}) \rightarrow \Delta(\bar{n})$

At this stage, only d-blocks and i-blocks are produced. The number of the non-regular blg.'s depends only on $\Gamma(a) \rightarrow \Delta(a)$.

(ii) $\forall$-left or $\exists$-right rules in which a bounded term t is of the form $S^i u$ (u is normal and $i > 0$).

In
$$\frac{\Gamma \rightarrow \Delta,\ A(t)}{\Gamma \rightarrow \Delta,\ \exists x A(x)} \quad \text{or} \quad \frac{A(t),\ \Gamma \rightarrow \Delta}{\forall x A(x),\ \Gamma \rightarrow \Delta},$$
if there are normal occurrences which include a bonded term occurrence t as their subterm and are of the form $S^j t$, then $\# S^j \# S^i \# u$ arise in the upper sequent and non-regular blg.'s $\# S^j \# S^i \#$ arise. Since $\delta_1(A(x))$ is at most $k_1 = \max\{k$, the number of bound variable occurrences in $\Gamma(a) \rightarrow \Delta(a)\}$, the number of these non-regular blg.'s $\leqq k_1$.

(iii) EQ-rules $\dfrac{t = u \supset St = Su,\ \Gamma \rightarrow \Delta}{\Gamma \rightarrow \Delta}$ in which t or u is of the form $S^i v$ (v is normal and not of the form Sw) and $i \geqq 0$.

In this case at most 2 non-regular blg.'s arise.

(iv) Critical rules $\dfrac{A, \Gamma \to \Delta}{\Gamma \to \Delta}$

In this case, if A contains normal occurrences $S^{i_1}S^{j_1}t_1, \cdots, S^{i_r}S^{j_r}t_r$ ($t_1, \cdots, t_r$ are normal, not of the form Sw) and S^{j_m}'s are explicit occurrences of S or $S^{j_m}t_m$'s are critical term occurrences, then non-regular blg.'s $\sharp S^{i_m}\sharp S^{j_m}\sharp$'s arise. Since $\delta(A) \leqq k$, the number of non-regular blg.'s arising from critical term occurrences is at most k. Here we cannot estimate the blg's arising from explicit S's. But we must observe that such a non-regular blg. consists of only one i-block (note that if $i_m \neq 0$, $S^{i_m}S^{j_m}t_m$ is a critical term) and for all i-blocks $\sharp S^j \sharp$'s, j's are bounded by some number which depends only on $\Gamma(a) \to \Delta(a)$ and schemata $\varepsilon_0, \cdots, \varepsilon_r$.

Now we trace sequents in $\tilde{\mathfrak{P}}_n$ from the end-sequent up to basic sequents and count the number of non-regular blg.'s.

A path in $\tilde{\mathfrak{P}}_n$ will mean a sequence of sequents $S_1, \cdots, S_r$ in $\tilde{\mathfrak{P}}_n$ such that (i) S_1 is a basic sequent, (ii) S_r is the end-sequent, and (iii) S_i is one of upper sequents of some inference rule (I_i) in $\tilde{\mathfrak{P}}_n$ and S_{i+1} is the lower sequent of I_i $(1 \leqq i < r)$. Since skeletons arising from $\mathfrak{P}_n$'s are finite, the number of paths in each $\tilde{\mathfrak{P}}_n$ is bounded uniformly in n, and further, for each path the number of $\forall$-left, $\exists$-right, EQ-rules and Critical rules in it is bounded uniformly in n. Now for a path in $\tilde{\mathfrak{P}}_n$, the number of non-regular blg.'s in S_r (the end-sequent) depends only on $\Gamma(a) \to \Delta(a)$.

In tracing from S_{i+1} up to S_i,

(i) if I_i is one of the structural rules or propositional rules or EQ-rules not for the function symbol S, then the number of non-regular blg.'s in S_i is the same as in S_{i+1}.

(ii) if I_i is one of $\forall$-left or $\exists$-right or EQ-rules for the function symbol, S, then the number of non-regular blg.'s in S_i is greater than in S_{i+1} by at most $\max\{k_1, 2\}$.

(iii) if I_i is a critical rule, then the number of non-regular blg.'s which arise by way of the critical terms in S_i is at most k. All the other non-regular blg.'s are one of the i-blocks $\sharp S^i \sharp$ $(i \leqq m)$, where m depends only on $\Gamma(a) \to \Delta(a)$ and schemata $\varepsilon_0, \cdots, \varepsilon_r$.

For any basic sequent in $\tilde{\mathfrak{P}}_n$, $\{(u_j^1, u_j^2)$: u_j^1 is $\boxed{1}$ v_j^1 and u_j^2 is $\boxed{2}$ v_j^2 and $\boxed{1}$ or or $\boxed{2}$ is non-regular$\}$ is divided into M_1 and M_2 such that (i) M_1 consists of the pairs such that $\boxed{1}$ and $\boxed{2}$ don't consits of only one i-block and (ii) M_2 consists of the pairs such that one of $\boxed{1}$ or $\boxed{2}$ consists of only one i-block. Clearly the number of the pairs in M_1 is bounded uniformly in n. For the pairs in M_2 with non-regular $\boxed{1}$ and $\boxed{2}$, the number of such ($\boxed{1}$, $\boxed{2}$)'s is also bounded uniformly in n. For the pairs with a regular $\boxed{1}$ or $\boxed{2}$, the equations arising from these are one of the followings: $b_i=\bar{i}$, $\bar{i}=b_i$ $(i\leqq m)$, where m is independent of n.

From the above considerations we can conclude that the number of the equations in Ω_n is bounded uniformly in n. This completes the proof of the claim. Q. E. D.

COROLLARY-1. *On the same assumptions as in the theorem,* $\vdash_{\boldsymbol{G}} \rightarrow \forall x A(x)$ *iff there is a* k *such that* $(\forall n)\overset{k}{\vdash_{\boldsymbol{G}_k}} \rightarrow A(\bar{n})$.

COROLLARY-2. (Parikh [2; Theorem 3])

$\vdash_{\boldsymbol{PA}^*} \forall x A(x)$ *iff there is a* k *such that* $(\forall n)\overset{k}{\vdash_{\boldsymbol{PA}^*}} A(\bar{n})$.

(proof) We can prove the claim for the following formulation of $\boldsymbol{PA}^*$. We omitt the EQ-rules and instead take the schema:

$$\forall \bar{x} \forall \bar{y}(x_1=y_1 \wedge \cdots \wedge x_n=y_n \wedge \sigma(x_1 \cdots x_n) \supset \sigma(y_1 \cdots y_n)\,.$$

Observe that $(\forall n)\overset{k}{\vdash_{\boldsymbol{PA}^*}} A(\bar{n})$ implies an existence of r such that $(\forall n)\overset{k}{\vdash_{\boldsymbol{PA}^*_r}} A(\bar{n})$. (This is because of the fact that $\boldsymbol{PA}^*$ has only one unary function symbol S. cf. Parikh [2: Theorem 2].) So if we take $k'=\max\{k, r\}$, then $(\forall n)\overset{k'}{\vdash_{\boldsymbol{PA}^*_{k'}}} A(\bar{n})$. Hence the result follows from the theorem. Q. E. D.

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