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A REMARK ON NONNEGATIVELY CURVED HOMOGENEOUS KÄHLER MANIFOLDS

By
Mitsuhiro ITOH

Introduction.

The aim of this paper is to show the following

THEOREM *If a compact connected homogeneous Kähler manifold (M, g) is of nonnegative curvature, then it is a Kählerian direct product of a flat complex torus (T, g_0) and a Hermitian symmetric space of compact type (M', g'); $(M, g) \cong (T, g_0) \times (M', g')$.*

Here, a Kähler manifold (M, g) is called homogeneous if the isometry group $I(M, g)$ acts on M transitively. And a Kähler manifold is said to be of nonnegative curvature when the sectional curvature K_σ is nonnegative for any plane section σ.

Hermitian symmetric spaces of compact type give typical examples for compact Kähler manifolds of nonnegative curvature (Helgason [2]).

For a Kähler manifold (M, g), the holomorphic bisectional curvature $H_{\sigma,\tau}$ for holomorphic plane sections σ and τ is defined by

$$H_{\sigma,\tau}=g(R(X, IX)IY, Y), \ \sigma=X\wedge IX, \ \tau=Y\wedge IY, \ g(X, X)=g(Y, Y)=1,$$

where R is the curvature tensor and I is the complex structure (Kobayashi and Nomizu [5]). Since $g(R(X, IX)IY, Y)=g(R(X, Y)Y, X)+g(R(X, IY)IY, X)$, the holomorphic bisectional curvature $H_{\sigma,\tau}$ is written by a sum of two sectional curvatures up to nonnegative constant factors. Thus, if a Kähler manifold is of nonnegative curvature, then the holomorphic bisectional curvature $H_{\sigma,\tau}$ is also nonnegative for any holomorphic plane sections σ and τ.

From a theorem of Matsushima [6], a compact connected homogeneous Kähler manifold (M, g) is a Kählerian direct product of a flat complex torus (T, g_0) and a Kähler C-space (M', g'); $(M, g)\cong(T, g_0)\times(M', g')$, where a Kähler C-space is by definition a compact simply connected homogenous Kähler manifold.

If the compact connected homogeneous Kähler manifold (M, g) is of nonnegative curvature, so is the Kähler C-space (M', g'). Therefore, Theorem is a direct conclusion of the following proposition.

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PROPOSITION *If a Kähler C-space is of nonnegative curvature, then it is a Hermitian symmetric space of compact type.*

Before we give a proof of Proposition, we will summarize structure of a Kähler C-space and geometrical concepts (an invariant Kähler metric, connection and curvature tensor) from Lie group theoretical point of view (Wang [7], Itoh [4]).

1. Structure of Kähler C-space

Let (M, g) be a Kähler C-space. Then M is described as $M=G_u/K=G/L$, where G_u is a compact connected semisimple subgroup of $I(M, g)$, G a connected complex semisimple Lie group which contains G_u as its maximal compact subgroup, L a connected closed parabolic subgroup of G and $K=G_u \cap L$.

Conversely, given a connected complex semisimple Lie group G and its connected closed parabolic subgroup L, a coset space G/L has structure of a Kähler C-space.

We will give a full detail of structure of a Kähler C-space. Let $\mathfrak{g}$ and $\mathfrak{h}$ be a complex semisimple Lie algebra and a Cartan subalgebra of $\mathfrak{g}$. The set of all nonzero roots with respect to $(\mathfrak{g}, \mathfrak{h})$ is denoted by Δ. $\Pi=\{\alpha_1,\cdots,\alpha_l\}$ $(l=\dim_c\mathfrak{h}=\operatorname{rank}\mathfrak{g})$ represents a fundamental root system of Δ. Any root is a linear combination of α_i, $i=1, \cdots, l$, over $\boldsymbol{Z}$; $\Delta\subset\boldsymbol{Z}\alpha_1+\cdots+\boldsymbol{Z}\alpha_l$. The lexicographic order is defined in $\boldsymbol{Z}\alpha_1+\cdots+\boldsymbol{Z}\alpha_l$ with respect to Π. Δ^+(or Δ^-) denotes the set of positive (or negative) roots. We fix a basis $\{H_j, E_\alpha;\ j=1, \cdots, l, \alpha\in\Delta\}$ of $\mathfrak{g}$, called Weyl's canonical basis; namely, $\{H_1, \cdots, H_l\}$ is a basis of $\mathfrak{h}$ and each E_α is a root vector for a root α such that $B(H, H_j)=\alpha_j(H)$ for $j=1, \cdots, l$, and for any $H\in\mathfrak{h}$, $B(E_\alpha, E_{-\alpha})=-1$ for $\alpha\in\Delta^+$ and $[E_\alpha, E_\beta]=N_{\alpha,\beta}E_{\alpha+\beta}$, $N_{\alpha,\beta}=N_{-\alpha,-\beta}\in\boldsymbol{R}$, where B is the Killing form of $\mathfrak{g}$.

Choose an arbitrary subset $\Phi(\neq\phi)$ of the fundamental root system Π. We define a complex subalgebra $\mathfrak{l}$ and a real subalgebra $\mathfrak{g}_u$ of $\mathfrak{g}$ by

$$\mathfrak{l}=\mathfrak{h}+\sum_{\alpha\in\Delta-\Delta^+(\Phi)}\boldsymbol{C}E_\alpha,$$

$$\mathfrak{g}_u=\sum_{j=1}^{l}\boldsymbol{R}\sqrt{-1}H_j+\sum_{\alpha\in\Delta^+}\{\boldsymbol{R}(E_\alpha+E_{-\alpha})+\boldsymbol{R}\sqrt{-1}(E_\alpha-E_{-\alpha})\},$$

where $\Delta^+(\Phi)=\{\alpha=\sum_{j=1}^{l}m_j\alpha_j\in\Delta^+;\ m_j>0$ for some $\alpha_j\in\Phi\}$. Since $\mathfrak{l}$ contains a Borel subalgebra of $\mathfrak{g}$, $\mathfrak{h}+\sum_{\alpha\in\Delta^-}\boldsymbol{C}E_\alpha$, and $B_{|\mathfrak{g}_u\times\mathfrak{g}_u}$ is negative definite, $\mathfrak{l}$ is a parabolic subalgebra of $\mathfrak{g}$ and $\mathfrak{g}_u$ is a compact real form of $\mathfrak{g}$.

Let G be a simply connected complex semisimple Lie group with Lie algebra $\mathfrak{g}$. Let L, G_u and K be connected closed subgroups of G corresponding to subalgebras $\mathfrak{l}$, $\mathfrak{g}_u$ and $\mathfrak{k}=\mathfrak{l}\cap\mathfrak{g}_u$ respectively. Then the imbedding $G_u\subset G$ induces the indentification $G_u/K=G/L$ and hence the coset space G_u/K has structure of a compact simply connected homogeneous complex manifold.

It is shown that the space G_u/K admits a Kähler metric g which is invariant under G_u. Then the Kähler manifold $(G_u/K, g)$ constructed from such a pair $(\mathfrak{g}, \Phi)$ is a Kähler C-space. We call it *a Kähler C-space associated with* $(\mathfrak{g}, \Phi)$.

It is known that any parabolic subalgebra of $\mathfrak{g}$ is conjugate with $\mathfrak{h}+\sum_{\alpha\in\Delta-\Delta^+(\Phi)} CE_\alpha$ for a certan Cartan subalgebra $\mathfrak{h}$ and a certain subset Φ of Π. Therefore, any Kähler C-space (M, g) can be written as a Kähler C-space $(G_u/K, g)$ associated with an appropriate pair $(\mathfrak{g}, \Phi)$.

Note that the second Betti number b_2 of a Kähler C-space associated with a pair $(\mathfrak{g}, \Phi)$ is equal to $\sharp\Phi$.

If we set $\mathfrak{m}=\sum_{\alpha\in\Delta^+(\Phi)}\{(\boldsymbol{R}(E_\alpha+E_{-\alpha})+\boldsymbol{R}\sqrt{-1}(E_\alpha-E_{-\alpha})\}$, then $\mathfrak{g}_u$ has a reductive decomposition, namely

$$\mathfrak{g}_u=\mathfrak{k}+\mathfrak{m},\ ad(\mathfrak{k})\mathfrak{m}\subset\mathfrak{m}.$$

We identify $\mathfrak{m}$ with the tangent space at the origin of G_u/K. The G_u-invariant complex structure I on the Kähler C-space $(G_u/K, g)$ satisfies on $\mathfrak{m}$

$$I(E_\alpha+E_{-\alpha})=\sqrt{-1}(E_\alpha-E_{-\alpha}),$$

$$I(\sqrt{-1}(E_\alpha-E_{-\alpha}))=-(E_\alpha+E_{-\alpha}).$$

The complexification $\mathfrak{m}^C$ of $\mathfrak{m}$ is decomposed into the sum of $\mathfrak{m}^+$ and $\mathfrak{m}^-$;

$$\mathfrak{m}^C=\mathfrak{m}^++\mathfrak{m}^-,\ \mathfrak{m}^+=\{X\in\mathfrak{m}^C,\ IX=\sqrt{-1}X\}=\sum_{\alpha\in\Delta^+(\Phi)} CE_\alpha,$$

$$\mathfrak{m}^-=\overline{\mathfrak{m}^+}.$$

Any G_u-invariant Kähler metric g can be written at the origin as

$$g=\sum_{\alpha\in\Delta^+(\Phi)} c_\alpha\omega^\alpha\cdot\omega^{\bar\alpha}$$

where c_α is a positive number for each $\alpha\in\Delta^+(\Phi)$, $c_{\alpha+\beta}=c_\alpha+c_\beta$ for α, β, $\alpha+\beta\in\Delta^+(\Phi)$ and $c_{\alpha+\gamma}=c_\alpha$ for α, $\alpha+\gamma\in\Delta^+(\Phi)$, $\gamma\in\Delta^+-\Delta^+(\Phi)$, and ω^α (or $\omega^{\bar\alpha}$) is the dual of E_α (or $E_{-\alpha}$).

We define a linear operator Λ, called a connection function associated with the invariant Riemannian connection $\nabla(\nabla g=0,\ \nabla I=0)$ as follows

$$\Lambda(X)Y=1/2[X, Y]_{\mathfrak{m}^C}+U(X, Y),\ X, Y\in\mathfrak{m}^C,$$

where $[X, Y]_{\mathfrak{m}^C}$ is the $\mathfrak{m}^C$-part of $[X, Y]$ and U is a symmetric bilinear mapping of $\mathfrak{m}^C\times\mathfrak{m}^C$ to $\mathfrak{m}^C$ defined by

$$g(U(X, Y), Z)=g([Z, X]_{\mathfrak{m}^C}, Y)+g(X, [Z, Y]_{\mathfrak{m}^C}),\ X, Y, Z\in\mathfrak{m}^C.$$

The curvature tensor R at the origin is described in the following way

$$R(X, Y)Z=[\Lambda(X), \Lambda(Y)]Z-\Lambda([X, Y]_{\mathfrak{m}^C})Z-[[X, Y]_{\mathfrak{k}^C}, Z],\ X, Y, Z\in\mathfrak{m}^C,$$

where $\mathfrak{k}^C$ is the complexification of $\mathfrak{k}$ and $[X, Y]_{\mathfrak{k}^C}$ denotes the $\mathfrak{k}^C$-part of $[X, Y]$.

The metric g, the connection function Λ and the curvature tensor R are considered as ones extended over complex numbers.

The holomorphic bisectional curvature $H_{\sigma,\tau}$ for holomorphic plane sections $\sigma=X\wedge IX$ and $\tau=Y\wedge IY$ at the origin is given as

$$H_{\sigma,\tau}=\frac{g(R(Z,\bar{Z})W,\bar{W})}{g(Z,\bar{Z})g(W,\bar{W})},\quad Z=\frac{1}{\sqrt{2}}(X-\sqrt{-1}IX),\quad W=\frac{1}{\sqrt{2}}(Y-\sqrt{-1}IY).$$

In order to verify Proposition it is sufficient to study the holomorphic bisectional curvature for a Kähler C-space of Betti number $b_2=1$ from the consideration stated later on (§ 2).

Henceforth, a Kähler C-space is assumed to be associated with a pair $(\mathfrak{g}, \alpha_i)$, where $\mathfrak{g}$ is a complex semisimple Lie algebra and α_i is a fundamental root.

Set

$$\Delta^+(\alpha_i)=\Delta_1^+(\alpha_i)\cup\Delta_2^+(\alpha_i)\cup\Delta_3^+(\alpha_i)\cup\cdots;$$

$$\Delta_k^+(\alpha_i)=\{\alpha=\sum_j m_j\alpha_j\in\Delta^+;\ m_i=k\},\quad k=1,2,\cdots$$

and

$$\mathfrak{m}^+=\mathfrak{m}^{+1}+\mathfrak{m}^{+2}+\mathfrak{m}^{+3}+\cdots;\quad \mathfrak{m}^{+k}=\sum_{\alpha\in\Delta_k^+(\alpha_i)} CE_\alpha,\ k=1,2,\cdots.$$

Then the invariant Kähler metric g and the connection function Λ satisfy

$$g=c\sum_k k\sum_{\alpha\in\Delta^+_k(\alpha_i)}\omega^\alpha\cdot\omega^{\bar{\alpha}},\quad c>0$$

and

$$\Lambda(X)Y=\frac{k}{j+k}[X,Y]_{\mathfrak{m}^+},\quad X\in\mathfrak{m}^{+j},\quad Y\in\mathfrak{m}^{+k},$$

$$\Lambda(\bar{X})Y=[\bar{X},Y]_{\mathfrak{m}^+},\quad X,Y\in\mathfrak{m}^+,\ \text{(Itoh [4])}.$$

2. Proof of Proposition.

Let (M, g) be a Kähler C-space which satisfies $H_{\sigma,\tau}\geqq 0$ for any holomorphic plane sections σ and τ. Since M is simply connected, we obtain the following de Rham decomposition; $(M, g)\cong(M_1, g_1)\times\cdots\times(M_r, g_r)$ as a Kählerian direct product. Each factor space (M_j, g_j) is an irreducible Kähler C-space. It also satisfies $H_{\sigma,\tau}\geqq 0$ for σ and τ. Since the Betti number b_2 is equal to 1 for a compact connected irreducible Kähler manifold of nonnegative holomorphic bisectional curvature (Itoh [3]), each (M_j, g_j) is a Kähler C-space of $b_2=1$. Therefore, it is associated with $(\mathfrak{g}, \alpha_i)$, where $\mathfrak{g}$ is complex semisimple and α_i is a fundamental root. $\mathfrak{g}$ is decomposed into the sum of simple ideals; $\mathfrak{g}=\mathfrak{g}_1+\cdots+\mathfrak{g}_s$. Since α_i is a fundamental root for some $\mathfrak{g}_k$, the parabolic subalgebra $\mathfrak{l}$ associated with α_i is written as $\mathfrak{l}=\mathfrak{g}_1+\mathfrak{g}_2+\cdots+\mathfrak{g}_{k-1}+\mathfrak{l}_k+\mathfrak{g}_{k+1}+\cdots+\mathfrak{g}_s$, where $\mathfrak{l}_k$ is a parabolic subalgebra associated with α_i in $\mathfrak{g}_k$. Hence, we may assume that each factor (M_j, g_j) is a Kähler C-space associated with a pair $(\mathfrak{g}, \alpha_i)$, where $\mathfrak{g}$ is complex simple and α_i is a fundamental root.

From this fact, it suffices for proving Proposition to show the following assertion.

ASSERTION *Let (M, g) be a Kähler C-space associated with a pair $(\mathfrak{g}, \alpha_i)$, where $\mathfrak{g}$ is complex simple and α_i is a fundamental root. If (M, g) satisfies $H_{\sigma,\tau}\geqq 0$ for any σ and τ, then (M, g) is a Hermitian symmetric space of compact type.*

Complex simple Lie algebras are fully classified. Algebras of type A, B, C, D, E, F and G exhaust thoroughly complex simple algebras. For all notations and basic concepts with respect to root systems of type A, B, C, D, E, F and G that will be used without comment, we refer to Bourbaki [1].

A pair $(\mathfrak{g}, \alpha_i)$ is called ***regular*** if it is one of the following;

$$(A_l, \alpha_i)_{1\leq i\leq l},\ (B_l, \alpha_i)_{i=1,l},\ (C_l, \alpha_i)_{i=1,l},\ (D_l, \alpha_i)_{i=1,l-1,l},$$
$$(E_6, \alpha_i)_{i=1,6},\ (E_7, \alpha_7) \text{ and } (G_2, \alpha_1).$$

A Kähler C-space of $b_2=1$ is Hermitian symmetric if and only if it is associated with a regular pair (Itoh [4]). Symmetric spaces of compact type have nonnegative sectional curvature ([2]). Therefore, in order to establish Assertion, it suffices to show that any Kähler C-space associated with a pair which is not regular has strictly negative holomorphic bisectional curvature.

Before proving this statement, we show the following lemma.

LEMMA *Let (M, g) be a Kähler C-space associated with a pair $(\mathfrak{g}, \alpha_i)$. If there exist roots α, $\beta\in\Delta_1^+(\alpha_i)$ such that $\alpha+\beta\in\Delta$ and $\alpha-\beta\notin\Delta$, then $H_{\sigma,\tau}<0$ for $\sigma=X\wedge IX$, $\tau=Y\wedge IY$* where $X=E_\alpha+E_{-\alpha}$, $Y=E_\beta+E_{-\beta}$.

PROOF of LEMMA Since $g(E_\alpha, E_{-\alpha})=g(E_\beta, E_{-\beta})=c$, $H_{\sigma,\tau}$ is written as

$$H_{\sigma,\tau}=\frac{1}{c^2}\{g([\Lambda(E_\alpha),\ \Lambda(E_{-\alpha})]E_\beta,E_{-\beta})-g(\Lambda([E_\alpha,E_{-\alpha}]_{\mathfrak{m}^C})E_\beta,\ E_{-\beta})$$
$$-g([[E_\alpha,E_{-\alpha}]_{\mathfrak{k}^C},\ E_\beta],\ E_{-\beta})\}.$$

As $[E_\alpha, E_{-\alpha}]\in\mathfrak{h}$ and $\Lambda(E_{-\alpha})E_\beta=[E_{-\alpha}, E_\beta]_{\mathfrak{m}^+}=0$, $H_{\sigma,\tau}$ is equal to $\frac{1}{c^2}\{-g(\Lambda(E_{-\alpha})\Lambda(E_\alpha)E_\beta, E_{-\beta})-g([[E_\alpha, E_{-\alpha}], E_\beta], E_{-\beta})\}$. Using the formulae in the last of §1, we have

$$H_{\sigma,\tau}=\frac{1}{c^2}\left\{\frac{1}{2}cB([E_{-\alpha},\ [E_\alpha,E_\beta]],\ E_{-\beta})+cB([[E_\alpha,E_{-\alpha}],\ E_\beta],\ E_{-\beta})\right\}.$$

Since the adjoint representation is skew-symmetric with respect to the Killing form B, we have

$$H_{\sigma,\tau}=-\frac{1}{2c}||[E_\alpha, E_\beta]||_B^2<0,$$

where $||Z||_B^2$ denotes $-B(Z,\bar{Z})$ for $Z\in\mathfrak{g}$

To verify Assertion we show the existence of roots α and β which satisfy the assumption of Lemma for each pair $(\mathfrak{g}, \alpha_i)$ that is not regular in the following list;

Type B: $(B_l, \alpha_i)_{1<i<l}$,

$\alpha=\alpha_i$,

$\beta=\alpha_1+\cdots+\alpha_i+2\alpha_{i+1}+\cdots+2\alpha_l$,

$\alpha+\beta=\alpha_1+\cdots+\alpha_{i-1}+2\alpha_i+\cdots+2\alpha_l\in\Delta$,

$\alpha-\beta=-(\alpha_1+\cdots+\alpha_{i-1}+2\alpha_{i+1}+\cdots+2\alpha_l)\notin\Delta$.

Type C: $(C_l, \alpha_i)_{1<i<l}$,

$\alpha=\alpha_i$,

$\beta=\alpha_1+\alpha_2+\cdots+\alpha_i+2\alpha_{i+1}+\cdots+2\alpha_{l-1}+\alpha_l$, $i<l-1$,

$=\alpha_1+\alpha_2+\cdots+\alpha_l$, $i=l-1$,

$\alpha+\beta=\alpha_1+\cdots+\alpha_{i-1}+2\alpha_i+\cdots+2\alpha_{l-1}+\alpha_l\in\Delta$, $i<l-1$,

$=\alpha_1+\cdots+\alpha_{l-2}+2\alpha_{l-1}+\alpha_l\in\Delta$, $i=l-1$,

$\alpha-\beta=-(\alpha_1+\cdots+\alpha_{i-1}+2\alpha_{i+1}+\cdots+2\alpha_{l-1}+\alpha_l)\notin\Delta$, $i<l-1$,

$=-(\alpha_1+\cdots+\alpha_{l-1}+\alpha_l)\notin\Delta$, $i=l-1$.

Type D: $(D_l, \alpha_i)_{1<i<l-1}$,

$\alpha=\alpha_i$,

$\beta=\alpha_1+\cdots+\alpha_i+2\alpha_{i+1}+\cdots+2\alpha_{l-2}+\alpha_{l-1}+\alpha_l$, $i<l-2$,

$=\alpha_1+\cdots+\alpha_l$, $i=l-2$,

$\alpha+\beta=\alpha_1+\cdots+\alpha_{i-1}+2\alpha_i+\cdots+2\alpha_{l-2}+\alpha_{l-1}+\alpha_l\in\Delta$, $i<l-2$,

$=\alpha_1+\cdots+\alpha_{l-3}+2\alpha_{l-2}+\alpha_{l-1}+\alpha_l\in\Delta$, $i=l-2$,

$\alpha-\beta=-(\alpha_1+\cdots+\alpha_{i-1}+2\alpha_{i+1}+\cdots+2\alpha_{l-2}+\alpha_{l-1}+\alpha_l)\notin\Delta$, $i<l-2$,

$=-(\alpha_1+\cdots+\alpha_{l-3}+\alpha_{l-1}+\alpha_l)\notin\Delta$, $i=l-2$.

Type E: (E_6, α_2), (E_7, α_2), (E_8, α_2),

$\alpha=\alpha_2$,

$\beta=\alpha_1+\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+\alpha_6$,

$\alpha+\beta=\alpha_1+2\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+\alpha_6\in\Delta$,

$\alpha-\beta=-(\alpha_1+2\alpha_3+3\alpha_4+2\alpha_5+\alpha_6)\notin\Delta$.

(E_6, α_3), (E_7, α_3), (E_8, α_3),

$\alpha=\alpha_3$,

$\beta=\alpha_1+\alpha_2+\alpha_3+2\alpha_4+2\alpha_5+\alpha_6$,

$\alpha+\beta=\alpha_1+\alpha_2+2\alpha_3+2\alpha_4+2\alpha_5+\alpha_6\in\Delta$,

$\alpha-\beta=-(\alpha_1+\alpha_2+2\alpha_4+2\alpha_5+\alpha_6)\notin\Delta.$

(E_6, α_4), (E_7, α_4), (E_8, α_4),

$\alpha=\alpha_4,$

$\beta=\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6,$

$\alpha+\beta=\alpha_1+\alpha_2+\alpha_3+2\alpha_4+\alpha_5+\alpha_6\in\Delta,$

$\alpha-\beta=-(\alpha_1+\alpha_2+\alpha_3+\alpha_5+\alpha_6)\notin\Delta.$

(E_6, α_5), (E_7, α_5), (E_8, α_5),

$\alpha=\alpha_5,$

$\beta=\alpha_1+\alpha_2+2\alpha_3+2\alpha_4+\alpha_5+\alpha_6,$

$\alpha+\beta=\alpha_1+\alpha_2+2\alpha_3+2\alpha_4+2\alpha_5+\alpha_6\in\Delta,$

$\alpha-\beta=-(\alpha_1+\alpha_2+2\alpha_3+2\alpha_4+\alpha_6)\notin\Delta.$

(E_7, α_1),

$\alpha=\alpha_1,$

$\beta=\alpha_1+2\alpha_2+3\alpha_3+4\alpha_4+3\alpha_5+2\alpha_6+\alpha_7,$

$\alpha+\beta=2\alpha_1+2\alpha_2+3\alpha_3+4\alpha_4+3\alpha_5+2\alpha_6+\alpha_7\in\Delta,$

$\alpha-\beta=-(2\alpha_2+3\alpha_3+4\alpha_4+3\alpha_5+2\alpha_6+\alpha_7)\notin\Delta.$

(E_7, α_6), (E_8, α_6),

$\alpha=\alpha_6,$

$\beta=\alpha_1+\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+\alpha_6+\alpha_7,$

$\alpha+\beta=\alpha_1+\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+2\alpha_6+\alpha_7\in\Delta,$

$\alpha-\beta=-(\alpha_1+\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+\alpha_7)\notin\Delta.$

(E_8, α_1),

$\alpha=\alpha_1,$

$\beta=\alpha_1+2\alpha_2+3\alpha_3+4\alpha_4+3\alpha_5+2\alpha_6+\alpha_7+\alpha_8,$

$\alpha+\beta=2\alpha_1+2\alpha_2+3\alpha_3+4\alpha_4+3\alpha_5+2\alpha_6+\alpha_7+\alpha_8\in\Delta,$

$\alpha-\beta=-(2\alpha_2+3\alpha_3+4\alpha_4+3\alpha_5+2\alpha_6+\alpha_7+\alpha_8)\notin\Delta.$

(E_8, α_7),

$\alpha=\alpha_7,$

$\beta=\alpha_1+2\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+2\alpha_6+\alpha_7+\alpha_8,$

$\alpha+\beta=\alpha_1+2\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+2\alpha_6+2\alpha_7+\alpha_8\in\Delta,$

$\alpha-\beta=-(\alpha_1+2\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+2\alpha_6+\alpha_8)\notin\Delta.$

(E_8, α_8),

$\alpha=\alpha_8,$

$\beta=2\alpha_1+3\alpha_2+4\alpha_3+6\alpha_4+5\alpha_5+4\alpha_6+3\alpha_7+\alpha_8,$

$\alpha+\beta=2\alpha_1+3\alpha_2+4\alpha_3+6\alpha_4+5\alpha_5+4\alpha_6+3\alpha_7+2\alpha_8\in\Delta,$

$\alpha-\beta=-(2\alpha_1+3\alpha_2+4\alpha_3+6\alpha_4+5\alpha_5+4\alpha_6+3\alpha_7)\notin\Delta.$

Type F: (F_4, α_1),

$\alpha=\alpha_1,$

$\beta=\alpha_1+3\alpha_2+4\alpha_3+2\alpha_4,$

$\alpha+\beta=2\alpha_1+3\alpha_2+4\alpha_3+2\alpha_4\in\Delta,$

$\alpha-\beta=-(3\alpha_2+4\alpha_3+2\alpha_4)\notin\Delta.$

$(F_4,\ \alpha_2),$

$\alpha=\alpha_2,$

$\beta=\alpha_1+\alpha_2+2\alpha_3+2\alpha_4,$

$\alpha+\beta=\alpha_1+2\alpha_2+2\alpha_3+2\alpha_4\in\Delta,$

$\alpha-\beta=-(\alpha_1+2\alpha_3+2\alpha_4)\notin\Delta.$

$(F_4,\ \alpha_3),$

$\alpha=\alpha_3,$

$\beta=\alpha_2+\alpha_3+\alpha_4,$

$\alpha+\beta=\alpha_2+2\alpha_3+\alpha_4\in\Delta,$

$\alpha-\beta=-(\alpha_2+\alpha_4)\notin\Delta.$

$(F_4,\ \alpha_4),$

$\alpha=\alpha_4,$

$\beta=\alpha_1+2\alpha_2+3\alpha_3+\alpha_4,$

$\alpha+\beta=\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4\in\Delta,$

$\alpha-\beta=-(\alpha_1+2\alpha_2+3\alpha_3)\notin\Delta.$

Type G: $(G_2,\ \alpha_2),$

$\alpha=\alpha_2,$

$\beta=3\alpha_1+\alpha_2,$

$\alpha+\beta=3\alpha_1+2\alpha_2\in\Delta,$

$\alpha-\beta=-3\alpha_1\notin\Delta.$

Thus, Proposition is verified.

Remark Observing the process of verification for Proposition, we can replace the assumption of nonnegative curvature in Theorem by assumption that the Kähler manifold is of nonnegative holomorphic bisectional curvature.

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Institute of Mathematics
The University of Tsukuba
Ibaraki, 300-31, Japan

Added in proof. Recently A. Gray [Compact Kähler manifolds with nonnegative sectional curvature, Invent. Math., 41 (1977) 33-43] has proved the following remarkable result; *Let (M, g) be a compact Kähler manifold of nonnegative sectional curvature. If it has constant scalar curvature, then it is locally symmetric.* Since a compact homogeneous Kähler manifold has constant scalar curvature, we can use this result to shorten the proof of our theorem.