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ON ITERATED WEIGHTED MEANS OF BOUNDED SEQUENCES AND UNIFORM DISTRIBUTION

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Abstract. It is proved that natural conditions on the weights p_n all weighted Hölder and Cesaro means are equivalent to a generalized Abel method for bounded sequences. Furthermore it is shown that the induced ∞-limitation methods are equivalent to a uniform limitation method. At last some applications to the theory of uniform distribution of sequences in compact spaces are given.

1. Introduction.

Let $P=(p_n)_{n=1}^{\infty}$ be a sequence of positive real numbers, such that

$$P_n=\sum_{k=1}^{n} p_k \tag{1}$$

tends to infinity (as $n\to\infty$). Then a real sequence $(x_n)_{n=1}^{\infty}$ is called limitable to x with respect to the limitation method M_p if

$$\lim_{n\to\infty}\frac{1}{P_n}\sum_{k=1}^{n} p_k x_k=x\ . \tag{2}$$

For short we will write $\lim x_n=x$ (M_p). This method is regular since $\lim_{n\to\infty} P_n=\infty$. If $p_n=1$ for all n, then M_p is the usual arithmetic mean or the first Hölder mean H_1. Therefore we can call the weighted arithmetic mean M_p weighted Hölder mean $H_1^{(P)}$ and the k times iterated weighted mean weighted Hölder mean of order k $H_k^{(P)}$ defined inductively by

$$\begin{aligned} &H_0^{(P)}(x_n)=x_n \\ &H_{r+1}^{(P)}(x_n)=M_p(H_r^{(P)}(x_n))=\frac{1}{P_n}\sum_{k=1}^{n} p_k H_r^{(P)}(x_k) \quad (r=0,1,\cdots). \end{aligned} \tag{3}$$

For simplicity we will write $\lim x_n=x$ $(H_r^{(P)})$ instead of $\lim_{n\to\infty} H_r^{(P)}(x_n)=x$. On the other hand M_p is the first Cesaro mean for $p_n=1$. Therefore we can introduce a weighted Cesaro mean by $C_r^{(P)}(x_n)=c_r^{(P)}(x_n)/P_{r,n}$, where

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$$(4)\quad \begin{aligned} &c_0^{(P)}(x_n)=x_n, && c_{r+1}^{(P)}(x_n)=\sum_{k=1}^{n} p_k c_r^{(P)}(x_k) \quad (r=0, 1, \cdots), \\ &P_{0,n}=1, && P_{r+1,n}=\sum_{k=1}^{n} p_k P_{r,k} \quad (r=0, 1, \cdots). \end{aligned}$$

A sequence $(x_n)_{n=1}^{\infty}$ is said to be limitable to x with respect to $C_r^{(P)}$ if

$$(5)\quad \lim_{n\to\infty} C_r^{(P)}(x_n)=x \qquad (\text{for short } \lim x_n=x\ (C_r^{(P)})).$$

It is easily verified that $\lim x_n=x\ (H_r^{(P)})$ (or $(C_r^{(P)})$) implies $\lim x_n=x\ (H_{r_1}^{(P)})$ (or $(C_{r_1}^{(P)})$) for $r\leqq r_1$. Therefore we can consider the related $H_\infty^{(P)}$ and $C_\infty^{(P)}$ limitation methods. A sequence $(x_n)_{n=1}^{\infty}$ is called limitable to x with respect to $H_\infty^{(P)}$ or $C_\infty^{(P)}$ if

$$(6)\quad \lim_{r\to\infty} \overline{\lim_{n\to\infty}}\, H_r^{(P)}(x_n)=\lim_{r\to\infty} \underline{\lim_{n\to\infty}}\, H_r^{(P)}(x_n)=x$$

or

$$(7)\quad \lim_{r\to\infty} \overline{\lim_{n\to\infty}}\, C_r^{(P)}(x_n)=\lim_{r\to\infty} \underline{\lim_{n\to\infty}}\, C_r^{(P)}(x_n)=x .$$

For short we will write $\lim x_n=x\ (H_\infty^{(P)})$ or $\lim x_n=x\ (C_\infty^{(P)})$.

It is also interesting to consider a generalized Abel method $A^{(P)}$. We will say $\lim x_n=x\ (A^{(P)})$ if

$$(8)\quad \lim_{s\to 0+} s\sum_{n=1}^{\infty} p_n x_n e^{-sP_n}=x .$$

Note that (8) is similar to the definition of the method J_p (see [7], [2]).

For $p_n=1/n$ the relations $H_r^{(P)}$, $H_\infty^{(P)}$ and $A^{(P)}$ have been discussed by P. Diaconis [4]. P. Schatte [14], [16], [17] and R.F. Tichy [23] developed many results concerning H_∞, especially in connection with the theory of uniform distribution. The intention of this paper is to discuss the relations between $H_r^{(P)}$, $C_r^{(P)}$, $H_\infty^{(P)}$, $C_\infty^{(P)}$, and $A^{(P)}$ for bounded sequences (section 2) and to give some applications to the theory of uniform distribution (section 3).

2. Summation theoretical theorems.

The generalization of the "classical" equivalence between Hölder and Cesaro means is

THEOREM 1. *Let $P=(p_n)_{n=1}^{\infty}$ be a real positive mean with $\lim_{n\to\infty} P_n=\infty$ and $\lim_{n\to\infty} p_n/P_n=0$. Then, for every positive integer r, $C_r^{(P)}$ is equivalent to $H_r^{(P)}$.*

Proof. It is easy to see that $C_{r+1}^{(P)}=M_qC_r^{(P)}$ where M_q is a weighted arithmetic mean with $q_n=p_nP_{r,n}$ and $Q_n=P_{r+1,n}$. A simple induction shows that

$$P_{r,n}\sim\frac{P_n^r}{r!}\qquad(n\to\infty),$$

indeed

$$P_{r+1,n}=\sum_{k=1}^{n}p_kP_{r,k}\sim\frac{1}{r!}\sum_{k=1}^{n}p_kP_k^r\sim\frac{1}{r!}\frac{1}{r+1}P_n^{r+1}.$$

Now Theorem II.20 of [13] yields that the weighted means M_q and M_p are equivalent, since

$$\frac{p_nQ_n}{q_nP_n}=\frac{P_{r+1,n}}{P_{r,n}P_n}\sim\frac{1}{r+1}.$$

Therefore $C_{r+1}^{(P)}(C_r^{(P)})^{-1}=M_q\cong M_p$. Since the limitation methods $C_r^{(P)}$, $C_{r+1}^{(P)}$, and M_p are regular and normal (see Theorem II.22 of [13]), this implies

$$C_{r+1}^{(P)}=C_{r+1}^{(P)}(C_r^{(P)})^{-1}C_r^{(P)}\cong M_pC_r^{(P)}=H_1^{(P)}C_r^{(P)}$$

which gives by induction $C_r^{(P)}\cong H_r^{(P)}$ for every positive integer r. ∎

Notice that $C_r^{(P)}$ and $C_{r+1}^{(P)}$ are not equivalent under these assumptions. But if the sequence is bounded we have

THEOREM 2. *Let $P=(p_n)_{n=1}^{\infty}$ be a real positive mean with $\lim_{n\to\infty}P_n=\infty$ and $\lim_{n\to\infty}p_n/P_n=0$. Then all weighted Cesaro and Hölder limitation methods $C_r^{(P)}$ and $H_r^{(P)}$ with $r\geqq1$ are equivalent for bounded sequences $(x_n)_{n=1}^{\infty}$.*

(For a proof of the equivalence of the weighted Hölder means see [3], Compare also with [15]. An alternate proof will be indicated in the sequel.)

Since $\lim_{n\to\infty}p_n/P_n=0$ it is not difficult to construct a sequence $(x_n)_{n=1}^{\infty}$ with $x_n\in\{-1,1\}$ such that

$$\left|\frac{1}{P_n}\sum_{k=1}^{n}p_kx_k\right|\leqq\sup_{k\geqq n}\frac{p_k}{P_k}.$$

Therefore we have $\lim x_n=0$ $(H_r^{(P)})$ for $r\geqq1$. Thus $H_r^{(P)}$ is not equivalent to usual convergence. But if $\underline{\lim}_{n\to\infty}p_n/P_n>0$ instead of $\lim_{n\to\infty}p_n/P_n=0$ $\lim x_n=x$ $(H_r^{(P)})$ (for some $r\geqq1$) implies $\lim_{n\to\infty}x_n=x$. Next we discuss the relation between $H_\infty^{(P)}$ and $C_\infty^{(P)}$.

THEOREM 3. *Let $P=(p_n)_{n=1}^{\infty}$ be a real positive mean with $\lim_{n\to\infty}P_n=\infty$ and $\lim_{n\to\infty}p_n/P_n=0$. Then $\lim x_n=x$ $(C_\infty^{(P)})$ and $\lim x_n=x$ $(H_\infty^{(P)})$ are equivalent to*

$$\lim_{n\to\infty}\sup_{k\ge 1}\left|\frac{1}{\log P_n}\sum_{k\le P_j\le kP_n}\frac{p_j}{P_j}x_j-x\right|=0 \tag{9}$$

for bounded sequences $(x_n)_{n=1}^{\infty}$.

Note that every $H_r^{(P)}$-limitable sequence is $H_{\infty}^{(P)}$-limitable. But with these theorems we can construct bounded sequences which are $H_{\infty}^{(P)}$-limitable but not $H_r^{(P)}$-limitable for any r. Set $x_n=e^{i\log P_n}=P_n^i$. Using Taylor's theorem we get

$$\frac{1}{i+1}P_n^{i+1}-\sum_{k=1}^{n}p_kx_k=\sum_{k=1}^{n}\left(\frac{1}{i+1}(P_k^{i+1}-P_{k-1}^{i+1})-p_kP_k^i\right)$$

$$=\mathcal{O}\left(\sum_{k=1}^{n}\frac{p_k^2}{P_k}\right)=o(P_n).$$

Therefore by Theorem 2 $(\mathrm{Re}(x_n))_{n=1}^{\infty}$ or $(\mathrm{Im}(x_n))_{n=1}^{\infty}$ are not $H_r^{(P)}$-limitable for any $r\ge 1$. On the other hand we have

$$\sum_{k\le P_j\le kP_n}\frac{p_j}{P_j}x_j=\sum_{k\le P_j\le kP_n}p_jP_j^{i-1}=\mathcal{O}(1)+\mathcal{O}\left(\sum_{k\le P_j\le kP_n}\frac{p_j}{P_j}\right)=\mathcal{O}(1)+o(\log P_n).$$

Thus by Theorem 3 $\lim\mathrm{Re}(x_n)=\lim\mathrm{Im}(x_n)=0$ $(H_{\infty}^{(P)})$.

For the proof of Theorem 3 we can proceed similarly to [4]. We need two Lemmata.

LEMMA 1. *Let* $P=(p_n)_{n=1}^{\infty}$ *be a real positive mean with* $\lim_{n\to\infty}P_n=\infty$ *and* $\lim_{n\to\infty}p_n/P_n=0$. *Then*

$$H_{r+1}^{(P)}(x_n)=\frac{1}{P_n r!}\sum_{k=1}^{n}p_k\log(P_n/p_k)^r x_k+o(1)\qquad(n\to\infty). \tag{10}$$

and

$$C_{r+1}^{(P)}(x_n)=\frac{r+1}{P_n^{r+1}}\sum_{k=1}^{n}p_k(P_n-P_k)^r x_k+o(1)\qquad(n\to\infty). \tag{11}$$

hold for every non-negative integer r *and for bounded sequences* $(x_n)_{n=1}^{\infty}$.

PROOF. For $r=0$ (10) and (11) are trivial. If (10) holds for some r we have

$$H_{r+2}^{(P)}(x)=\frac{1}{P_n}\sum_{k=1}^{n}p_kH_{r+1}^{(P)}(x_k)=\frac{1}{P_n r!}\sum_{k=1}^{n}\frac{p_k}{P_k}\sum_{j=1}^{k}p_j\log(P_k/P_j)^r x_j+o(1)$$

$$=\frac{1}{P_n r!}\sum_{j=1}^{n}p_jx_j\sum_{k=j}^{n}\frac{p_k}{P_k}\log(P_k/P_j)^r+o(1).$$

Since

$$\int_{P_j}^{P_n}\frac{\log(x/P_j)^r}{x}dx=\frac{\log(P_n/P_j)^{r+1}}{r+1},$$

the function $\log(x/P_j)^r/x$ is monotonely increasing for $P_j \leqq x \leqq P_j e^r$ and monotonely decreasing for $x \geqq P_j e^r$, and $\lim_{n\to\infty} P_n/P_{n+1}=1$ we get for arbitrary $\varepsilon>0$ and $n \geqq j \geqq N(\varepsilon)$

$$(1-\varepsilon)\sum_{k=j}^{n}\frac{p_k}{P_k}\log(P_k/P_j)^r-2\frac{r^r p_i}{P_i}\leqq\int_{P_j}^{P_n}\frac{\log(x/P_j)^r}{x}dx$$

$$\leqq(1+\varepsilon)\sum_{k=j}^{n}\frac{p_k}{P_k}\log(P_n/P_j)^r+2\frac{r^r p_i}{P_i},$$

where $P_{i-1} \leqq P_j e^r < P_i$. Hence

$$\frac{1}{P_n r!}\sum_{j=1}^{n}p_j x_j \sum_{k=j}^{n}\frac{p_k}{P_k}\log(P_k/P_j)^r-\frac{1}{P_n(r+1)!}\sum_{j=1}^{n}p_j x_j\frac{\log(P_n/P_j)^{r+1}}{r+1}$$

$$=\mathcal{O}_\varepsilon\left(\frac{1}{P_n}\right)+\mathcal{O}(\varepsilon)+o_\varepsilon(1),$$

which concludes the proof of Lemma 1, since (11) can be proved similarly. ∎

LEMMA 2. *Under the same conditions as in Lemma* 1

$$\sum_{k=1}^{n}\frac{p_k}{P_k}x_k=\mathcal{O}(1) \qquad (n\to\infty) \tag{12}$$

implies $\lim x_n=0$ $(H_\infty^{(P)})$ *and* $\lim x_n=0$ $(C_\infty^{(P)})$ *for bounded sequences* $(x_n)_{n=1}^{\infty}$.

PROOF. Set $F(x)=x\log(1/x)^r/r!$. Then by (10)

$$H_{r+1}^{(P)}(x_n)=\sum_{k=1}^{n}\frac{p_k x_k}{P_k}F(P_k/P_n)+o_r(1)$$

$$=\sum_{k=1}^{n-1}(F(P_k/P_n)-F(P_{k+1}/P_n))\sum_{j=1}^{n}\frac{p_j}{P_j}x_j+o_r(1)$$

yields

$$H_{r+1}^{(P)}(x_n)=\mathcal{O}(F(e^{-r}))+o_r(1),$$

since $F(x)$ is monotonely increasing for $0<x\leqq e^{-r}$ and monotonely decreasing for $e^{-r}\leqq x\leqq 1$. Thus

$$\lim_{r\to\infty}\overline{\lim_{n\to\infty}}|H_{r+1}^{(P)}(x_n)|=0.$$

$\lim x_n=0$ $(C_\infty^{(P)})$ can be shown similarly. ∎

Now we can prove Theorem 3. First suppose $\lim x_n=0$ $(H_\infty^{(P)})$. It is no loss of generality to assume $x=0$. By induction we get for every $r\geqq 1$

$$\sum_{k\leq P_j\leq kP_n}\frac{p_j}{P_j}H_r^{(P)}(x_j)=\sum_{k\leq P_j\leq kP_n}\frac{p_j}{P_j}x_j+o(\log P_n)$$

uniformly in k. Choose r and $N(\varepsilon)$ such that $|H_r^{(P)}(x_n)|<\varepsilon$ for $n\geqq N(\varepsilon)$. Thus

$$\sum_{k\leq P_j\leq kP_n} \frac{p_j}{P_j} x_j = o(\log P_n) + O_\varepsilon(1) + \varepsilon \log P_n$$

uniformly in k. The arguments for $\lim x_n = 0$ $(C_\infty^{(P)})$ are similar. Thus the first part of the proof is finished. Now suppose that (9) holds with $x=0$. Choose $N(\varepsilon)$ such that

$$\sup_{k\geq 1} \left| \frac{1}{\log P_n} \sum_{k\leq P_j\leq kP_n} \frac{p_j}{P_j} x_j \right| < \varepsilon$$

for $n \geqq N(\varepsilon)$ and define y_k by

$$y_k = \sum_{P_n^l \leq P_j < P_n^{l+1}} \frac{p_j}{P_j} x_j \Big/ \sum_{P_n^l \leq P_j < P_n^{l+1}} \frac{p_j}{P_j}$$

for $P_n^l \leqq p_k < P_n^{l+1}$. Thus for every $r \geqq 1$

$$\varlimsup_{k\to\infty} |H_r^{(P)}) y_k)| \leqq \varlimsup_{k\to\infty} |y_k| = O(\varepsilon) \quad \text{and} \quad \varlimsup_{k\to\infty} C_r^{(P)}(y_k) = O(\varepsilon) .$$

Trivially the sequence $z_n = y_n - x_n$ satisfies the assumption of Lemma 2. Thus $\lim z_n = 0$ $(H_\infty^{(P)})$ and $\lim z_n = 0$ $(C_\infty^{(P)})$ or

$$\varlimsup_{n\to\infty} |H_r^{(P)}(z_n)| < \varepsilon \quad \text{and} \quad \varlimsup_{n\to\infty} |C_r^{(P)}(z_n)| < \varepsilon$$

for $r \geqq r(\varepsilon)$. Hence

$$\varlimsup_{n\to\infty} |H_r^{(P)}(x_n)| = O(\varepsilon) \quad \text{and} \quad \varlimsup_{n\to\infty} |C_r^{(P)}(x_n)| = O(\varepsilon)$$

for $r \geqq r(\varepsilon)$ which implies $\lim x_n = 0$ $(H_\infty^{(P)})$ and $\lim x_n = 0$ $(C_\infty^{(P)})$. This completes the proof of Theorem 3.

In the following we study the limitation method $A^{(P)}$. We will prove

THEOREM 4. *Let $P = (p_n)_{n=1}^\infty$ be a real positive mean with $\lim_{n\to\infty} P_n = \infty$ and $\lim_{n\to\infty} p_n/P_n = 0$. Then $A^{(P)}$ is equivalent to $C_r^{(P)}$ and $H_r^{(P)}$ $(r \geqq 1)$ for bounded sequences.*

It is sufficient to prove that $A^{(P)}$ is equivalent to M_p for bounded sequences. But this is an immediate consequence of the following Lemmata.

LEMMA 3. *Let $P = (p_n)_{n=1}^\infty$ be a real positive mean with $\lim_{n\to\infty} P_n = \infty$ and $\lim_{n\to\infty} p_n/P_n = 0$. Then $\lim_{n\to\infty} x_n = x$ implies $\lim x_n = x$ $(A^{(P)})$. ($A^{(P)}$ is regular.*

Proof. First we show $\lim 1 = 1$ $(A^{(P)})$. Since $\lim_{n\to\infty} P_n/P_{n-1} = 1$ we have $P_n \leqq P_{n-1}(1+\varepsilon)$ for $n \geqq N(\varepsilon)$. Thus

$$\int_{P_{n-1}}^{P_n} \varepsilon^{-(1+\varepsilon)st}dt \leqq p_n e^{-(1+\varepsilon)sP_{n-1}} \leqq p_n e^{-sP_n} \leqq \int_{P_{n-1}}^{P_n} e^{-st}dt$$

yields

$$\frac{1}{1+\varepsilon}+sK(\varepsilon) \leqq s\sum_{n=1}^{\infty} p_n e^{-sP_n} \leqq 1\,,$$

which implies $\lim 1=1$ $(A^{(P)})$. If $\lim\limits_{n\to\infty} x_n=0$ we have for arbitrary $\varepsilon>0$

$$\left| s\sum_{n=1}^{\infty} p_n x_n e^{-sP_n}\right| \leqq sK(\varepsilon)+s\varepsilon\int_0^{\infty} e^{-st}dt=sK(\varepsilon)+\varepsilon\,.$$

Therefore $A^{(P)}$ is regular. ■

LEMMA 4. *Let* $P=(p_n)_{n=1}^{\infty}$ *be as in Lemma* 3. *If the sequence* $(x_n)_{n=1}^{\infty}$ *is bounded below or above then* $\lim x_n=x$ $(A^{(P)})$ *implies* $\lim x_n=x$ (M_p).

PROOF. The proof is a direct application of Karamata's method. We can assume $x_n>0$. If $f(x)=x^k$, $k\geqq 0$, we have

$$\lim_{s\to 0+} s\sum_{n=1}^{\infty} p_n x_n e^{-sP_n} f(e^{-sP_n})=x\int_0^1 f(t)dt\,. \tag{13}$$

Thus (13) holds for all Riemann-integrable functions $f(x)$. Now set $f(x)=0$ $0\leqq x<e^{-1}$ and $f(x)=1/x$ for $e^{-1}\leqq x\leqq 1$ and $s=1/P_N$. Then (13) implies $\lim x_n$ $=x$ (M_p). (Compare also with [8] and [21].) ■

LEMMA 5. *Let* $P=(p_n)_{n=1}^{\infty}$ *be as in Lemma* 3. *If the sequence* $(x_n)_{n=1}^{\infty}$ *is bounded then* $\lim x_n=x$ (M_p) *implies* $\lim x_n=x$ $(A^{(P)})$.

PROOF. Let $\lim x_n=x$ (M_p). It is no loss of generality to assume $x_n>0$. Set $y_n=c_1^{(P)}(x_n)$. Then we get by similar methods as in the proof of Lemma 3

$$\lim_{s\to 0+} s^2\sum_{n=1}^{\infty} p_n y_n e^{-sP_n}=x \tag{14}$$

By partial summation we have

$$\sum_{k=1}^{\infty} p_k y_k e^{-sP_k}=\sum_{n=1}^{\infty} p_n x_n \sum_{k=n}^{\infty} p_k e^{-sP_k}\,.$$

Now we have for $n\geqq N(\varepsilon)$

$$\frac{e^{-s(1+\varepsilon)P_{n-1}}}{s(1+\varepsilon)}=\int_{P_{n-1}}^{\infty} e^{-s(1+\varepsilon)t}dt \leqq \sum_{k=n}^{\infty} p_k e^{-sP_k}$$

$$\leqq \int_{P_n}^{\infty} e^{-st}dt+p_n e^{-sP_n}=\frac{e^{-sP_n}}{s}+p_n e^{\ sP_n}\,,$$

which implies

$$sK(\varepsilon)+\frac{s}{1+\varepsilon}\sum_{n=1}^{\infty}p_n x_n e^{-s(1+\varepsilon)P_n}\leqq s^2\sum_{k=1}^{\infty}p_k y_k e^{-sP_k}$$

$$\leqq s\sum_{n=1}^{\infty}p_n x_n e^{-sP_n}+s^2\sum_{n=1}^{\infty}p_n^2 x_n e^{-sP_n}\,.$$

Since

$$\lim_{s\to 0+}s^2\sum_{n=1}^{\infty}p_n^2 x_n e^{-sP_n}=0\,,$$

(14) and (15) imply $\lim x_n=x\ (A^{(P)})$. ■

REMARK. Note that a refinement of Lemma 5 gives an alternate proof of Theorem 2. If $\lim x_n=x\ (C_r^{(P)})$ for some $r\geqq 1$ we get by similar methods

$$\lim_{s\to 0+}s^{r+1}\sum_{n=1}^{\infty}p_n c_r^{(P)}(x_n)e^{-sP_n}=x$$

and then by induction for $0\leqq k\leqq r$

$$\lim_{s\to 0+}s^{k+1}\sum_{n=1}^{\infty}p_n c_k^{(P)}(x_n)e^{-sP_n}=x\,.$$

Therefore $\lim x_n=x\ (A^{(P)})$.

3. Uniform distribution

Let X be a compact metric space and μ a positive normalized Borel measure on X. Then a sequence $(x_n)_{n=1}^{\infty}$, $x_n\in X$, is said to be uniformly distributed with respect to μ and a real positive mean $P=(p_n)_{n=1}^{\infty}$ if

$$\lim_{n\to\infty}\frac{1}{P_n}\sum_{k=1}^{n}p_k f(x_k)=\int_X f\,d\mu \tag{16}$$

holds for all real-valued continuous functions f on X. Note that (16) can be read as

$$\lim f(x_n)=\int_X f\,d\mu\ \ (H_1^{(P)})\,. \tag{17}$$

Therefore it is a natural generalisation to consider $H_r^{(P)}$-uniformly distributed ($H_r^{(P)}$-u.d.) and $C_r^{(P)}$-u.d. sequences which are defined by the relations

$$\lim f(x_n)=\int_X f\,d\mu\ (H_r^{(P)})\quad\text{and}\quad \lim f(x_n)=\int_X f\,d\mu\ (C_r^{(P)}) \tag{18}$$

for all continuous $f: X\to\boldsymbol{R}$. Niederreiter [12] has proved that the condition $\lim_{n\to\infty} p_n/P_n=0$ is necessary and sufficient for the existance of a $H_1^{(P)}$-u.d. sequence $(x_n)_{n=1}^{\infty}$ if $\lim P_n=\infty$ and μ is not concentrated on one point. For weighted Hölder and Cesaro means we can prove

THEOREM 5. *Let X be a compact metric space, μ a normalized Borel measure on X with $\mu(\{x\})=0$ for all $x\in X$, and $P=(p_n)_{n=1}^{\infty}$ a real positive mean. Then there exists an $H_r^{(P)}$-u.d. (or $C_r^{(P)}$-d.d.) sequence $(r\geqq 1)$ if and only if $\lim_{n\to\infty} P_n=\infty$ and $\lim_{n\to\infty} p_n/P_n=0$.*

PROOF. If $\lim_{n\to\infty} P_n=\infty$ and $\lim_{n\to\infty} p_n/P_n=0$ there exists on $H_1^{(P)}$-u.d. sequence which is $H_r^{(P)}$-u.d. and $C_r^{(P)}$-u.d. for $r\geqq 1$, too.

Now suppose that $(x_n)_{n=1}^{\infty}$ is $H_r^{(P)}$-u.d. (or $C_r^{(P)}$-n.d.) for some $r\geqq 1$. First we want to show that $\lim_{n\to\infty} P_n=P_{\infty}<\infty$ is impossible. If $f(x)\geqq 0$ for all $x\in X$ it is easy to derive for every $k\geqq 1$

$$\varliminf_{n\to\infty} H_r^{(P)}(f(x_n))\geqq \frac{p_k}{P_{\infty}} f(x_k) \quad \left(\text{or } \varliminf_{n\to\infty} C_r^{(P)}(f(x_n))\geqq \frac{p_k}{(r-1)!P_{\infty}} f(x_k)\right).$$

Now we construct a continuous function $f: X\to \boldsymbol{R}$ with $\int_X f\,d\mu=1$ and $f(x)p_k/(P_{\infty}(r-1)!)\geqq 2$ for some $k\geqq 1$. Denote $d(\cdot,\cdot)$ a metric on X and $B(x, r)=\{y\in X \mid d(x, y)<r\}$ the open ball with centre x and radius r. If $\mu(B(x_n, r_n))=0$ for some sequence $(r_n)_{n=1}^{\infty}$ of positive numbers there is an Urysohn function $f: X\to\boldsymbol{R}$ with $f(x_n)=0$, $n\geqq 1$, and $\int_X f\,d\mu<0$, which is impossible. Thus there is some $k\geqq 1$ such that $\mu(B(x_k, r))>0$ for all $r>0$. Furthermore $\lim_{r\to 0}\mu(B(x_k, r))=0$ since $\mu(\{x_k\})=0$. Now construct an Urysohn function $\bar{f}: X\to\boldsymbol{R}$ with $\bar{f}(x_k)=1$ and $\int_X f\,d\mu\leqq p_k/(2(r-1)!P_{\infty})$. Thus $f(x)=\bar{f}(x)/\int_X f\,d\mu$ satisfies $\int_X f\,d\mu=1$ and $f(x_k)p_k/(P_{\infty}(r-1)!)\geqq 2$. But this is a contradiction to the assumption that $(x_n)_{n=1}^{\infty}$ is $H_r^{(P)}$-u.d. (or $C_r^{(P)}$-u.d.). Now suppose that $\varlimsup_{n\to\infty} p_n/P_n>0$. Then there is an increasing sequence $(n_k)_{k=1}^{\infty}$ of possitive integers such that $p_{nk}/P_{nk}>\varepsilon$ for some $\varepsilon>0$. Let x_0 be an accumulation point of the subsequence $(x_{n_k})_{k=1}^{\infty}$. Now construct a continuous function $f: X\to\boldsymbol{R}$ with $f(x)\geqq 0$, $\int_X f\,d\mu\leqq 1$, and $f(y)\geqq 2\varepsilon^{-r}$ for all $y\in K(x_0, r)$ for some $r>0$. Then the relation

$$H_r^{(P)}(f(x_n))\geqq\left(\frac{p_n}{P_n}\right)^r f(x_n) \quad \left(\text{or } C_r^{(P)}(f(x_n))\geqq\left(\frac{p_n}{P_n}\right)^r f(x_n)\right)$$

is again a contradiction to the assumption that $(x_n)_{n=1}^{\infty}$ is $H_r^{(P)}$-u.d. (or $C_r^{(P)}$-u.d.). ∎

Therefore the conditions $\lim_{n\to\infty} P_n=\infty$ and $\lim_{n\to\infty} p_n/P_n=0$ seem to be natural for the application of Theorem 2 to uniform distribution. Since every continuous function $f: X\to\boldsymbol{R}$ on a compact space X is bounded we get under the usual assumptions on $P=(p_n)_{n=1}^{\infty}$ that a sequence $(x_n)_{n=1}^{\infty}$ is $H_r^{(P)}$-u.d. (or $C_r^{(P)}$-u.d.) with respect to μ if and only if $(x_n)_{n=1}^{\infty}$ is u.d. with respect to μ and P.

As a generalisation of an Abel unform distribution introduced and discussed by E. Hlawka [9] and H. Niederreiter [11] we can define that a sequence $(x_n)_{n=1}^{\infty}$ in a compact metric space X is $A^{(P)}$-u.d. with respect to μ if

$$\lim f(x_n)=\int_X f\,d\mu \quad (A^{(P)}) \tag{19}$$

holds for every continuous function $f: X\to \mathbf{R}$. Under the usual assumptions on P this definition is again equivalent to (16). In the case of uniform distribution modulo 1 similar theorems to those in [9] and [11] can be deduced easily by verbally the same arguments used there.

Theorem 3 is interesting in relation to well distribution with respect to weighted means. (Note that there are many possibilities to generalize the usual well distribution to weighted means. See e.g. Tichy [22], Goto and Kano [6], and Schatte [18], [19],]20].) We use Schatte's concept. Define $L(k, n)$ by

$$\sum_{j=k+1}^{L(k,n)} p_j \leqq P_n < \sum_{j=k+1}^{L(k,n)+1} p_j\,. \tag{20}$$

Then a sequence $(x_n)_{n=1}^{\infty}$ is said to be well distributed (for short w.d.) with respect to μ and $P=(p_n)_{n=1}^{\infty}$ if

$$\lim_{n\to\infty} \sup_{k\geqq 1} \left| \frac{1}{P_n} \sum_{j=k+1}^{L(k,n)} p_j f(x_j) - \int_X f\,d\mu \right| = 0 \tag{21}$$

holds for every continuous function $f: X\to \mathbf{R}$.

It motivates an $H_\infty^{(P)}$- and a $C_\infty^{(P)}$-uniform distribution. A sequence is called $H_\infty^{(P)}$-u.d. (or $C_\infty^{(P)}$-u.d.) with respect to μ if

$$\lim f(x_n)=\int_X f\,d\mu \quad (H_\infty^{(P)}) \quad \left(\text{or } \lim f(x_n)=\int_X f\,d\mu \quad (C_\infty^{(P)})\right) \tag{22}$$

holds for every continuous function $f: X\to \mathbf{R}$.

First we remark that every $H_1^{(P)}$-u.d. sequence is $H_\infty^{(P)}$-u.d. and $C_\infty^{(P)}$-u.d. if $\lim_{n\to\infty} P_n=\infty$. But the converse statement is not true. As above it can be shown (by Weyl's criterion) that the sequence $x_n=\log P_n$ is not $H_1^{(P)}$-u.d. but $H_\infty^{(P)}$-u.d. and $C_\infty^{(P)}$-u.d. modulo 1. Thus we get such an example for arbitrary compact metric spaces X and normalized Borel measures μ by Hedrlin's lifting method [1].

Next we use Theorem 3 to get another equivalence. Under the usual assumptions on $P=(p_n)_{n=1}^{\infty}$ a sequence $(x_n)_{n=1}^{\infty}$ is $H_{\infty}^{(P)}$-u.d. or $C_{\infty}^{(P)}$-u.d. if and only if it is w.d. with respect to $Q=(p_n/P_n)_{n=1}$. Furthermore we can read Theorem 3 the other way round. A sequence $(x_n)_{n=1}^{\infty}$ is w.d. with respect to $Q=(q_n)_{n=1}^{\infty}$ if and only if it is $H_{\infty}^{(P)}$-u.d. of $C_{\infty}^{(P)}$-u.d. with respect to $P=(q_n e^{Q_n})_{n=1}^{\infty}$ if $\lim_{n\to\infty} q_n=0$ and $\lim_{n\to\infty} Q_n=\infty$ $(Q_n=\sum_{k=1}^{n} q_k)$.

In [5] metric theorems concerning the weighted well distribution with respect to the infinite product measure μ_{∞} are established. Comparing these with Theorem 3 we obtain

THEOREM 6. *Let X be a compact metric space, μ a normalized positive Borel measure on X, and $P=(p_n)_{n=1}^{\infty}$ a real positive mean with $\lim_{n\to\infty} P_n=\infty$ and $\lim_{n\to\infty} p_n/P_n=0$. If*

$$\overline{\lim_{n\to\infty}} \frac{p_n}{P_n} \log n<\infty \quad \textit{and} \quad \lim_{n\to\infty} \frac{\log P_n}{\log n}=\infty$$

or if p_n/P_n is monotonely decreasing and

$$\lim_{n\to\infty} \frac{p_n}{P_n} \sqrt{n} \log n (\log\log n)^{3/2+\varepsilon}=0$$

for some $\varepsilon>0$ then μ_{∞}-almost all sequences $(x_n)_{n=1}^{\infty}$ are $H_{\infty}^{(P)}$-u.d. or $C_{\infty}^{(P)}$-u.d. If μ is not concentrated on one point and

$$\lim_{n\to\infty} \frac{p_n}{P_n} \log n=\infty$$

then μ_{∞}-almost no sequences are $H_{\infty}^{(P)}$-u.d. or $C_{\infty}^{(P)}$-u.d.

For example consider the sequence of weights

$$p_n=\frac{1}{\log(n+1)} e^{n/\log(n+1)}.$$

Then p_n is monotone (for $n\geqq 4$) and $\lim_{n\to\infty} \log(n)\cdot p_n/P_n=1$. Therefore μ_{∞}-almost all sequences are $H_{\infty}^{(P)}$-u.d. (or $C_{\infty}^{(P)}$-u.d.) but $P=(p_n)_{n=1}^{\infty}$ does not satisfy Hill's condition

$$\sum_{n=1}^{\infty} e^{-\delta P_n^2/\sum_{k=1}^{n} p_k^2}<\infty \qquad \text{for all } \delta>0.$$

Thus μ_{∞}-almost no sequences are u.d. with respect to P since Hill's condition is a criterion for the Borel property in the case of monotone weights p_n (see [10]).

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