TSUKUBA J. MATH.
No. 8 No. 2 (1984), 227–253

HIGHER R-DERIVATIONS OF SPECIAL SUBRINGS OF MATRIX RINGS

By

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1. Introduction.

Let R be a ring with identity and P be a special subring of $M_n(R)$ ([7]), i.e. P is of the form

$$P=\{A\in M_n(R);\, A_{ij}=0 \text{ for } (i, j)\notin\rho\}\ ,$$

where ρ is a (reflexive and transitive) relation on the set $\{1, 2, \cdots, n\}$, and $M_n(R)$ is the ring of $n\times n$ matrices over R.

In this paper we study the group $D_s^R(P)$ of all R-derivations of order s ([5], [8]—[11]) of P. We prove (Theorem 5.3) that every element $d\in D_s^R(P)$ has a unique representation of the form $d=d^{(1)}*d^{(2)}$, where $d^{(1)}$ is an inner derivation in $D_s^R(P)$ ([8]), and $d^{(2)}$ is an element of a certain abelian subgroup of $D_s^R(P)$ whose simple description is given in Section 3 (by $*$ we denote the multiplication in the group $D_s^R(P)$). This theorem plays a basic role in our further considerations.

Moreover, in Section 4, we give some necessary and suffiecient conditions for a ring P to have all R-derivations (all derivations) of order s of P to be inner.

In Sections 7, 8, 9 we investigate s'-integrable R-derivations of order s (where $s<s'$) i.e. such R-derivations of order s which can be extended to R-derivations of order s' (comp. [4]). We show in Example 7.4 that, in general, there are non-integrable R-derivations of P. We prove (Theorem 9.6) that if the homology group $H_1(\Gamma)$ of the simplicial complex Γ of the relation ρ (Section 2) is free abelian, then every usual R-derivation is 3-integrable, and if, in addition, $H_2(\Gamma)=0$ then every R-derivation of order s is s'-integrable for any $s<s'$ (Theorem 8.6).

At the end of this paper, we formulate three open problems.

2. Preliminaries.

Throughout this paper R is a ring with identity, n is a fixed natural number and ρ is a reflexive and transitive relation on the set $I_n=\{1, 2, \cdots, n\}$.

Received October 25, 1983.

We denote by $M_n(R)$ the ring of $n\times n$ matrices over R and by $Z(R)$ the center of R.

Moreover, we use the following conventions:

S=a segment of $N=\{0, 1, \cdots\}$, that is, $S=N$ or $S=\{0, 1, \cdots, k\}$ for some integer $k\geqq 0$

$s=\sup(S)\leqq\infty$,

$A_{ij}=ij$-coefficient of a matrix A,

E^{ij}=the element of the standard basis of $M_n(R)$,

$\bar{r}$=the diagonal matrix whose all coefficients on the diagonal are equal to $r\in R$,

$M_n(R)_\rho$=the set $\{A\in M_n(R)\,;\, A_{ij}=0 \text{ for } (i, j)\notin\rho\}$.

It is clear, that $M_n(R)_\rho$ is a subring of $M_n(R)$. (Conversely, if σ is a reflexive relation on I_n and $M_n(R)_\sigma$ is a subring of $M_n(R)$, then σ is transitive). We say that the subring $P=M_n(R)_\rho$ of $M_n(R)$ is *special with the relation* ρ.

Let P be an arbitrary ring with identity. A sequence $d=(d_m)_{m\in S}$ of mappings $d_m: P\to P$ is called a *derivation of order* s *of* P (see [5], [8], [9], [10], [11]) if the sollowing properties are satisfied:

(1) $d_m(a+b)=d_m(a)+d_m(b)$,

(2) $d_m(ab)=\sum_{i+j=m} d_i(a)d_j(b)$,

(3) $d_0(a)=a$,

for all $m\in S$ and $a, b\in P$.

The set $D_s(P)$ of all derivations of order s of P is a group under the multiplication $*$ defined by the formula

$$(d*d')_m=\sum_{i+j=m} d_i\circ d'_j,$$

wehre $d, d'\in D_s(P)$ and $m\in S$ ([9], [10], [4]).

If $a\in P$ and $k\in S\setminus\{0\}$ then by $[a, k]$ we denote the element of $D_s(P)$ defined by

$$[a, k]_m(x)=\begin{cases} x, & \text{if } m=0, \\ 0, & \text{if } k\nmid m, \\ a^r x-a^{r-1}xa, & \text{if } m=kr>0, \end{cases}$$

for $m\in S$, $x\in P$ ([8]).

If $\underline{a}=(a_m)_{m\in S\setminus\{0\}}$ is a sequence of elements of P then by $\Delta(\underline{a})$ we denote the *inner derivation of order* s *of* P with respect to $\underline{a}$ ([8]), i.e., $\Delta(\underline{a})$ is a derivation of order s of P such that

$$\Delta(\underline{a})_m=([a_1, 1]*\cdots*[a_m, m])_m$$

for all $m\in S$. The set of inner derivations of order s of P, denoted by $ID_s(P)$, is a normal subgroup of $D_s(P)$ ([8] Corollary 3.3).

Recall that the *usual derivation* of P is an additive mapping $\delta: P\to P$ such that $\delta(ab)=\delta(a)b+a\delta(b)$, for all a, $b\in P$.

The set of usual derivations of P corresponds bijectively to the set $D_1(P)$, namely if $d\in D_s(P)$ then d_1 is an usual derivation of P.

We now assume that P is a special subring of $M_n(R)$ with the relation ρ.

Observe that we can extend every derivation of order s of R to a derivation of order s of P.

Indeed, if $\delta\in D_s(R)$ then the sequence $d=(d_m)_{m\in S}$ of mappings $d_m: P\to P$ defined by $d_m(A)_{ij}=\delta_m(A_{ij})$ (for $A\in P$, $m\in S$) is a derivation of order s of P such that $d_m(\bar{r})=\overline{\delta_m(r)}$ for any $r\in R$, $m\in S$.

Look also on a generalization of the above fact.

EXAMPLE 2.1. Let $\bar{\rho}$ be the smallest equivalence relation on I_n containing ρ, T a fixed set of representatives of equivalence classes of $\bar{\rho}$, and $v: I_n\to T$ the mapping defined by:

$$v(p)=t \quad \text{iff} \quad p\bar{\rho}t\,.$$

Moreover, let $\underline{d}=(d^{(t)})_{t\in T}$ be a sequence of elements of $D_s(R)$. Consider the sequence $\Theta(\underline{d})=(d_m)_{m\in S}$ of mappings from P to P defined as follows

$$d_m(A)_{ij}=d_m^{(v(i))}(A_{ij})$$

for all $m\in S$, $A\in P$.

It is easy to verify that $\Theta(\underline{d})$ belongs to $D_s(P)$.

If a derivation $d\in D_s(P)$ satisfies following equivalent two conditions:

(4) $d_m(\bar{r}A)=\bar{r}d_m(A)$ for all $m\in S$, $r\in R$, $A\in P$,

(5) $d_m(\bar{r})=0$ for all $m\in S\setminus\{0\}$, $r\in R$,

then d is called *R-derivation of order s of P*, and the set of all such derivations is denoted by $D_s^R(P)$.

We define similarly an *usual R-derivation*, an *inner R-derivation* and the set $ID_s^S(P)$. It is clear, that $D_s^R(P)$ is a subgroup of $D_s(P)$, and (by [8] Corollary 3.3) $ID_s^R(P)$ is a normal subgroup of $D_s^R(P)$. An inner derivation $\Delta(\underline{A})$, where $\underline{A}=(A^{(m)})_{m\in S\setminus\{0\}}$ is a sequence of matrices of P, belongs to $ID_s^R(P)$ iff $A^{(m)}\in M_n(Z(R))$ for any m.

LEMMA 2.2. *If* $d\in D_s^R(P)$ *then* $d_m(E^{pq})_{ij}\in Z(R)$ *for any* $m\in S$ *and all* $i, j, p, q \in I_n$ *such that* $p\rho q$.

PROOF. Let $r\in R$. Sinɔe $\bar{r}E^{pq}-E^{pq}\bar{r}=0$ then

$$\begin{aligned} 0&=d_m(\bar{r}E^{pq}-E^{pq}\bar{r})_{ij}\\ &=\sum_{u+v=m}(d_u(\bar{r})d_v(E^{pq})-d_u(E^{pq})d_v(\bar{r}))_{ij}\\ &=(\bar{r}d_m(E^{pq})-d_m(E^{pq})\bar{r})_{ij}\\ &rd_m(E^{pq})_{ij}-d_m(E^{pq})_{ij}r \end{aligned}$$

Usual derivations and usual R-derivations of P are investigated in [6], [1], [2], [7]. In this paper (Section 5) we give a description of the group $D_s^R(P)$.

Let $s<\infty$, and S' be a segment of N such that $S\subsetneqq S'$. We say (comp. [4]) that an R-derivation $d\in D_s^R(P)$ is *s'-integrable* (where $s'=\sup(S')\leqq\infty$) if there exists an R-derivation $d'\in D_{s'}^R(P)$ such that $d'_m=d_m$ for all $m\in S$. We will study such derivations in Sections 7, 8, 9.

Now we will define the graph Γ of the relation ρ. Let $\sim$ be the equivalence relation on I_n defined by:

$$x\sim y \quad \text{iff} \quad x\rho y \quad \text{and} \quad y\rho x\,.$$

Denote by $[x]$ the equivalence class of $x\in I_n$ with respect to $\sim$, and let I'_n be the set of all equivalence classes. We define a relation ρ' of partial order on I_n as follows:

$$[x]\rho'[y] \quad \text{iff} \quad x\rho y\,.$$

We will denote the pair (I'_n, ρ') by Γ (or $\Gamma(\rho)$) and calle it the *graph* of ρ. Elements of I'_n we calle *vertices* of Γ and pairs (a, b), where $a\rho' b$ and $a\neq b$, *arrows* of Γ.

Let us imbed the set of the vertices of Γ in an Euclidean space of a sufficiently high dimension so that the vertices will be linearly independent.

If $a_0, a_1, \cdots, a_k$ are elements of I'_n such that $a_i\rho' a_{i+1}$ and $a_i\neq a_{i+1}$ for $i=0, 1, \cdots, k-1$, then by $(a_0, a_1, \cdots, a_k)$ we denote the k-dimensional simplex with vertices $a_0, \cdots, a_k$ ([3]). The union of all 0, 1, 2 or 3-dimensional such simplicies we will denote also by Γ. Therefore, Γ is a simplicial complex of dimension $\leqq 3$.

Let $C_k(\Gamma)$, for $k=0, 1, 2, 3$, be the free abelian group whose free generetors are k-dimensional simplicies of the complex Γ. We have the following standard complex of abelian groups:

$$0 \longrightarrow C_3(\Gamma) \xrightarrow{\partial_3} C_2(\Gamma) \xrightarrow{\partial_2} C_1(\Gamma) \xrightarrow{\partial_1} C_0(\Gamma) \longrightarrow 0,$$

where

$$\partial_1(a, b)=(b)-(a),$$

$$\partial_2(a, b, c)=(b, c)-(a, c)+(a, b),$$

$$\partial_3(a, b, c, d)=(b, c, d)-(a, c, d)+(a, b, d)-(a, b, c).$$

Then $H_1(\Gamma)=\operatorname{Ker}\partial_1/\operatorname{Im}\partial_2$, $H_2(\Gamma)=\operatorname{Ker}\partial_2/\operatorname{Im}\partial_3$ and (by the Künneth formulas)

$$H^1(\Gamma, G)=\operatorname{Hom}(H_1(\Gamma), G)$$

for an arbitrary abelian group G (see [3]).

In the sequel P denotes a special subring of $M_n(R)$ with the relation ρ.

3. Transitive mappings.

Recall from [7] that a mapping $\varphi:\rho\to Z(R)$ is called *transitive* if $\varphi(p, r)=\varphi(p, q)+\varphi(q, r)$ for $p\rho q$, $q\rho r$. In this paper such mappings will be called *usual transitive mappings* from ρ to R.

DEFINITION 3.1. A sequence $f=(f_m)_{m\in S}$ of mappings $f_m:\rho\to Z(R)$ is called a *transitive mapping of order s from ρ to R* if the following properties are satisfied:

(a) $f_0(p, q)=1$ for all $p\rho q$,
(b) $f_m(p, r)=\sum_{i+j=m} f_i(p, q)f_j(q, r)$ for all $m\in S$ and $p\rho q\rho r$.

We denote by $TM_s(\rho, R)$ the set of transitive mappings of order s from ρ to R.

By the above definition it follows that if $f\in TM_s(\rho, R)$ then

$$f_1(p, r)-f_1(p, q)-f_1(q, r)=0,$$

i.e. f_1 is an usual transitive mapping from ρ to R, and

$$f_2(p, r)-f_2(p, q)-f_2(q, r)=f_1(p, q)f_1(q, r),$$

$$f_3(p, r)-f_3(p, q)-f_3(q, r)=f_1(p, q)f_2(q, r)+f_2(p, q)f_1(q, r)$$

for all $p\rho q\rho r$.

It is easy to prove

LEMMA 3.2. (1) *$f_m(p, p)=0$, for all $p\in I_n$, $m\in S\setminus\{0\}$.*
(2) *If $p\rho q$ and $q\rho p$, and $f_2(p, q)=\cdots=f_m(p, q)=0$ for some $m\geqq 2$, then*

$f_k(p, q)=(-1)^k f_1(p, q)^k=f_1(q, p)^k$ for $k=0, \cdots, m$.

EXAMPLE 3.3. If $Q\subseteqq R$ and $\varphi: \rho\to Z(R)$ is an usual transitive mapping then the sequence $(f_m)_{m\in S'}$ where $f_m(p, q)=(m!)^{-1}\ \varphi(p, q)^m$, is a transitive mapping of order s from ρ to R.

EXAMPLE 3.4. Let

$$\rho=\begin{tikzcd} 1 \arrow[rr] \arrow[dr] & & 2 \\ & 3 \arrow[ur] & \end{tikzcd}\ .$$

Put $f_m(1, 2)=f_m(1, 3)=1$ and $f_m(2, 3)=0$ for all $m\in S\setminus\{0\}$. Then $f=(f_m)_{m\in S}$ belongs to $TM_s(\rho, R)$.

EXAMPLE 3.5. Let

$$\rho=\begin{tikzcd} 1 \arrow[r] \arrow[d] & 3 \\ 4 & 2 \arrow[l] \arrow[u] \end{tikzcd}\ .$$

If f_m, for any $m\in S\setminus\{0\}$, is an arbitrary mapping from ρ to $Z(R)$ then $(f_m)_{m\in S}$ is a transitive mapping of order s from ρ to R.

Let $f, g\in TM_s(\rho, R)$. Denote by $f*g$ the sequence $(h_m)_{m\in S}$ of mappings from ρ to $Z(R)$ defined by

$$h_m(p, q)=\sum_{i+j=m} f_i(p, q)g_j(p, q)$$

for all $m\in S$ and $p\rho q$.

Then $f*g$ belongs to $TM_s(\rho, R)$ and it is easy to check that the set $TM_s(\rho, R)$, under the multiplication $*$, is an abelian group.

For every $f\in TM_s(\rho, R)$ we will denote by Δ^f the sequence $(\Delta_m^f)_{m\in S}$ of mappings $\Delta_m^f: P\to P$ defined by the following formula

$$\Delta_m^f(A)_{pq}=f_m(p, q)A_{pq},$$

for all $A\in P$ and $p\rho q$.

Then we have

LEMMA 3.6. *The sequence Δ^f is an R-derivation of order s of P.*

PROOF. Every Δ_m^f is obviously an R-additive mapping. Let $A, B\in P$ and

$p\rho q$. Then

$$\begin{aligned}\Big(\sum_{k=0}^{m}\Delta_k^f(A)\Delta_{m-k}^f(B)\Big)_{pq} &= \sum_{k=0}^{m}\sum_{i=1}^{n}\Delta_k^f(A)_{pi}\Delta_{m-k}^f(B)_{iq}\\ &= \sum_{k=0}^{m}\sum_{i=1}^{n} f_k(p, i)f_{m-k}(i, q)A_{pi}B_{iq}\\ &= \sum_{i=1}^{n} f_m(p, q)A_{pi}B_{iq}\\ &= f_m(p, q)(AB)_{pq}\\ &= \Delta_m^f(AB)_{pq}\,.\end{aligned}$$

Therefore

$$\Delta_m^f(AB)=\sum_{k=0}^{m}\Delta_k^f(A)\Delta_{m-k}^f(B)\,,$$

for all $m\in S$ and A, $B\in P$.

PROPOSITION 3.7. *The mapping* $f\mapsto\Delta^f$ *is a group monomorphism from* $TM_s(\rho, R)$ *to* $D_s^R(P)$.

PROOF. The condition $\Delta^{f*g}=\Delta^f*\Delta^g$ follows from definition of multiplications. Suppose now that $\Delta^f=\Delta^g$ for some $f, g\in TM_s(\rho, R)$. Then, for $p\rho q$ and $m\in S$, we have

$$f_m(p, q)=\Delta_m^f(E^{pq})_{pq}=\Delta_m^g(E^{pq})_{pq}=g_m(p, q)\,,$$

i.e. $f=g$.

4. Inner derivations.

Recall from [7] that if f is an usual transitive mapping from ρ to R then f is called *trivial* iff there exists a mapping $\sigma: I_n\to Z(R)$ such that $f(p, q)=\sigma(p)-\sigma(q)$ for all $p\rho q$. We say that the relation ρ is *regular over* R iff every usual transitive mapping from ρ to R is trivial.

Combining [8] Theorem 4.2 with results of the paper [7] we obtain the following two theorems

THEOREM 4.1. *Let* P *be a special subring of* $M_n(R)$ *with the relation* ρ. *The following conditions are equivalent*:

(1) *Every* R*-derivation of order* s *of* P *is inner,*
(2) *Every usual* R*-derivation of* P *is inner,*
(3) *The relation* ρ *is regular over* $Z(R)$,
(4) *The relation* ρ' *is regular over* $Z(R)$,

(5) $H^1(\Gamma(\rho), Z(R))=0$.

THEOREM 4.2. *Let P be a special subring of $M_n(R)$ with the relation ρ. Denote by w, w_s, u, u' the following sentences:*

w="Every usual derivation of R is inner",
w_s="Every derivation of order s of R is inner",
u="The relation ρ is regular over $Z(R)$",
u'="The relation ρ' is regular over $Z(R)$".

Then the following conditions are equivalent:

(1) *Every derivation of order s of P is inner,*
(2) *Every usual derivation of P is inner,*
(3) *w and u,*
(4) *w_s and u,*
(5) *w and u',*
(6) *w_s and u',*
(7) *w and $H^1(\Gamma(\rho), Z(R))=0$,*
(8) *w_s and $H^1(\Gamma(\rho), Z(R))=0$.*

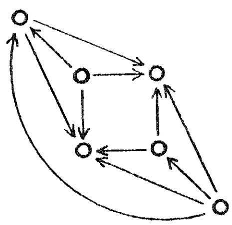
EXAMPLE 4.3. If $P=M_n(R)_\rho$ where

a) $n \leqq 3$, or

b) the graph $\Gamma(\rho)$ is a tree, or

c) the graph $\Gamma(\rho)$ is a conne (i.e. there exists $b \in I_n$ such that $b\rho a$ or $a\rho b$ for any $a \in I_n$) in particular $P=M_n(R)$ or P is the ring of triangular $n \times n$ matrices over R, or

d) the graph $\Gamma(\rho)$ is of the form



then every R-derivation (or every derivation, if every usual derivation of R is inner) of order s of P is inner (see [7]).

5. The group $D_s^R(P)$.

In this section we give a description of the group $D_s^R(P)$.

We start from the following two lemmas.

LEMMA 5.1. *Let* $d\in D_s^R(P)$, $m\in S\setminus\{0\}$. *Assume that* $d_k(E^{qq})_{pq}=0$ *for* $k=1, 2, \cdots, m$ *and all* $p\neq q$. *Then*

(i) $d_k(E^{pp})_{pp}=0$ *for* $k=1, 2, \cdots, m$ *and any* $p\in I_n$,

and

(ii) $d_k(E^{ij})_{pq}=0$ *for* $k=1, 2, \cdots, m$ *and all* $i\rho j$, $p\rho q$ *such that* $(p, q)\neq(i, j)$.

PROOF. (by induction with respect to m). If $m=1$ then this lemma follows from [7] Lemma 3.1. Let $m>1$ and suppose that the conditions (i) and (ii) hold for any $k<m$. We show that then

(1) $d_m(E^{ij})_{pq}=0$ for $i\neq p$, $j\neq q$,

(2) $d_m(E^{pp})_{pp}=0$ for any $p\in I_n$,

(3) $d_m(E^{pp})_{pj}=0$ for $p\neq j$,

(4) $d_m(E^{pq})_{iq}=0$ for $p\neq i$,

(5) $d_m(E^{pq})_{pj}=0$ for $q\neq j$.

For example we verify (1) and (2). The proofs of the conditions (3)-(5) are similar.

(1) Let $i\neq p$, $j\neq q$, and $p\rho q$, $i\rho j$. Then

$$\begin{aligned} d_m(E^{ij})_{pq}&=d_m(E^{ij}E^{jj})_{pq}\\ &=\sum_{k+l=m}(d_k(E^{ij})d_l(E^{jj}))_{pq}\\ &=\sum_{k+l=m}\sum_r d_k(E^{ij})_{pr}d_l(E^{jj})_{rq}. \end{aligned}$$

Hence, by induction, we have

$$\begin{aligned} d_m(E^{ij})_{pq}&=\sum_r(d_0(E^{ij})_{pr}d_m(E^{jj})_{rq}+d_m(E^{ij})_{pr}d_0(E^{jj})_{rq})\\ &=\sum_r(0d_m(E^{ij})_{rq}+d_m(E^{ij})_{pr}0)=0. \end{aligned}$$

(2) Let $p\in I_n$. Then

$$\begin{aligned} d_m(E^{pp})_{pp}&=d(E^{pp}E^{pp})_{pp}\\ &=\sum_{i+j=m}(d_i(E^{pp})d_j(E^{pp}))_{pp}\\ &=\sum_{i+j=m}\sum_r d_i(E^{pp})_{pr}d_j(E^{pp})_{rp}\\ &=\sum_r(d_0(E^{pp})_{pr}d_m(E^{pp})_{rp}+d_m(E^{pp})_{pr}d_0(E^{pp})_{rp})\\ &=d_m(E^{pp})_{pp}+d_m(E^{pp})_{pp}. \end{aligned}$$

Hence $d_m(E^{pp})_{pp}=0$.

LEMMA 5.2. *Let* $d\in D_s^R(P)$. *Assume that* $d_m(E^{qq})_{pq}=0$ *for all* $m\in S\setminus\{0\}$ *and all* $p\rho q$. *Then the sequence* $f=(f_m)_{m\in S}$ *of mappings from* ρ *to* R *defined by* $f_m(p, q)=d_m(E^{pq})_{pq}$ *for* $p\rho q$ *is a transitive mapping of order* s *from* ρ *to* R.

PROOF. Lemma 2.2 implies that $f_m(p, q)\in Z(R)$ for all $p\rho q$. Now let $p\rho q\rho r$, $m\in S$. By Lemma 5.1 we have

$$\begin{aligned} f_m(p, r)&=d_m(E^{pr})_{pr}=d_m(E^{pq}E^{qr})_{pr}\\ &=(\sum_{i+j=m} d_i(E^{pq})d_j(E^{qr}))_{pr}\\ &=\sum_t\sum_{i+j=m} d_i(E^{pq})_{pt}d_j(E^{qr})_{tr}\\ &=\sum_{i+j=m} d_i(E^{pq})_{pq}d_j(E^{qr})_{qr}\\ &=\sum_{i+j=m} f_i(p, q)f_j(q, r)\,, \end{aligned}$$

i.e. $f\in TM_s(\rho, R)$.

Now we can prove the following

THEOREM 5.3. *Let* P *be a special subring of* $M_n(R)$ *with the relation* ρ. *Every* R*-derivation* d *of order* s *of* P *has a unique representation*:

(0) $d=\Delta(\underline{A})*\Delta^f$,

where

(1) $\underline{A}=(A^{(m)})_{m\in S\setminus\{0\}}$ *is a sequence of matrices* $A^{(m)}\in P\cap M_n(Z(R))$ *such that* $A_{ii}^{(m)}=0$ *for* $i=1, 2, \cdots, n$,

(2) f *is a transitive mapping of order* s *from* ρ *to* R.

PROOF. (I). Let $d\in D_s^R(P)$. We define matrices $A^{(1)}, A^{(2)}, \cdots$ inductively as follows:

$$A_{pq}^{(1)}=d_1(E^{qq})_{pq}\,,$$

and

$$A_{pq}^{(m+1)}=d_{m+1}^{(m)}(E^{qq})_{pq} \qquad \text{for} \quad 1\leqq m<s\,,$$

where

$$d^{(m)}=([A^{(1)}, 1]*\cdots*[A^{(m)}, m])^{-1}*d\,.$$

Put $\delta=(\delta_m)_{m\in S}$, where $\delta_0=id_P$ and $\delta_m=d_m^{(m)}$ for $m\geqq 1$. Let $\underline{A}=(A^{(m)})_{m\in S\setminus\{0\}}$ and let $f=(f_m)_{m\in S}$ be the sequence of mappings from ρ to R defined by

$$f_m(p, q)=\delta_m(E^{pq})_{pq}$$

for all $m\in S$, $p\rho q$.

We show that $\underline{A}$ and f satisfy conditions (0), (1) and (2) of this theorem. Observe first that

a) $d_k^{(m)}=d_k^{(k)}$ for any $k\leqq m$,

b) δ is an R-derivation of order s of P,

c) $d=\Delta(\underline{A})*\delta$.

Now we prove that

d) $\delta_m(E^{qq})_{pq}=0$ for $m\in S\setminus\{0\}$ and $p\neq q$.

In fact, for $m=1$ we have

$$\begin{aligned}\delta_1(E^{qq})_{pq}&=d_1^{(1)}(E^{qq})_{pq}\\&=([A^{(1)},1]^{-1}*d)_1(E^{qq})_{pq}\\&=-[A^{(1)},1]_1(E^{qq})_{pq}+d_1(E^{qq})_{pq}\\&=-(A^{(1)}E^{qq}-E^{qq}A^{(1)})_{pq}+A_{pq}^{(1)}\\&=-A_{pq}^{(1)}+A_{pq}^{(1)}=0\end{aligned}$$

and, if $m>1$ then

$$\begin{aligned}\delta_m(E^{qq})_{pq}&=d_m^{(m)}(E^{qq})_{pq}\\&=([A^{(m)},m]^{-1}*d^{(m-1)})_m(E^{qq})_{pq}\\&=(\sum_{i+j=m}[A^{(m)},m]_i^{-1}\circ d_j^{(m-1)})(E^{qq})_{pq}\\&=[A^{(m)},m]_m^{-1}(E^{qq})_{pq}+\left(\sum_{i=1}^{m-1}Od_i^{(m-1)}\right)(E^{qq})_{pq}+d_m^{(m-1)}(E^{qq})_{pq}\\&=-(A^{(m)}E^{qq}-E^{qq}A^{(m)})_{pq}+A_{pq}^{(m)}\\&=-A_{pq}^{(m)}+A_{pq}^{(m)}=0.\end{aligned}$$

Using b), d), a) and Lemma 5.1 we have

e) $A_{pp}^{(m)}=d_m^{(m-1)}=d_m^{(m)}(E^{pp})_{pp}=0$ for $m\geqq 2$.

Moreover, $A_{pp}^{(1)}=0$, since

$$A_{pp}^{(1)}=d_1(E^{pp})_{pp}=d_1(E^{pp}E^{pp})_{pp}=A_{pp}^{(1)}+A_{pp}^{(1)}.$$

Observe also that

f) $A^{(m)}\in M_n(Z(R))\cap P$ (by Lemma 2.2),

and

g) f is a transitive mapping of order s from ρ to R (by b), d) and Lemma 5.2).

It remains to show that

h) $\delta=\Delta^f$.

If $X\in P$, $m\in S$ and $p\rho q$ then

$$\delta_m(X)_{pq}=\delta_m(\sum_{i,j}\overline{X}_{ij}E^{ij})_{pq}$$

$$=(\sum_{i,j}\bar{X}_{ij}\delta_m(E^{ij}))_{pq}$$

$$=\sum_{i,j}X_{ij}\delta_m(E^{ij})_{pq}$$

$$=X_{pq}\delta_m(E^{pq})_{pq} \qquad \text{(by d) and Lemma 5.1)}$$

$$=X_{pq}f_m(p, q)$$

$$=\Delta_m^f(X)_{pq}, \qquad \text{i.e,} \quad \delta=\Delta^f.$$

(II). Suppose that

$$\Delta(\underline{A})*\Delta^f=\Delta(\underline{B})*\Delta^g,$$

where $\underline{A}$, f and $\underline{B}$, g satisfy conditions (1) and (2).

Then, for $p\neq q$,

$$A_{pq}^{(1)}=(\Delta(\underline{A})*\Delta^f)_1(E^{qq})_{pq}=(\Delta(\underline{B})*\Delta^g)_1(E^{qq})_{pq}=B_{pq}^{(1)}.$$

So $A^{(1)}=B^{(1)}$.

Suppose that $A^{(1)}=B^{(1)}, \cdots, A^{(m)}=B^{(m)}$ for some $m<s$. Then

$$\begin{aligned}\Delta(0, \cdots 0, A^{(m+1)}, A^{(m+2)}, \cdots)*\Delta^f&=([A^{(1)}, 1]*\cdots*[A^{(m)}, m])^{-1}*\Delta(\underline{A})*\Delta^f\\ &=([B^{(1)}, 1]*\cdots*[B^{(m)}, m])^{-1}*\Delta(\underline{B})*\Delta^g\\ &=\Delta(0, \cdots, 0, B^{(m+1)}, B^{(m+2)}, \cdots)*\Delta^g,\end{aligned}$$

hence

$$\begin{aligned}A_{pq}^{(m+1)}&=(\Delta(0, \cdots, 0, A^{(m+1)}, A^{(m+2)}, \cdots)*\Delta^f)_{m+1}(E^{qq})_{pq}\\ &=(\Delta(0, \cdots, 0, B^{(m+1)}, B^{(m+2)}, \cdots)*\Delta^g)_{m+1}(E^{qq})_{pq}\\ &=B_{pq}^{(m+1)} \qquad \text{for} \quad p\neq q,\end{aligned}$$

and hence

$$A^{(m+1)}=B^{(m+1)}.$$

Therefore, by induction, $\underline{A}=\underline{B}$.

Further we have

$$\begin{aligned}\Delta^f&=\Delta(\underline{A})^{-1}*(\Delta(\underline{A})*\Delta^f)\\ &=\Delta(\underline{B})^{-1}*(\Delta(\underline{B})*\Delta^g)=\Delta^g\end{aligned}$$

hence, by Proposition 3.7, we obtain that $f=g$. This completes the proof.

6. Corollaries to Theorem 5.3.

Let S' be a segment of N such that $S\subset S'$ and let $s'=\sup(S')\leqq\infty$. We say that a transitive mapping $f\in TM_s(\rho, R)$ is *s'-integrable* if there exists a transitive mapping $f'\in TM_{s'}(\rho, R)$ such that $f'_m=f_m$ for all $m\in S$.

As an immediate consequence of Theorem 5.3 we have

COROLLARY 6.1. *The following conditions are equivalent:*

(1) *Every R-derivation of order s of P is s'-integrable,*
(2) *Every transitive mapping of order s from ρ to R is s'-integrable.*

If U is an ideal in P, then $U=[U_{ij}]$, where U_{ij} are ideals of R for any i, j (see [7] Lemma 2.1). Therefore from Theorem 5.3 we get

COROLLARY 6.2. *If $d\in D_s^R(P)$ and U is an ideal in P then $d_m(U)\subseteqq U$ for all $m\in S$.*

Observe also that from Theorem 5.3 follows

COROLLARY 6.3. *If $d\in D_s^R(P)$ and C is the center of P, then $d_m(C)=0$ for all $m\in S\setminus\{0\}$.*

Denote by $I(P)$ the set of all matrices $A\in P$ such that $A_{pp}=0$ for all $p\in I_n$. It is easy to verify the following two lemmas.

LEMMA 6.4. *The folowing conditions are equivalent:*

(1) *$I(P)$ is an ideal in P,*
(2) *$I(P)$ is a left-ideal in P,*
(3) *$I(P)$ is a right-ideal in P,*
(4) *$AB\in I(P)$ for all $A, B\in I(P)$,*
(5) *$AB-BA\in I(P)$ for all $A, B\in I(P)$,*
(6) *$AB-BA\in I(P)$ for all $A\in I(P)$, $B\in P$,*
(7) *The relation ρ is partial order.*

LEMMA 6.5 *The following two conditions are equivalent:*

(1) *$AB=0$ for all $A, B\in I(P)$,*
(2) *There do not exist three different elements $a, b, c\in I_n$ such that $a\rho b\rho c$.*

Combining Lemma 6.4 with Theorem 5.3 and Lemma 3.2(1) we obtain

COROLLARY 6.6. *Let $d\in D_s^R(P)$. If the relation ρ is a partial order then $d_m(P)\subseteqq I(P)$ for all $m\in S\setminus\{0\}$.*

We end this section with

COROLLARY 6.7. *Assume that there do not exist three different elements $a, b, c\in I_n$ such that $a\rho b\rho c$. Let $d=(d_m)_{m\in S}$ be a sequence of mappings from P to*

P such thot $d_0=id_P$.

Then d is an R-derivation of order s of P if and only if every mapping d_m (for $m\in S\setminus\{0\}$) is an usual R-derivation of P.

PROOF. If $d\in D_s^R(P)$ then, by Corollary 6.6 and Lemma 6.5, $d_i(A)d_j(B)=0$ for $i>0$ or $j>0$ and any $A, B\in P$. Therefore $d_m(AB)=Ad_m(B)+d_m(A)B$, for any $m\in S\setminus\{0\}$ and $A, B\in P$. Conversely, if any d_m is an usual R-derivation of P then, by Corollary 6.6, $d_m(A)\subseteqq I(P)$ for any $A\in P$, hence, by Lemma 6.5, $d_i(A)d_j(B)=0$ for any $A, B\in P$ and $i>0$ or $j>0$. Therefore

$$\begin{aligned} d_m(AB) &= Ad_m(B)+d_m(A)B \\ &= \sum_{i+j=m} d_i(A)d_j(B)\,, \qquad \text{i.e.} \quad d\in D_s^R(P)\,. \end{aligned}$$

7. Integrable R-derivations.

Let S' be a segment of N such that $S\subset S'$ and let $s'=\sup(S')\leqq\infty$.

In the sequel we shall study s'-integrable R-derivations of order s of P.

In this section, we give some examples of such R-derivations and we show that in general there are non-integrable R-derivations.

Notice first that, by Corllary 6.1, we may reduce our investigations and to study only s'-integrable transitive mappings of order s from ρ to R.

Observe also, that it suffices to consider the case where ρ is a partial order. It follows from the following

LEMMA 7.1. *The following conditions are equivalent:*

(1) *Every transitive mapping of order s from ρ to R is s'-integrable,*

(2) *Every transitive mapping of order s from ρ' to R is s'-integrable.*

PROOF. Denote by W some fixed set of representatives of the cosets with respect to $\sim$.

(1)$\Rightarrow$(2). Let $g\in TM_s(\rho', R)$. Consider the sequence $f=(f_m)_{m\in S}$ of mappings from ρ to $Z(R)$ defined by $f_m(x, y)=g_m([x], [y])$ for all $m\in S$ and $x\rho y$. If $x\rho y\rho z$ then $[x]\rho'[y]\rho'[z]$ and we have

$$\begin{aligned} f_m(x, z) &= g_m([x], [z]) \\ &= \sum_{i+j=m} g_i([x], [y])g_j([y], [z]) \\ &= \sum_{i+j=m} f_i(x, y)f_j(y, z) \end{aligned}$$

for all $m\in S$. Therefore $f\in TM_s(\rho, R)$, and, by (1), there exists $f'\in TM_{s'}(\rho, R)$

such that $f'_m = f_m$ for all $m \in S$.

Put $g'_i([a], [b]) = f'_i(a, b)$ for $i \in S'$ and $a, b \in W$.

Then $g' = (g'_i)_{i \in S'}$ is a transitive mapping of order s' from ρ' to R. Indeed, if $[a]\rho'[b]\rho'[c]$, then $a\rho b\rho c$ and we have

$$\begin{aligned} g'_i([a], [c]) &= f'_i(a, c) \\ &= \sum_{p+q=i} f'_p(a, b) f'_q(b, c) \\ &= \sum_{p+q=i} g'_p([a], [b]) g'_q([b], [c]) \quad \text{for all } i \in S'. \end{aligned}$$

Moreover, if $m \in S$, $[a]\rho'[b]$ then

$$g'_m([a], [b]) = f'_m(a, b) = f_m(a, b) = g_m([a], [b]),$$

i.e. $g'_m = g_m$ for all $m \in S$.

(2)$\Rightarrow$(1). Let $f \in TM_s(\rho, R)$. We define the element $g \in TM_s(\rho', R)$ by

$$g_m([a], [b]) = f_m(a, b),$$

where $m \in S$ and $a, b \in W$.

Let g' be such an element in $TM_{s'}(\rho', R)$ that $g'_m = g_m$ for all $m \in S$. We shall construct (by induction) a sequence $f' \in TM_{s'}(\rho, R)$ such that

(i) $\qquad f'_m = f_m \quad$ for all $m \in S$,

and

(ii) $\qquad f'_k(a, b) = g'_k([a], [b]) \quad$ for all $a, b \in W$ and $k \in S'$.

If $t \leqq s$ then we put $f'_t = f_t$.

Now let $s \leqq t < s'$ and assuume that $(f'_0, f'_1, \cdots, f'_t) \in TM_t(\rho, R)$ and the mappings $f'_0, f'_1, \cdots, f'_t$ satisfy the condition (ii). If $x\rho y$ then we put

$$\begin{aligned} f'_{t+1}(x, y) &= g'_{t+1}([a], [b]) \\ &= \sum_{i=1}^{t} f'_i(x, a) f'_{t+1-i}(a, y) \\ &\quad - \sum_{i=1}^{t} f'_i(y, b) f'_{t+1-i}(b, y) \\ &\quad + \sum_{i=1}^{t} f'_i(a, b) f'_{t+1-i}(b, y), \end{aligned}$$

where a, b are elements of W such that $x \sim a$, $y \sim b$. Lemma 3.2 implies that $f'_{t+1}(a, b) = g'_{t+1}([a], [b])$ for $a, b \in W$.

It remains to show that

$$f'_{t+1}(x, z)-f'_{t+1}(x, y)-f'_{t+1}(y, z)=\sum_{i=1}^{t} f'_i(x, y)f'_{t+1-i}(y, z)$$

for $x\rho y\rho z$.

For this purpose we introduce the following notices:

$$(x_1, x_2, x_3)=\sum_{i=1}^{t} f'_i(x_1, x_2)f'_{t+1-i}(x_2, x_3) \quad \text{for} \quad x_1\rho x_2\rho x_3,$$

$$A(x_1, x_2, x_3, x_4)=(x_2, x_3, x_4)-(x_1, x_3, x_4)$$
$$+(x_1, x_2, x_4)-(x_1, x_2, x_3) \quad \text{for} \quad x_1\rho x_2\rho x_3\rho x_4.$$

Observe that

(iii) $$A(x_1, x_2, x_3, x_4)=0.$$

In fact,

$$\begin{aligned} A(x_1, x_2, x_3, x_4) &= -\sum_{i=1}^{t}(f'_i(x_1, x_3)-f'_i(x_2, x_3))f'_{t+1-i}(x_3, x_4) \\ &\quad +\sum_{i=1}^{t} f'_i(x_1, x_2)(f'_{t+1-i}(x_2, x_4)-f'_{t+1-i}(x_2, x_3)) \\ &= -\sum_{i=1}^{t} f'_i(x_1, x_2)f'_{t+1-i}(x_3, x_4) \\ &\quad -\sum f'_p(x_1, x_2)f'_q(x_2, x_3)f'_r(x_3, x_4) \\ &\quad +\sum_{i=1}^{t} f'_i(x_1, x_2)f'_{t+1-i}(x_3, x_4) \\ &\quad +\sum f'_p(x_1, x_2)f'_q(x_2, x_3)f'_r(x_3, x_4) \\ &=0. \end{aligned}$$

Observe also that if a, b, c are such elements of W that $a\rho b\rho c$ then, by (ii), we have

(iv) $$g'_{t+1}([a], [c])-g'_{t+1}([a], [b])-g'_{t+1}([b], [c])=(a, b, c).$$

In fact, since $g'\in TM_{s'}(\rho', R)$ we have

$$\begin{aligned} & g'_{t+1}([a], [c])-g'_{t+1}([a], [b])-g'_{t+1}([b], [c]) \\ &=\sum_{i=1}^{t} g'_i([a], [b])g'_{t+1-i}([b], [c]) \\ &=\sum_{i=1}^{t} f'_i(a, b)f'_{t+1-i}(b, c) \\ &=(a, b, c). \end{aligned}$$

Now, let $x\rho y\rho z$ and let a, b, c be such elements of W that $a\sim x$, $b\sim y$, $c\sim z$. Then, by (iii), (iv) and by the fact thet $(y, y, z)=0$ (Lemma 3.2) we obtain

$$
\begin{aligned}
&f'_{t+1}(x, z)-f'_{t+1}(x, y)-f'_{t+1}(y, z)\\
&\quad=(a, b, c)\\
&\qquad+(x, a, z)-(z, c, z)+(a, c, z)\\
&\qquad-(x, a, y)+(y, b, y)-(a, b, y)\\
&\qquad-(y, b, z)+(z, c, z)-(b, c, z)\\
&\quad=((a, y, z)-(x, y, z)+(x, a, z)-(x, a, y))\\
&\qquad-((b, c, z)-(a, c, z)+(a, b, z)-(a, b, c))\\
&\qquad+((b, y, z)-(a, y, z)+(a, b, z)-(a, b, y))\\
&\qquad-((b, y, z)-(y, y, z)+(y, b, z)-(y, b, y))\\
&\qquad+(x, y, z)-(y, y, z)\\
&\quad=A(x, a, y, z)-A(a, b, c, z)+A(a, b, y, z)-A(y, b, y, z)\\
&\qquad+(x, y, z)-(y, y, z)\\
&\quad=(x, y, z)-(y, y, z)\\
&\quad=(x, y, z)\,.
\end{aligned}
$$

This completes the proof.

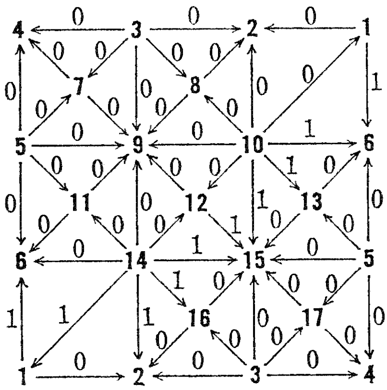
EXAMPLE 7.2. Let P be such as in Example 4.3. Since $D_s^R(P)=ID_s^R(P)$ then every R-derivation of order s of P is s'-integrable (for any s').

EXAMPLE 7.3. Let $P=M_4(R)$, where

$$\rho=\begin{tikzcd} 1 \arrow[r] \arrow[d] & 3 \\ 4 & 2 \arrow[l] \arrow[u] \end{tikzcd} \quad \text{i.e.} \quad P=\begin{bmatrix} R & 0 & R & R \\ 0 & R & R & R \\ 0 & 0 & R & 0 \\ 0 & 0 & 0 & R \end{bmatrix}.$$

There exist R-derivations of order s of P which are not inner ([7]). But, by Corollary 6.1 and Example 3.5, every R-derivation of order s of P is s'-integrable, for any $s'\leqq\infty$ (see also Corollary 6.7).

EXAMPLE 7.4. Consider the following relation ρ on the set I_{17}



(see [7] Section 5).

Let $R=Z_2$ and let $f_1:\rho\to Z_2$ be the usual transitive mapping from ρ to Z_2 defined by the numbers at the arrows (for example $f_1(14,1)=1$, $f_1(10,2)=0$).

Let $f_0(a,b)=1$ for all $a\rho b$. Then $f=(f_0, f_1)$ is a transitive mapping of order 1 from ρ to Z_2. We show that f is not 2-integrable. Suppose that there exists $f_2:\rho\to Z_2$ such that

$$f_2(a,c)=f_2(a,b)+f_2(b,c)+f_1(a,b)f_1(b,c)\,,$$

for any $a\rho b\rho c$.

Denote $f_2(a,b)$ by (a,b). Then we have

$$\begin{aligned}1&=f_1(14,1)f_1(1,6)\\&=(14,6)+(14,1)+(1,6)\\&=[(14,12)+(10,12)+(10,1)+(1,2)+(3,2)+(3,4)+(5,4)+(5,6)]\\&\quad+[(1,2)+(3,2)+(3,4)+(5,4)+(5,6)+(1,6)+(10,1)+(10,12)+(14,12)]+(1,6)\\&=0.\end{aligned}$$

The above example and Corollary 6.1 show that there exist non-integrable R-derivations of P.

8. A necessary condition for s′-integrability.

Let $\Gamma=\Gamma(\rho)=(I'_n, \rho')$ be the graph of the relation ρ (see Section 2), and $f\in TM_s(\rho', R)$.

If a, b, c are such elements in I'_n that $a\rho' b\rho' c$ then by $t(a,b,c)$ we denote the element $(a,c)-(a,b)-(b,c)$ of $C_1(\Gamma)$, and by $\bar{f}_{m+1}(a,b,c)$, for $m\in S$, we denote the element

$$\sum_{i=1}^{m} f_i(a,b)f_{m+1-i}(b,c)$$

of $Z(R)$.

For example:

$$\bar{f}_1(a, b, c)=0,$$

$$\bar{f}_2(a, b, c)=f_1(a, b)f_1(b, c),$$

$$\bar{f}_3(a, b, c)=f_1(a, b)f_2(b, c)+f_2(a, b)f_1(b, c).$$

Consider the following equality (in the group $C_1(\Gamma)$):

$$(*) \qquad \sum_{i=1}^{k} z_i t(a_i, b_i, c_i)=0,$$

where $k\in N$, $z_1, \cdots, z_k\in Z$ and $a_i\rho' b_i\rho' c_i$ for $i=1, 2, \cdots, k$.

DEFINITION 8.1. Let $s<\infty$. We say that Γ is an *s-graph over* R if for any transitive mapping f of order s from ρ' to R and for any equality of the form $(*)$ holds

$$\sum_{i=1}^{k} z_i \bar{f}_{s+1}(a_i, b_i, c_i)=0.$$

For example, Γ is a 1-graph over R if for every usual transitive mapping $\varphi: \rho'\to Z(R)$ and for every equality $(*)$ holds

$$\sum_{i=1}^{k} z_i \varphi(a_i, b_i)\varphi(b_i, c_i)=0,$$

and Γ is a 2-graph over R if for every $f=(f_0, f_1, f_2)\in TM_2(\rho', R)$ and for every equality $(*)$ holds

$$\sum_{i=0}^{k} z_i(f_1(a_i, b_i)f_2(b_i, c_i)+f_2(a_i, b_i)f_1(b_i, c_i))=0.$$

In Section 9 we prove that every graph Γ is a 1-graph and is a 2-graph over an arbitrary ring R.

EXAMPLE 8.2. Let

$$\rho= \begin{tikzcd} & 4 & \\ & 2 \arrow[u] \arrow[dr] & \\ 1 \arrow[uur] \arrow[ur] \arrow[rr] & & 3 \arrow[uul] \end{tikzcd} .$$

We show that $\Gamma=(I_4, \rho)$ is an s-graph over an arbitrary ring R, for any $s\in N$.

Observe, that for Γ we have only one equality of the form $(*)$. Namely,

$$[(1,4)-(1,2)-(2,4)]-[(1,3)-(1,2)-(2,3)]$$

$$+[(2,4)-(2,3)-(3,4)]-[(1,4)-(1,3)-(3,4)]=0\,,$$

$$\text{i.e.}\quad t(1,2,4)-t(1,2,3)+t(2,3,4)-t(1,3,4)=0\,.$$

If $s\in N$, $f\in TM_s(\rho,\ R)$, then we have

$$\begin{aligned}
&\bar{f}_{s+1}(1,2,4)-\bar{f}_{s+1}(1,2,3)+\bar{f}_{s+1}(2,3,4)-\bar{f}_{s+1}(1,3,4)\\
&=\sum_{k=1}^{s}[f_k(1,2)f_{s+1-k}(2,4)-f_k(1,2)f_{s+1-k}(2,3)\\
&\quad+f_k(2,3)f_{s+1-k}(3,4)-f_k(1,3)f_{s+1-k}(3,4)]\\
&=\sum_{k=1}^{s}f_k(1,2)\Big(f_{s+1-k}(3,4)+\sum_{\substack{p+q=s-k+1\\ p\geqq 1,q\geqq 1}}f_p(3,4)f_q(2,3)\Big)\\
&\quad-\sum_{k=1}^{s}\Big(f_k(1,2)+\sum_{\substack{p+q=k\\ p\geqq 1,q\geqq 1}}f_p(1,2)f_q(2,3)\Big)f_{s+1-k}(3,4)=0\,.
\end{aligned}$$

Now we prove a necessary condition for any R-derivation of order s of P to be $(s+1)$-integrable.

Proposition 8.3. *Let $P=M_n(R)_\rho$. If every R-derivation of order s of P is $(s+1)$-integrable then $\Gamma=\Gamma(\rho)$ is an s-graph.*

Proof. Consider in $C_1(\Gamma)$ the equality of the form $(*)$ and let $f\in TM_s(\rho',\ R)$. There exists, by Corollary 6.1 and Lemma 7.1, a transitive mapping $f'\in TM_{s+1}(\rho',\ R)$ such that $f'_m=f_m$ for all $m=0,\ 1,\ \cdots,\ s$. Observe that, for $i=1,\ 2,\ \cdots,\ k$, we have

$$f'_{s+1}(a_i,\ c_i)-f'_{s+1}(a_i,\ b_i)-f'_{s+1}(b_i,\ c_i)=\bar{f}_{s+1}(a_i,\ b_i,\ c_i)\,.$$

Let $\varphi: C_1(\Gamma)\to Z(R)$ be the group homomorphism defined (for free generators) by $\varphi(a,\ b)=f'_{s+1}(a,\ b)$.

Then we have

$$\begin{aligned}
\sum_{i=1}^{k}z_i\bar{f}_{s+1}(a_i,\ b_i,\ c_i)&=\sum_{i=1}^{k}z_i(f'_{s+1}(a_i,\ c_i)-f'_{s+1}(a_i,\ b_i)-f'_{s+1}(b_i,\ c_i))\\
&=\sum_{i=1}^{k}z_i(\varphi(a_i,\ b_i)-\varphi(a_i,\ b_i)-\varphi(b_i,\ c_i))\\
&=\varphi\Big(\sum_{i=1}^{k}z_it(a_i,\ b_i,\ c_i)\Big)\\
&=\varphi(0)\\
&=0\,.
\end{aligned}$$

This completes the proof.

We obtain some examples of s-graphs by the following

LEMMA 8.4. *If $H_2(\Gamma)=0$ then Γ is an s-graph over R for any natural s.*

PROOF. Suppose that in $C_1(\Gamma)$ the equality (*) holds, and let $f\in TM_s(\rho', R)$. We must to show that $\sum_{i=1}^{k} z_i\bar{f}_{s+1}(a_i, b_i, c_i)=0$.

Consider the group homomorphism $\varphi: C_2(\Gamma)\to R$ defined for free-generators by $\varphi(a, b, c,)=\bar{f}_{s+1}(a, b, c)$. Since $\sum_{i=1}^{k} z_i(a_i, b_i, c_i)\in\operatorname{Ker}\partial_2$ and $\operatorname{Ker}\partial_2=\operatorname{Im}\partial_3$ (see Section 2) then

$$\sum_{i=1}^{k} z_i(a_i, b_i, c_i)=\sum_{j=1}^{l} u_j[(x_j, y_j, w_j)-(x_j, y_j, t_j)+(x_j, w_j, t_j)-(y_j, w_j, t_j)]$$

for some $u_1, \cdots, u_l\in Z$ and $x_j\rho' y_j\rho' w_j\rho' t_j$, $j=1, 2, \cdots, l$.

Therefore, by Example 8.2, we have

$$\begin{aligned}\sum_{i=1}^{k} z_i\bar{f}_{s+1}(a_i, b_i, c_i)&=\varphi\Big(\sum_{i=1}^{k} z_i(a_i, b_i, c_i)\Big)\\ &=\sum_{j=1}^{l} u_j[\bar{f}_{s+1}(x_j, y_j, w_j)-\bar{f}_{s+1}(x_j, y_j, t_j)\\ &\quad+\bar{f}_{s+1}(x_j, w_j, t_j)-\bar{f}_{s+1}(y_j, w_j, t_j)]\\ &=\sum_{j=1}^{l} u_j 0=0.\end{aligned}$$

This completes the proof.

REMARK 8.5. The necessary condition for any R-derivation of order s of P to be $(s+1)$-integrable given in Proposition 8.3 is not sufficient. For example. let Γ be such as in Example 7.4. Then Γ is one-dimensional triangulation of the projective plane, and therefore $H_2(\Gamma)=0$ (see [3]). So, by Lemma 8.4, Γ is a 1-graph over Z_2. But, by Example 7.4, there exists an R-derivation d of order 1 of $P=M_n(R)_\rho$ (where $R=Z_2$) such that d is not 2-integrable.

THEOREM 8.6. *Let P be a special subring of $M_n(R)$ with the relation ρ, and let $\Gamma=\Gamma(\rho)$ and $s<s'\leqq\infty$. If $H_2(\Gamma)=0$ and $H_1(\Gamma)$ is a free abelian group then every R-derivation of order s of P is s'-integrable.*

PROOF. It follows from Corollary 6.1 and Lemma 7.1 that it is sufficient to prove that every transitive mapping of order s from ρ' to R is $(s+1)$-integrable.

Let $f\in TM_s(\rho', R)$ and consider a group homomorphism $\varphi:\operatorname{Im}\partial_2\to Z(R)$ defined (for generators) by $\varphi(\partial_2(a, b, c))=-\bar{f}_{s+1}(a, b, c)$. Observe that, by Lemma 8.4, φ is a well defined mapping. Since $H_1(\Gamma)$ is free then φ we can extend to a group homomorphism $\varphi':\operatorname{Ker}\partial_1\to Z(R)$. Further, by [7] Lemma 5.5, we can extend φ' to a group homomorphism $\varphi'': C_1(\Gamma)\to Z(R)$. Put $f_{s+1}(a, b)=\varphi''(a, b)$ for all $a\rho' b$. We show that, for any $a\rho' b\rho' c$, holds

$$f_{s+1}(a, c)=\sum_{i+j=s+1} f_i(a, b)f_j(b, c)$$

$$=f_{s+1}(a, b)+f_{s+1}(b, c)+\sum_{i=1}^{s} f_i(a, b)f_{s+1-i}(b, c).$$

In fact

$$\begin{aligned} f_{s+1}(a, b)&-f_{s+1}(a, b)-f_{s+1}(b, c)\\ &=\varphi''(a, c)-\varphi''(a, b)-\varphi''(b, c)\\ &=-\varphi''(\partial_2(a, b, c))\\ &=-\varphi(\partial_2(a, b, c))\\ &=\bar{f}_{s+1}(a, b, c)\\ &=\sum_{i=1}^{s} f_i(a, b)f_{s+1-i}(b, c). \end{aligned}$$

Therefore $(1, f_1, \cdots, f_s, f_{s+1})$ is a transitive mapping of order $(s+1)$ from ρ' to R, i.e. f is $(s+1)$-integrable. This completes the proof.

9. s-graphs.

In this section, using some additional properties of s-graphs, we describe (for fixed $s<s'$) a new class of special subrings of $M_n(R)$ in which every R-derivation of order s is s'-integrable.

Let $\Gamma=(I'_n, \rho')$ be the graph of the relation ρ and let $W(\Gamma)=Z[X_{(a,b)}\,;a\rho' b]$ be the ring of polynomials over Z in commuting indeterminates, one for each pair (a, b), where $a\rho' b$. Denote by $T(\Gamma)$ the ring $W(\Gamma)/I(\Gamma)$, where $I(\Gamma)$ is the ideal in $W(\Gamma)$ generated by all elements of the form

$$X_{(a,c)}-X_{(a,b)}-X_{(b,c)}$$

for $a\rho' b\rho' c$.

Moreover, denote by $\langle a, b\rangle$ the coset of the element $X_{(a,b)}$ in $T(\Gamma)$.

The following lemma plays a basic role in our further considerations.

LEMMA 9.1. *Let n be a power of a prime number p. If in the proup $C_1(\Gamma)$ holds the equality of the form* (*), *then in the ring $T(\Gamma)$ the following equality holds*

$$\sum_{i=1}^{k} z_i \sum_{j=1}^{n-1}(1/p)\binom{n}{j}\langle a_i, b_i\rangle^j\langle b_i, c_i\rangle^{n-j}=0.$$

PROOF. Observe that the equality (*) is equivalent to an equality of the form

$$(**)\qquad \sum_{i=1}^{u}(a'_i, c'_i)+\sum_{j=1}^{v}((a''_j, b''_j)+(b''_j, c''_j))$$

$$= \sum_{j=1}^{v}(a''_j,\ c''_j) + \sum_{i=1}^{u}((a'_i,\ b'_i)+(b'_i,\ c'_i)),$$

where $a'_i\rho' b'_i\rho' c'_i$, $a''_j\rho' b''_j\rho' c''_j$ for some integers u, v and $i=1, \cdots, u$, $j=1, \cdots, v$.

Hence it suffices to prove that, in the ring $T(\Gamma)$, we have

$$(***)\qquad \sum_{i=1}^{u}\sum_{k=1}^{n-1}(1/p)\binom{n}{k}\langle a'_i,\ b'_i\rangle^k\langle b'_i,\ c'_i\rangle^{n-k}$$

$$= \sum_{j=1}^{v}\sum_{k=1}^{n-1}(1/p)\binom{n}{k}\langle a''_j,\ b''_j\rangle^k\langle b''_j,\ c''_j\rangle^{n-k}.$$

Let α, $\beta : C_1(\Gamma)\to W(\Gamma)$ be the group homomorphisms defined, for free generators, as follows:

$$\alpha(a,\ b)=X_{(a,b)}$$

and

$$\beta(a,\ b)=X^n_{(a,b)}.$$

Further we denote $X_{(a,b)}$ by $(a,\ b)$ (for all $a\rho' b$).

Applying α to the equality $(**)$ we obtain the equality $(**)$ in the ring $W(\Gamma)$.

Applying β to the equality $(**)$ we obtain the following equality in $W(\Gamma)$:

$$(1)\qquad \sum_{i=1}^{u}(a'_i,\ c'_i)^n + \sum_{j=1}^{v}((a''_j,\ b''_j)^n+(b''_j,\ c''_j)^n)$$

$$= \sum_{j=1}^{v}(a''_j,\ c''_j)^n + \sum_{i=1}^{u}((a'_i,\ b'_i)^n+(b'_i,\ c'_i)^n).$$

Let

$$A_i=(a'_i,\ c'_i),$$

$$B_i=(a'_i,\ b'_i)+(b'_i,\ c'_i) \qquad \text{for}\quad i=1,\ 2,\ \cdots,\ u,$$

and

$$C_j=(a''_j,\ c''_j),$$

$$D_j=(a''_j,\ b''_j)+(b''_j,\ c''_j) \qquad \text{for}\quad j=1,\ 2,\ \cdots,\ v.$$

Rise both sides of the equality $(**)$ in $W(\Gamma)$ to the n-th power and apply (1). Then we have

$$(2)\qquad \sum_{i=1}^{u}\sum_{k=1}^{n-1}\binom{n}{k}(a'_i,\ b'_i)^k(b'_i,\ c'_i)^{n-k} - \sum_{j=1}^{v}\sum_{k=1}^{n-1}\binom{n}{k}(a''_j,\ b''_j)^k(b''_j,\ c''_j)^{n-k}$$

$$= \sum_{\substack{i_1+\cdots+i_u=n\\ i_1,\cdots,i_u\neq n}}(i_1,\ \cdots,\ i_u)\{A_1^{i_1}\cdots A_u^{i_u} - B_1^{i_1}\cdots B_u^{i_u}]$$

$$+ \sum_{\substack{j_1+\cdots+j_v=n\\ j_1,\cdots,j_v\neq n}}(j_1,\ \cdots,\ j_v)[D_1^{j_1}\cdots D_v^{j_v} - C_1^{j_1}\cdots C_v^{j_v}]$$

$$+ \sum_{k=1}^{n-1}\binom{u}{k}\left[\left(\sum_{i=1}^{u}A_i\right)^k\left(\sum_{j=1}^{v}D_j\right)^{n-k} - \left(\sum_{i=1}^{u}B_i\right)^k\left(\sum_{j=1}^{v}C_j\right)^{n-k}\right],$$

where $(i_1, \cdots, i_u)$, $(j_1, \cdots j_v)$ are Newton symbols, i.e.

$$(n_1, \cdots, n_k)=\frac{(n_1+\cdots+n_k)!}{n_1!\cdots n_k!} \quad \text{for integers } n_1, \cdots, n_k \geqq 0.$$

Since n is a power of a prime number p then every Newton symbol in the equality (2) is divisible by p, and therefore, since $W(\Gamma)$ is a ring with no Z-torsion, we can divide both sides of the equality (2) by p. We obtain the new equality in $W(\Gamma)$, we denote it by (3).

Observe, that the right side of the equality (3) is an element of the ideal $I(\Gamma)$. Therefore, in the ring $T(\Gamma)$, we have the equality (***). This completes the proof.

As a consequence of Lemma 9.1 we obtain

THEOREM 9.2. *Every graph Γ is a 1-graph over an arbitrary ring R.*

Observe, that this theorem is obvious if R is a 2-torsion-free ring. In fact. Let $f_1: \rho' \to Z(R)$ be an usual transitive mapping and suppose that in $C_1(\Gamma)$ the equality of the form (*) holds. Consider the group homomorphism $\varphi: C_1(\Gamma) \to Z(R)$ such that $\varphi(a, b)=f_1(a, b)^2$, for all $a\rho' b$. Then we have

$$\begin{aligned}
&2\sum_{i=1}^{k} z_i f_1(a_i, b_i) f_1(b_i, c_i)\\
&\quad =\sum_{i=1}^{k} z_i[(f_1(a_i, b_i)+f_1(b_i, c_i))^2-f_1^2(a_i, b_i)-f_1^2(b_i, c_i)]\\
&\quad =\sum_{i=1}^{k} z_i[\varphi(a_i, c_i)-\varphi(a_i, b_i)-\varphi(b_i, c_i)]\\
&\quad =\varphi\Big(\sum_{i=1}^{k} z_i t(a_i, b_i, c_i)\Big)\\
&\quad =\varphi(0)\\
&\quad =0.
\end{aligned}$$

PROOF OF THEOREM 9.2. Let $f \in TM_1(\rho', R)$ and suppose that in $C_1(\Gamma)$ the equality of the form (*) holds. Let $h: W(\Gamma) \to Z(R)$ be the ring homomorphism such that $h(X_{(a,b)})=f_1(a, b)$ for all $a\rho' b$. Since f_1 is an usual transitive mapping then h induces a ring homomorphism $\bar{h}: T(\Gamma) \to Z(R)$ such that $\bar{h}(\langle a, b\rangle)=f_1(a, b)$. From Lemma 9.1, for $n=2$, we have

$$\begin{aligned}
\sum_{i=1}^{k} z_i f_1(a_i, b_i) f_1(b_i, c_i) &= \bar{h}\Big(\sum_{i=1}^{k} z_i \langle a_i, b_i\rangle\langle b_i, c_i\rangle\Big)\\
&=\bar{h}(0)=0.
\end{aligned}$$

This completes the proof.

LEMMA 9.3. *If in $C_1(\Gamma)$ the equality* (*) *holds then in the ring $T(\Gamma)$ we have*

$$\sum_{i=1}^{k} z_i\langle a_i, b_i\rangle\langle b_i, c_i\rangle\langle a_i, c_i\rangle=0 .$$

PROOF. From Lemma 9.1, for $n=3$, we get

$$0=\sum_{i=1}^{k} z_i(\langle a_i, b_i\rangle^2\langle b_i, c_i\rangle+\langle a_i, b_i\rangle\langle b_i, c_i\rangle^2)$$

$$=\sum_{i=1}^{k} z_i\langle a_i, b_i\rangle\langle b_i, c_i\rangle(\langle a_i, b_i\rangle+\langle b_i, c_i\rangle)$$

$$=\sum_{i=1}^{k} z_i\langle a_i, b_i\rangle\langle b_i, c_i\rangle\langle a_i, c_i\rangle .$$

THEOREM 9.4. *Every graph Γ is a 2-graph over an arbitrary ring R.*

PROOF. Let $f\in TM_2(\rho', R)$ and suppose that in $C_1(\Gamma)$ holds (*). Consider the group homomorphism $\varphi: C_1(\Gamma)\to Z(R)$ such that

$$\varphi(a, b)=f_1(a, b)f_2(a, b)$$

for all $a\rho' b$.

Then we have

$$\begin{aligned}0&=\varphi(0)\\&=\sum_{i=1}^{k} z_i(\varphi(a_i, c_i)-\varphi(a_i, b_i)-\varphi(b_i, c_i))\\&=\sum_{i=1}^{k} z_i[(f_1(a_i, b_i)+f_1(b_i, c_i))(f_2(a_i, b_i)+f_2(b_i, c_i)\\&\quad+f_1(a_i, b_i)f_1(b_i, c_i))-f_1(a_i, b_i)f_2(b_i, c_i)]\\&=\sum_{i=1}^{k} z_i[f_2(a_i, b_i)f_1(b_i, c_i)+f_1(a_i, b_i)f_2(b_i, c_i)]\\&\quad+\sum_{i=1}^{k} z_i f_1(a_i, b_i)f_1(b_i, c_i)f_1(a_i, c_i) .\end{aligned}$$

Since, by Lemma 9.3,

$$\sum_{i=1}^{k} z_i f_1(a_i, b_i)f_1(b_i, c_i)f_1(a_i, c_i)=0$$

then

$$\sum_{i=1}^{k} z_i[f_2(a_i, b_i)f_1(b_i, c_i)+f_1(a_i, b_i)f_2(b_i, c_i)]=0 .$$

This completes the proof.

Using a similar method we can prove the following

THEOREM 9.5. *Let Γ be a graph and R be a ring.*

a) *If R is 2-torsion-free then Γ is a 3-graph over R,*
b) *Γ is a 4-graph over R,*
c) *If R is 6-torsion-free then Γ is a 5-graph over R,*
d) *Γ is a 6-graph.*

Using the above theorems and arguments from the proof of Theorem 8.6 we obtain

THEORREM 9.6. *Let P be a special subring of $M_n(R)$ with the relation ρ. Assume that the homology group $H(\Gamma(\rho))$ is free abelian. Then*

(1) *Every R-derivation of order $s<3$ of P is 3-integrable.*

(2) *If R is 2-torsion-free then every R-derivation of order $s<5$ of P is 5-integrable.*

(3) *If R is 3!-torsion-free then every R-derivation of order $s<7$ of P is 7-integrable.*

We end this paper with the following open problems:

1). Let $\Gamma=(I_n, \rho)$ be a fixed graph (i.e. ρ is a partial ordering relation on I_n) and let $s<s'$. Suppose that for every R any R-derivation of order s of $M_n(R)_\rho$ is s'-integrable. Is $H_1(\Gamma)$ a free group?

2). Find numbers n, s, a ring R, and a partial order ρ on I_n such that the graph $\Gamma=(I_n, \rho)$ is not s-graph over R.

3). Is every graph a 3-graph over an arbitrary ring?

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