TSUKUBA J. MATH.
Vol. 8 No. 2 (1984). 199—208

ASYMPTOTIC RISK COMPARISON OF SOME ESTIMATORS FOR BIVARIATE NORMAL COVARIANCE MATRIX

By

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Abstract. For Selliah's [5] and Stein's [3] loss functions, Sharma and Krishnamoorthy [6] have considered orthogonal equivariant estimators Ψ. In this paper, we obtain asymptotic risk expressions for Ψ and make a numerical comparison with those for Haff's [2] and Sugiura-Fujimoto [7] estimators. It is observed that Ψ is uniformly better than Haff's estimator and better than Sugiura-Fujimoto estimator except in a small region of the parameter space.

1. Introduction.

Let S have a Wishart distribution with unknown matrix Σ and n degrees of freedom and the problem be one of estimating Σ under Selliah's [5] loss function $L_1(\Sigma, \hat{\Sigma})=\mathrm{tr}(\hat{\Sigma}\Sigma^{-1}-I)^2$ or Stein's [3] loss function $L_2(\Sigma, \hat{\Sigma})=\mathrm{tr}(\hat{\Sigma}\Sigma^{-1})-\log|\hat{\Sigma}\Sigma^{-1}|-p$. The usual estimator of Σ is S/n. Through invariance considerations Selliah [5] and Stein [3] obtained the best lower triangular equivariant estimators φ_1 and φ_2 respectively which are better than any multiple aS of S. From Kiefer's [4] theorem their estimators are also minimax. Haff [2] using an identity for a Wishart matrix found an unbiased estimator of the difference between the risks of aS and the estimators proposed by him and hence proved that his estimators are better. Sugiura and Fujimoto [7] suggested some empirical Bayes estimators and derived asymptotic risk expressions for the best lower triangular equivariant estimators φ_i, Haff's [2] estimators and their own estimators. From their results, it is clear that neither Haff's nor Sugiura-Fujimoto estimators are uniformly better than φ_i. Following a suggestion by Eaton [1], Sharma and Krishnamoorthy [6] have obtained orthogonal equivariant estimator Ψ_i dominating φ_i under the loss L_i when Σ is 2×2.

In this paper, we derive asymptotic risk expressions for Ψ_i $(i=1, 2)$. Numerical comparison of the asymptotic risks in Sections 3 and 4 shows that Ψ_i is uniformly better than the corresponding Haff's estimator; this is consistent with the conclusions from the Monte-Carlo study in Sharma and Krishnamoorthy [ibid].

Received September 21, 1983.

It is also better than Sugiura-Fujimoto estimator except in a small region of the parameter space.

2. Asymptotic Risk Expressions of $\Psi_1(S)$ and $\Psi_2(S)$.

In this section, we derive asymptotic risk expressions of $\Psi_1(S)$ and $\Psi_2(S)$ under the loss functions L_1 and L_2 respectively. The estimator $\Psi_1(S)$ of Σ, when Σ is 2×2 is given by

$$\Psi_1(S)=a_1S+b_1\frac{S^{1/2}}{\text{tr } S^{-1/2}} \tag{2.1}$$

where, if $S=\Gamma D_\lambda \Gamma'$ with Γ orthogonal and $D_\lambda=\text{diag}(\lambda_1, \lambda_2)$, $S^{1/2}=\Gamma D_\lambda^{1/2}\Gamma'$ with $D_\lambda^{1/2}=\text{diag}(\sqrt{\lambda_1}, \sqrt{\lambda_2})$, $S^{-1/2}=(S^{1/2})^{-1}$, and

$$a_1=(n^2+n+2)/(n^3+5n^2+6n+4)\,, \quad b_1=2n/(n^3+5n^2+6n+4)\,.$$

To find the risk expression of $\Psi_1(S)$, we need the following lemma. For the proof, see Sharma and Krishnamoorthy [6, Sec. 3].

LEMMA 2.1. *Let* $S\sim W_2(\Sigma, n)$, *Then*

$$E\left[\frac{|S|^{1/2}}{\text{tr } S^{1/2}}\text{tr}(S^{1/2}\Sigma^{-1})\right]=(n-1)$$

$$E\left[\frac{|S|^{1/2}}{\text{tr } S^{1/2}}\text{tr}(S\Sigma^{-1}S^{1/2}\Sigma^{-1})\right]=(n+2)(n-1)$$

$$E\left[\frac{|S|}{(\text{tr } S^{1/2})^2}\text{tr}(S^{1/2}\Sigma^{-1}S^{1/2}\Sigma^{-1})\right]=(n+1)\left(nE\frac{\text{tr } S}{(\text{tr } S^{1/2})^2}-1\right).$$

Denoting the risk under L_i by R_i and using Lemma 2.1, we have

$$\begin{aligned} &R_1(\Sigma, \Psi_1(S))-R_1(\Sigma, a_1S) \\ &=b_1^2(n+1)\left[nE\frac{\text{tr } S}{(\text{tr } S^{1/2})^2}-1\right]+2b_1(n-1)[a_1(n+2)-1]\,. \end{aligned} \tag{2.2}$$

To find the asymptotic expansion of $E[(\text{tr } S)/(\text{tr } S^{1/2})^2]$, we need the following identity used in Sugiura and Fujimoto [ibid]:

For any analytic function $f(S)$,

$$Ef(S/n)=f(\Sigma)+n^{-1}\text{tr}(\Sigma\partial)^2 f(\Lambda)|_{\Lambda=\Sigma}+O(n^{-2}) \tag{2.3}$$

where ∂ is a matrix of differential operators and its (i, j) element is given by $(1/2)(1+\delta_{ij})\left(\frac{\partial}{\partial\lambda_{ij}}\right)$ for $\Lambda=(\lambda_{ij})$ and $\delta_{ij}=1$ or 0 according as $i=j$ or $i\neq j$. It is convenient to write

$$(2.4)\qquad \frac{1}{n}\operatorname{tr}(\Sigma\partial)^2 f(\Lambda)|_{\Lambda=\Sigma}=\frac{1}{n}\Sigma\sigma_{rs}\sigma_{tu}\partial_{st}\partial_{ur}f(\Lambda)|_{\Lambda=\Sigma},$$

so that the operator ∂ acts on $f(\Lambda)$ only.

Since, for S of order 2×2,

$$(\operatorname{tr} S)/(\operatorname{tr} S^{1/2})^2=(1+2|S|^{1/2}/\operatorname{tr} S)^{-1},$$

we have, from (2.2) and (2.3),

$$(2.5)\qquad E\frac{\operatorname{tr} S}{(\operatorname{tr} S^{1/2})^2}=\left(1+\frac{2|\Sigma|^{1/2}}{\operatorname{tr}\Sigma}\right)^{-1}$$
$$+\frac{1}{n}\Sigma\sigma_{rs}\sigma_{tu}\partial_{st}\partial_{ur}\left(1+\frac{2|\Lambda|^{1/2}}{\operatorname{tr}\Lambda}\right)^{-1}|_{\Lambda=\Sigma}+0(n^{-2}).$$

After evaluating the second term on the right hand side of (2.5), it can be seen that

$$(2.6)\qquad E\frac{\operatorname{tr} S}{(\operatorname{tr} S^{1/2})^2}=\left(1+\frac{2|\Sigma|^{1/2}}{\operatorname{tr}\Sigma}\right)^{-1}+\frac{|\Sigma|^{1/2}}{n\operatorname{tr}\Sigma}\left(1+\frac{2|\Sigma|^{1/2}}{\operatorname{tr}\Sigma}\right)^{-2}$$
$$\times\left[4-4\left(1+\frac{2|\Sigma|^{1/2}}{\operatorname{tr}\Sigma}\right)^{-1}\frac{|\Sigma|^{1/2}}{\operatorname{tr}\Sigma}+8\left(1+\frac{2|\Sigma|^{1/2}}{\operatorname{tr}\Sigma}\right)^{-1}\right.$$
$$\left.\times\frac{|\Sigma|^{1/2}\operatorname{tr}\Sigma^2}{(\operatorname{tr}\Sigma)^3}-4\frac{\operatorname{tr}\Sigma^2}{(\operatorname{tr}\Sigma)^2}\right]+0(n^{-2}).$$

Since Ψ_1 is orthogonal and scale equivariant take $\Sigma=\operatorname{diag}(1, c)$, $c>0$, without loss of generality. Let $c^*=(1+2\sqrt{c}/(1+c))^{-1}$. Then, (2.6) can be written as

$$(2.7)\quad E\frac{\operatorname{tr} S}{(\operatorname{tr} S^{1/2})^2}=c^*+\frac{1}{n}\left(\frac{\sqrt{c}}{1+c}\right)(c^*)^2\left[4-4c^*\frac{\sqrt{c}}{1+c}\left(1-2\frac{(1+c)^2}{(1+c)^2}\right)-4\frac{1+c^2}{(1+c)^2}\right]$$
$$+0(n^{-2}).$$

Thus, using (2.7) and

$$R_1(\Sigma,\ a_1S)=2n(n+3)a_1^2-4na_1+2,$$

we get from (2.2)

$$(2.8)\qquad R_1(\Sigma,\ \Psi_1(S))=2n(n+3)a_1^2-4na_1+2+2b_1(n-1)[a_1(n+2)-1]$$
$$+b_1^2(n+1)(n\times(\text{approx.})-1)+0(n^{-4}),$$

where (approx.) is the expression given in (2.7).

Next, let us find the asymptotic risk expression of $\Psi_2(S)$ under L_2. For $\Psi_2(S)$ given as

$$a_2S+b_2\frac{S^{1/2}}{\operatorname{tr} S^{-1/2}}\quad\text{with}\quad a_2=1/(n+1)\quad\text{and}\quad b_2=2/(n^2-1),$$

in Sharma and Krishnamoorthy [ibid], we have

(2.9) $$R_2(\Sigma, \Psi_2(S))=E\ \mathrm{tr}\Big(a_2S+\frac{b_2S^{1/2}}{\mathrm{tr}\ S^{-1/2}}\Big)\Sigma^{-1}-E\log\Big|\Big(a_2S+\frac{b_2S^{1/2}}{\mathrm{tr}\ S^{-1/2}}\Big)\Sigma^{-1}\Big|-2$$

$$=a_2E\ \mathrm{tr}\ S\Sigma^{-1}+b_2E\frac{1}{\mathrm{tr}\ S^{-1/2}}\mathrm{tr}\ S^{1/2}\Sigma^{-1}-2\log a_2$$

$$-E\log|S\Sigma^{-1}|-E\log|I+(b_2/a_2)S^{-1/2}/\mathrm{tr}\ S^{-1/2}|-2\,.$$

Using the relations

$$E\log|S\Sigma^{-1}|=\sum_{j=1}^{p}E\log(\chi^2_{n-j+1})$$

and

(2.10) $$E\log|I+(b_2/a_2)S^{-1/2}/\mathrm{tr}\ S^{-1/2}|$$

$$=(b_2/a_2)-[b_2^2/(2a_2^2)]E\ \mathrm{tr}\ S/(\mathrm{tr}\ S^{1/2})^2+0(n^{-3})$$

and Lemma 2.1, (2.9) can be written as

(2.11) $$R_2(\Sigma, \Psi_2(S))=2\log(n+1)-\sum_{j=1}^{2}E\log(\chi^2_{n-j+1})$$

$$-b_2/a_2+[b_2^2/(2a_2^2)]E\ \mathrm{tr}\ S/(\mathrm{tr}\ S^{1/2})^2.$$

Using digamma function $\zeta(x)=\Gamma'(x)/\Gamma(x)$, we have

(2.12) $$\sum_{j=1}^{2}E\log(\chi^2_{n-j+1})=2\log 2+\sum_{j=1}^{2}\zeta\Big(\frac{n-j+1}{2}\Big)\,.$$

Thus, from (2.11), (2.12) and (2.7),

(2.13) $$R_2(\Sigma, \Psi_2(S))=2\log\Big(\frac{n+1}{2}\Big)-\zeta(n/2)-\zeta\Big(\frac{n-1}{2}\Big)-2/(n-1)$$

$$+\frac{2}{(n-1)^2}\times(\text{approx.})+0(n^{-4})\,,$$

where, as before, (approx.) is the expression given in (2.7).

3. Comparison of the Estimators under the Loss L_1.

Haff [2] has considered estimators of $\Sigma(pxp)$ of the form

(3.1) $$\Sigma_{1H}=a\Big(S+\frac{b}{\mathrm{tr}\ S^{-1}}I\Big),\quad a=(n+p+1)^{-1},\quad 0<b\leqq 2(p-1)/(n-p+3)\,,$$

which dominate the best multiple aS of S. Sugiura and Fujimoto [ibid] have derived an asymptotic risk expression of Σ_{1H} under the loss L_1:

(3.2) $$R_1(\Sigma, \Sigma_{1H})-R_1(\Sigma, aS)$$

$$=\frac{2b}{(n+p+1)^2}\Big[-n(p+1)+\Big\{2n-4p-4-bn(p+1)+\frac{bn^2}{2}\Big\}\frac{\mathrm{tr}\ \Sigma^{-2}}{(\mathrm{tr}\ \Sigma^{-1})^2}$$

$$+(p+1)^2-\frac{8\,\mathrm{tr}\,\Sigma^{-3}}{(\mathrm{tr}\,\Sigma^{-1})^3}+3(bn+4)\frac{(\mathrm{tr}\,\Sigma^{-2})^2}{(\mathrm{tr}\,\Sigma^{-1})^4}\Big]+0(n^{-4})\,.$$

The asymptotic value of b which minimizes an upper bound of the expression (3.2) is $b_0=(p-1)(n+p-3)/n^2$. Also, we know that $R_1(\Sigma,\ aS)=\frac{p(p+1)}{n+p+1}$. Thus, for $p=2$ and $b=b_0$, when $\Sigma=\mathrm{diag}(1,\ c)$,

$$R_1(\Sigma,\ \Sigma_{1H})=\frac{6}{n+3}+\frac{2(n-1)}{n^2(n+3)^2}\Big[-3n+\frac{(5n-1)(n-6)}{2n}\cdot\frac{(1+c^2)}{(1+c)^2}$$

$$+9-\frac{8(1+c^3)}{(1+c)^3}+\frac{3(5n-1)}{n}\cdot\frac{(1+c^2)^2}{(1+c)^4}\Big]+0(n^{-4})\,.$$

Notice that Σ_{1H} is orthogonal and scale equivariant like Ψ_i, so that Σ can be taken to be diag $(1,\ c)$ without loss of generality.

Sugiura and Fujimoto [ibid], on empirical Bayes arguments, suggested an estimator of the form

$$\Sigma_{1SF}=a\Big(S+\frac{b\,\mathrm{tr}\,S^{-1}C}{\mathrm{tr}(S^{-1}C)^2}C\Big)\,, \tag{3.4}$$

where $a=(n+p+1)^{-1}$, $0<b\leqq 2(b-1)/n$ and C any positive definite matrix. It is asymptotically better than aS and if $b=0(n^{-1})$ and $C=I$,

(3.5) $\quad R_1(\Sigma,\ \Sigma_{1SF})-R_1(\Sigma,\ aS)$

$$=\frac{2b}{(n+p+1)^2}\Big[-2n+n\Big(\frac{bn}{2}-p-1\Big)E\frac{(\mathrm{tr}\,S^{-1})^2}{\mathrm{tr}\,S^{-2}}+4nE\frac{\mathrm{tr}\,S^{-1}\,\mathrm{tr}\,S^{-3}}{(\mathrm{tr}\,S^{-2})^2}+4p+6$$

$$+\{(p+1)^2-6-bn(2p+3)/2\}\frac{(\mathrm{tr}\,\Sigma^{-1})^2}{\mathrm{tr}\,\Sigma^{-2}}-16\frac{\mathrm{tr}\,\Sigma^{-4}}{(\mathrm{tr}\,\Sigma^{-2})^2}$$

$$-4(bn+2p+4)\frac{\mathrm{tr}\,\Sigma^{-1}\,\mathrm{tr}\,\Sigma^{-3}}{(\mathrm{tr}\,\Sigma^{-2})^2}+32\frac{\mathrm{tr}\,\Sigma^{-1}\,\mathrm{tr}\,\Sigma^{-5}}{(\mathrm{tr}\,\Sigma^{-2})^3}$$

$$+8bn\frac{(\mathrm{tr}\,\Sigma^{-1})^2\,\mathrm{tr}\,\Sigma^{-4}}{(\mathrm{tr}\,\Sigma^{-2})^3}-\frac{bn(\mathrm{tr}\,\Sigma^{-1})^4}{(\mathrm{tr}\,\Sigma^{-2})^2}\Big]+0(n^{-4})\,.$$

The asymptotic value of b, which minimizes an upper bound of the above expression, is $b_0=(p-1)(n+p-3)/n^2$. Also, they established

$$E\frac{(\mathrm{tr}\,S^{-1})^2}{\mathrm{tr}\,S^{-2}}=(\mathrm{tr}\,\Sigma^{-1})^2/\mathrm{tr}\,\Sigma^{-2}+\frac{1}{n}\Big[\frac{8\,\mathrm{tr}\,\Sigma^{-4}(\mathrm{tr}\,\Sigma^{-1})^2}{(\mathrm{tr}\,\Sigma^{-2})^3}-\frac{(\mathrm{tr}\,\Sigma^{-1})^4}{(\mathrm{tr}\,\Sigma^{-2})^2} \tag{3.6}$$

$$-\frac{8\,\mathrm{tr}\,\Sigma^{-3}\,\mathrm{tr}\,\Sigma^{-1}}{(\mathrm{tr}\,\Sigma^{-2})^2}-\frac{(\mathrm{tr}\,\Sigma^{-1})^2}{\mathrm{tr}\,\Sigma^{-2}}+2\Big]+0(n^{-2})=SF1(\mathrm{say})\,.$$

and

$$E\Big(\frac{\mathrm{tr}\,S^{-1}\,\mathrm{tr}\,S^{-3}}{(\mathrm{tr}\,S^{-2})^2}\Big)=\frac{\mathrm{tr}\,\Sigma^{-1}\,\mathrm{tr}\,\Sigma^{-3}}{(\mathrm{tr}\,\Sigma^{-2})^2}+\frac{1}{n}\Big[24\frac{\mathrm{tr}\,\Sigma^{-1}\,\mathrm{tr}\,\Sigma^{-3}\,\mathrm{tr}\,\Sigma^{-4}}{(\mathrm{tr}\,\Sigma^{-2})^4} \tag{3.7}$$

$$-\frac{2}{(\mathrm{tr}\,\Sigma^{-2})^3}\{(\mathrm{tr}\,\Sigma^{-1})^3\,\mathrm{tr}\,\Sigma^{-3}+12\,\mathrm{tr}\,\Sigma^{-1}\,\mathrm{tr}\,\Sigma^{-5}+4(\mathrm{tr}\,\Sigma^{-3})^2\}$$

$$+\frac{1}{(\mathrm{tr}\,\Sigma^{-2})^2}\{\mathrm{tr}\,\Sigma^{-1}\,\mathrm{tr}\,\Sigma^{-3}+6\,\mathrm{tr}\,\Sigma^{-4}\}+\frac{3(\mathrm{tr}\,\Sigma^{-1})^2}{\mathrm{tr}\,\Sigma^{-2}}\Bigg]+0(n^{-2})$$

$$=SF2(\text{say}),$$

using (2.3).

As Σ_{1SF} is orthogonal and scale equivariant, we can take Σ to be a matrix of the form $\mathrm{diag}(1, c_2, \cdots, c_p)$. For $p=2$, $b=b_0$, $C=I$, and $\Sigma=\mathrm{diag}(1, c)$, (3.2)-(3.7) imply

$$R_1(\Sigma,\ \Sigma_{1SF})=R_1(\Sigma,\ aS)+\frac{2(n-1)}{n^2(n+3)^2} \tag{3.8}$$

$$\times\Bigg[-2n-\frac{5n+1}{2}(SF1)+4n(SF2)+14-\frac{n-7}{2n}\cdot\frac{(1+c)^2}{1+c^2}$$

$$+\{-16(1+c^4)(1+c^2)-4(9n-1)(1+c)(1+c^2)(1+c^3)/n$$

$$+32(1+c)(1+c^5)+8(n-1)(1+c)^2(1+c^4)/n$$

$$-(n-1)(1+c)^4(1+c^2)/n\}/(1+c^2)^3\Bigg]+0(n^{-4}).$$

We have done numerical calculation of the asymptotic risks of Ψ_1, Σ_{1H} and Σ_{1SF} for various values of c and n. The Tables 1(a)-(e) are for $c\geqq 1$; scale equivariance of the estimators implies that the values at c and c^{-1} are the same. The estimator Ψ_1 is seen to be uniformly better than Σ_{1H} and better than Σ_{1SF} except at $c=1$, 2 for $n\geqq 20$.

Tables 1(a)-(e). Asymptotic comparison of the estimators Ψ_1, Σ_{1H} and Σ_{1SF}. Columns (1), (2) and (3) give the values of $R_1(\Sigma,\Psi_1)$, $R_1(\Sigma,\Sigma_{1H})-R_1(\Sigma,\Psi_1)$ and $R_1(\Sigma,\ \Sigma_{1SF})-R_1(\Sigma,\ \Psi_1)$ respectively for $\Sigma=\mathrm{diag}(1,\ c)$.

(a); $n=10$

c	(1)	(2)	(3)
1	.44414	.00203	−.00721
2	.44438	.00258	−.00608
3	.44473	.00336	−.00114
4	.44505	.00403	.00252
5	.44533	.00457	.00472
10	.44636	.00603	.00793
20	.44747	.00677	.00815
30	.44811	.00689	.00781
40	.44854	.00686	.00749
50	.44887	.00679	.00723
100	.44976	.00642	.00643
200	.45049	.00597	.00574

(b); $n=20$

c	(1)	(2)	(3)
1	.25441	.00072	−.00316
2	.25450	.00101	−.00151
3	.25463	.00138	.00046
4	.25475	.00167	.00163
5	.25487	.00189	.00228
10	.25526	.00243	.00317
20	.25568	.00266	.00317
30	.25592	.00267	.00303
40	.25608	.00264	.00291
50	.25620	.00261	.00281
100	.25653	.00245	.00252
200	.25680	.00227	.00226

(c); $n=40$

c	(1)	(2)	(3)
1	.13754	.00023	−.00100
2	.13757	.00033	−.00034
3	.13761	.00046	.00028
4	.13765	.00055	.00061
5	.13768	.00062	.00079
10	.13780	.00078	.00102
20	.13793	.00084	.00100
30	.13801	.00084	.00096
40	.13806	.00083	.00092
50	.13810	.00082	.00089
100	.13820	.00077	.00080
200	.13828	.00071	.00072

(d); $n=80$

c	(1)	(2)	(3)
1	.07173	.00007	−.00028
2	.07174	.00010	−.00008
3	.07175	.00013	.00010
4	.07176	.00016	.00018
5	.07177	.00018	.00023
10	.07181	.00022	.00029
20	.07184	.00024	.00028
30	.07186	.00024	.00027
40	.07188	.00024	.00026
50	.07189	.00023	.00025
100	.07192	.00022	.00023
200	.07194	.00020	.00020

(e); $n=160$

c	(1)	(2)	(3)
1	.03666	.00002	−.00007
2	.03666	.00003	−.00002
3	.03666	.00004	.00003
4	.03667	.00004	.00005
5	.03667	.00005	.00006
10	.03668	.00006	.00008
20	.03669	.00006	.00008
30	.03670	.00006	.00007
40	.03670	.00006	.00007
50	.03670	.00006	.00007
100	.03671	.00006	.00006
200	.03672	.00005	.00005

4. Comparison of the Estimators under the Loss L_2.

Haff [2] has shown that an estimator of Σ of order $p\times p$ of the form

$$\Sigma_{2H}=a\left(S+\frac{b}{\mathrm{tr}\, S^{-1}}I\right),\quad a=1/n,\quad 0<b\leqq\frac{2(p-1)}{n},$$

is uniformly better than the best multiple S/n of S under the loss function L_2. It has been proved by Sugiura and Fujimoto [ibid] that

$$(4.1)\qquad R_2(\Sigma,\ \Sigma_{2H})-R_2(\Sigma,\ aS)$$
$$=\frac{b}{n}\left[E\frac{\mathrm{tr}\,\Sigma^{-1}}{\mathrm{tr}\,S^{-1}}-n+\frac{bn}{2}E\frac{\mathrm{tr}\,S^{-2}}{(\mathrm{tr}\,S^{-1})^2}-\frac{b^2n}{3}\cdot\frac{\mathrm{tr}\,\Sigma^{-3}}{(\mathrm{tr}\,\Sigma^{-1})^3}\right]+0(n^{-4}),$$

$$E\frac{\operatorname{tr}\Sigma^{-1}}{\operatorname{tr}S^{-1}}=n-p-1+2E\frac{\operatorname{tr}S^{-2}}{(\operatorname{tr}S^{-1})^2} \tag{4.2}$$

and

$$E\frac{\operatorname{tr}S^{-2}}{(\operatorname{tr}S^{-1})^2}=\frac{\operatorname{tr}\Sigma^{-2}}{(\operatorname{tr}\Sigma^{-1})^2}+\frac{1}{n}\Big[\frac{6(\operatorname{tr}\Sigma^{-2})^2}{(\operatorname{tr}\Sigma^{-1})^4}-\frac{8\operatorname{tr}\Sigma^{-3}}{(\operatorname{tr}\Sigma^{-1})^3}+\frac{\operatorname{tr}\Sigma^{-2}}{(\operatorname{tr}\Sigma^{-1})^2}+1\Big]+0(n^{-2}). \tag{4.3}$$

An optimal value of b, in the sense that it minimizes an upper bound of $R_2(\Sigma, \Sigma_{2H})-R_2(\Sigma, aS)$ is $b_0=(p-1)/n$ and we shall take $b=b_0$ in the estimator Σ_{2H}.

Thus, for $p=2$, $\Sigma=\operatorname{diag}(1, c)$ and $b=n^{-1}$, from (4.1) we get

$$R_2(\Sigma, \Sigma_{2H})=R_2(\Sigma, aS)-\frac{3}{n^2}+\frac{5}{2n^2}\cdot\frac{1+c^2}{(1+c)^2}+\frac{5}{2n^3}\Big[\frac{6(1+c^2)^2}{(1+c)^4}-\frac{8(1+c^3)}{(1+c)^3}+\frac{(1+c^2)}{(1+c)^2}+1\Big]-\frac{(1+c^3)}{3n^3(1+c)^3}+0(n^{-4}). \tag{4.4}$$

The Sugiura-Fujimoto estimator Σ_{2SF} dominating S/n is

$$\frac{1}{n}\Big(S+b\frac{\operatorname{tr}CS^{-1}}{\operatorname{tr}(CS^{-1})^2}C\Big),\quad 0<b\leqq 2(p-1)/n\,, \tag{4.5}$$

where C is an arbitrary positive definite matrix. We take $C=I$ and $b=(p-1)/n$. This value of b is optimal in the sense that it minimizes an upper bound of the risk difference

$$R_2(\Sigma, \Sigma_{2SF})-R_2(\Sigma, S/n)=\frac{b}{n}\Big[\Big(\frac{nb}{2}-p-1\Big)E\frac{(\operatorname{tr}S^{-1})^2}{\operatorname{tr}S^{-2}}-2+4E\frac{\operatorname{tr}S^{-3}\operatorname{tr}S^{-1}}{(\operatorname{tr}S^{-2})^2}\Big]-\frac{b^3}{3}\cdot\frac{(\operatorname{tr}\Sigma^{-1})^3\operatorname{tr}\Sigma^{-3}}{(\operatorname{tr}\Sigma^{-2})^3}. \tag{4.6}$$

For (4.6), we have taken $C=I$ and b arbitrary, when $p=2$, $\Sigma=\operatorname{diag}(1, c)$ and $b=n^{-1}$, it can be seen from (3.6) and (3.7) that

$$\begin{aligned}R_2(\Sigma, \Sigma_{2SF})={}&R_2(\Sigma, S/n)-\frac{5(1+c)^2}{2n^2(1+c^2)}+\frac{4(1+c)(1+c^3)}{n^2(1+c^2)^2}-\frac{2}{n^2}\\&-\frac{5}{2n^3}\Big[8\frac{(1+c^4)(1+c)^2}{(1+c^2)^3}-\frac{(1+c)^4}{(1+c^2)^2}-\frac{8(1+c^3)(1+c)}{(1+c^2)^2}-\frac{(1+c)^2}{(1+c^2)}+2\Big]\\&+\frac{4}{n^3}\Big[\frac{24(1+c)(1+c^3)(1+c^4)}{(1+c^2)^4}-\frac{2}{(1+c^2)^3}\\&\times\{(1+c)^3(1+c^3)+12(1+c)(1+c^5)+4(1+c^3)^2\}\end{aligned} \tag{4.7}$$

$$+\frac{1}{(1+c^2)^2}\{(1+c)(1+c^3)+6(1+c^4)\}+3\frac{(1+c)^2}{1+c^2}\Big]$$

$$-\frac{(1+c)^3(1+c^3)}{3n^3(1+c^2)^3}+0(n^{-4}).$$

The values of $R_2(\Sigma, \Psi_2)$, $R_2(\Sigma, \Sigma_{2H})-R_2(\Sigma, \Psi_2)$ and $R_2(\Sigma, \Sigma_{2SF})-R_2(\Sigma, \Psi_2)$ have been calculated from (2.13) and (4.7) and presented in Tables 2(a)-(e) for various values of $c\geqq 1$ and n. We find that Ψ_2 is uniformly better than Σ_{2H} for all values of c and n under consideration. The estimator Ψ_2 is seen to be better than Σ_{2SF} for all c and $n=10$. However, it is worse than Σ_{2SF} at $c=1$, 2 and $n=20$, 40, 80, 160. For L_1 also one can expect the same result because $L_1/2\to L_2$ as $n\to\infty$ (see, for example, Haff [2]). Incidentally, "c greater than or equal to 1" is no restriction because of the scale equivariance of the estimators.

Tables 2(a)-(e). Asymptotic comparison of the estimators Ψ_2, Σ_{2H} and Σ_{2SF}. Columns (1), (2) and (3) give the values of $R_2(\Sigma, \Psi_2)$, $R_2(\Sigma, \Sigma_{2H})-R_2(\Sigma, \Psi_2)$ and $R_2(\Sigma, \Sigma_{2SF})-R_2(\Sigma, \Psi_2)$ respectively for $\Sigma=\text{diag}(1, c)$.

(a) ; $n=10$

c	(1)	(2)	(3)
1	.30525	.00356	.00597
2	.30558	.00394	.00101
3	.30605	.00448	.00201
4	.30649	.00496	.00408
5	.30688	.00533	.00558
10	.30829	.00621	.00792
20	.30982	.00641	.00783
30	.31070	.00623	.00732
40	.31129	.00601	.00688
50	.31173	.00581	.00653
100	.31296	.00507	.00546
200	.31397	.00432	.00452

(b) ; $n=20$

c	(1)	(2)	(3)
1	.15084	.00077	−.00049
2	.15091	.00096	−.00012
3	.15103	.00119	.00078
4	.15113	.00137	.00141
5	.15122	.00151	.00179
10	.15155	.00181	.00232
20	.15191	.00191	.00228
30	.15211	.00188	.00216
40	.15225	.00184	.00206
50	.15235	.00179	.00198
100	.15263	.00163	.00173
200	.15286	.00147	.00152

(c) ; $n=40$

c	(1)	(2)	(3)
1	.07516	.00018	−.00037
2	.07517	.00023	−.00008
3	.07520	.00030	.00023
4	.07523	.00035	.00039
5	.07525	.00040	.00050
10	.07533	.00048	.00062
20	.07541	.00051	.00061
30	.07546	.00051	.00058
40	.07550	.00050	.00055
50	.07552	.00049	.00053
100	.07559	.00045	.00047
200	.07564	.00041	.00042

(d) ; $n=80$

c	(1)	(2)	(3)
1	.03753	.00004	−.00012
2	.03754	.00006	−.00003
3	.03754	.00008	.00006
4	.03755	.00009	.00010
5	.03756	.00010	.00013
10	.03758	.00012	.00016
20	.03760	.00013	.00016
30	.03761	.00013	.00015
40	.03762	.00013	.00014
50	.03762	.00013	.00014
100	.03764	.00012	.00012
200	.03765	.00011	.00011

(e); $n=160$

c	(1)	(2)	(3)
1	.01876	.00001	−.00004
2	.01876	.00001	−.00001
3	.01876	.00002	.00002
4	.01876	.00002	.00003
5	.01876	.00003	.00003
10	.01877	.00003	.00004
20	.01877	.00003	.00004
30	.01878	.00003	.00004
40	.01878	.00003	.00004
50	.01878	.00003	.00004
100	.01878	.00003	.00003
200	.01879	.00003	.00003

Acknowledgement: The authors are thankful to the referee for pointing out certain computational errors in the original version of the paper.

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