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ON ALMOST-PRIMES IN ARITHMETIC PROGRESSIONS

By

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1. Introdnction.

Let P_r denote integers with at most r prime factors counted according to multiplicity. In 1975, Y, Motohashi [9] showed that there exists a P_3 such that

$$P_3 \equiv a \pmod{q}, \qquad P_3 \ll q(\log q)^{70}$$

for any fixed non-zero integer a and almost-all moduli q with $(q, a)=1$. His argument based upon the weighted linear sieve and the Brun-Titchmarsh theorem on average, which are due to H. E. Richert [10] and C. Hooley [4], respectively. H. Iwaniec's fundamental works [6, 7], therefore, suggest possibilities of an improvement upon the above result. In this paper we present an estimation for P_2. We shall prove the following

THEOREM. *Let Q be a large parameter and a be any fixed integer, $0<|a|\leqq Q$. Then, except possibly for $O(Q/\log Q)$ moduli q with $(q, a)=1$ and $Q<q\leqq 2Q$, there exists a P_2 such that*

$$P_2 \equiv a \pmod{q}, \qquad P_2 \leqq \tau(a)q(\log q)^7$$

where the implied O-constant is absolute and τ denotes the divisor function.

Our proof of Theorem is performed by a simple modification of the argument in our previous paper [8], in which the dual problem is considered. In fact, the numerical work in the main term from sieve estimate is identical. Succeeding to Hooley's investigation [4] we treat the remainder terms with the same manner as in [8]. Our main lemma (see Lemma 1 below) is weaker than E. Fouvry's works [1-3] in its scope; however, it will be found to be suitable for an application to the weighted sieve.

We use the standard notation in number theory. Especially, $\bar{r}$, used in either r/s or congruence $(\bmod q)$, means that $\bar{r}r\equiv 1 \pmod{s}$. $\sum_{x=1}^{y}{}^{*}$ stands for the summation with restriction $(x, y)=1$. $n\sim N$ means $N\leqq N_1<n\leqq N_2\leqq 2N$ for some N_1 and N_2. ε denotes a small positive constant and the constants implied in the

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symbols $\ll$ and O may depend only on ε.

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2. Proof of Theorem.

Firstly we state the inequality for P_2. This follows from Richert-Chen-Iwaniec's work, see [6]. For $q\leqq x$ and $(q, a)=1$, put

$$\mathcal{A}=\{n : n\leqq x,\ n\equiv a \pmod q),\ (n, a)=1\},$$

$$\mathcal{A}_d=\{n : n\in\mathcal{A},\ n\equiv 0 \pmod d)\},$$

$$\mathcal{P}=\{p : p \nmid q\}, \qquad \omega(d)=\begin{cases} 1, & (d, a)=1 \\ 0, & (d, a)>1, \end{cases}$$

and

$$r(\mathcal{A}, d)=|\mathcal{A}_d|-\frac{\phi(a)}{a}\frac{x}{q}\frac{\omega(d)}{d}. \tag{2.1}$$

Let α, α, v be the parameters such that

$$\frac{1}{\alpha}<u<v, \quad \frac{2}{\alpha}\leqq v\leqq\frac{4}{u}, \quad u<3. \tag{2.2}$$

Write

$$D=x^{\alpha}, \qquad y=x^{1/u}, \qquad z=x^{1/v}.$$

Then the following inequality is valid.

$$\begin{aligned} \Pi_2(x : q, a)&=|\{P_2 : P_2\leqq x,\ P_2\equiv a \pmod q)\}| \\ &>\prod_{\substack{p<z \\ p\nmid q}}\left(1-\frac{\omega(p)}{p}\right)\frac{x}{q}\{C(\alpha, u, v)-E\}+\sum_{(d,q)=1}\lambda_d r(\mathcal{A}, d)-\sum_{z\leqq p<y}\sum_{\substack{n\in\mathcal{A} \\ p^2|n}}1 \\ &=\Pi+E_2(q)+E_1(q), \quad \text{say,} \end{aligned} \tag{2.3}$$

where $C(\alpha, u, v)$ is some constant, E is a very small quantity and the sequence $(\lambda_d)=(\lambda_d(D))$ has the properties:

$$\lambda_d=0 \quad \text{if } d\geqq D.$$

$$|\lambda_d|\leqq\mu^2(d).$$

and for any $M>y$, $N>1$, $MN=D$,

$$\lambda_d=\sum_{l\leqq(\log D)^2}\sum_{m\leqq M}\sum_{\substack{n\leqq N \\ mn=d}} a_m(l, M, N)b_n(l, M, N) \tag{2.4}$$

with $|a_m|$, $|b_n|\leqq 1$.

We next choose $\alpha=11/20-6\varepsilon$, $1/u=1/2-8\varepsilon$, $1/v=\alpha/4+11/80-2\varepsilon$, then the condition (2.2) is satisfied and, for sufficiently large x and small ε,

$$ve^{-\gamma}\{C(\alpha, u, v)-E\}>\frac{1}{200},$$

which is caluculated in [8]. Hence

$$\Pi>\frac{1}{200}\frac{x}{q\log x}. \tag{2.5}$$

We turn to $E_2(q)$. We use the following lemma:

LEMMA 1. *Let* $2Q<x$, $Q^{1/4}x^{-1/5}>x^{\varepsilon}$. *If* $M\leqq Q^{1/2}x^{-4\varepsilon}$ *and* $N\leqq Q^{1/4}x^{-1/5}$, *then*

$$\sum_{\substack{Q<q\leqq 2Q\\(q,a)=1}}\left|\sum_{(d,q)=1}\lambda_d r(\mathcal{A}, d)\right|^2\ll\tau(a)x(\log x)^4+x(a)^2Q+\frac{x^{2-\varepsilon/2}}{Q}.$$

We postpone the proof of lemma 1 until the final sectien. Now we set $x=\tau(a)Q(\log Q)^7$, $M=x^{1/2-5\varepsilon}>y=x^{1/u}=x^{1/2-8\varepsilon}$ and $N=x^{11/20-\varepsilon}$, then $M\leqq Q^{1/2}x^{-4\varepsilon}$ and $N\leqq Q^{1/4}x^{-1/5}$. Lemma 1 yields

$$\begin{aligned}\sum_{\substack{Q<q<\leqq 2Q\\(q,a)=1}}|E_2(q)|^2&\ll\tau(a)x(\log x)^4+\tau(a)^2Q+\frac{x^{2-\varepsilon/2}}{Q}\\&\ll\left(\frac{x}{q}\right)^2Q\frac{\tau(a)Q(\log Q)^4}{x}+\left(\frac{x}{q}\right)^2Q^{1-\varepsilon/2}\\&\ll\left(\frac{x}{q}\right)^2Q(\log Q)^{-3}.\end{aligned} \tag{2.6}$$

Moreover we have

$$\begin{aligned}\sum_{\substack{Q<q\leqq 2Q\\(q,a)=1}}|E_1(q)|&\leqq\sum_q\sum_{x^{2\varepsilon}\leqq p\leqq x^{1/2}}\sum_{\substack{a<n\leqq x\\n\equiv a\,(q)\\p^2|n}}1\\&\leqq\sum_{x^{2\varepsilon}\leqq p\leqq x^{1/2}}\sum_{\substack{a<n\leqq x\\p^2|n}}\tau(n-a)\\&\ll x^{\varepsilon}\sum_{p\geqq x^{2\varepsilon}}\frac{x}{p^2}\\&\ll x^{1-\varepsilon}\ll\left(\frac{x}{Q}\right)Q(\log Q)^{-2}.\end{aligned} \tag{2.7}$$

In conjunction with (2.3), (2.5), (2.6) and (2.7) we obtain

$$\Pi_2(\tau(a)Q(\log Q)^7:q, a)>\frac{3Cx}{q\log x}+E_1(q)+E_2(q) \tag{2.8}$$

where C is a positive absolute constant and $E_j(q)$ $(j=1, 2)$ satisfy (2.7) and (2.6), respectively.

We proceed to the proof of Theorem. Put $x(t)=\tau(a)t(\log t)^7$ and $\mathcal{E}=\{q:Q<q\leqq 2Q,\ (q, a)=1,\ \Pi_2(x(q):q, a)=0\}$. We shall deduce $|\mathcal{E}|\ll Q(\log Q)^{-1}$,

from which our Theorem will follow. Now, by (2.8), we have

$$\Pi_2(x(q):q,\ a)\geqq\Pi_2(x(Q):q,\ a)>\frac{3Cx}{q\log x}+E_1(q)+E_2(q).$$

For all $q\in\mathcal{E}$ we then see that

$$|E_1(q)|>\frac{Cx}{q\log x}\quad\text{or}\quad|E_2(q)|>\frac{Cx}{q\log x}.$$

Hence

$$\mathcal{E}\subset\mathcal{E}_1\cup\mathcal{E}_2, \tag{2.9}$$

where $\mathcal{E}_j=\left\{q:Q<q\leqq 2Q,\ (q,\ a)=1,\ |E_j(q)|>\dfrac{Cx}{q\log x}\right\}$. Furthermore,

$$|\mathcal{E}_j|\Big(\frac{Cx}{2Q\log x}\Big)^j<\sum_{q\in\mathcal{E}_j}|E_j(q)|^j\leqq\sum_{\substack{Q<q\leqq 2Q\\(q,a)=1}}|E_j(q)|^j.$$

Combining this with (2.6), (2.7) and (2.9) we get

$$\begin{aligned}|\mathcal{E}|&\leqq\sum_{j=1,2}\Big(\frac{Q\log Q}{x}\Big)^j\cdot\Big(\frac{x}{Q}\Big)^jQ(\log Q)^{-j-1}\\&\ll Q(\log Q)^{-1},\end{aligned}$$

as required.

3. Proof of Lemma 1.

In this section we follow the argument of [4] with a minor modification. We use the following elementary lemma: *If* $(c,\ d)=1$, *then*

$$\sum_{\substack{A<m\leqq B\\m\equiv e(d)\\(m,c)=1}}1=\frac{\phi(c)}{c}\frac{B-A}{d}+O(\tau(c)). \tag{3.1}$$

By the definition (2.1) of $r(\mathcal{A},\ d)$ we have

$$\sum_{(d,q)=1}\lambda_d r(\mathcal{A},\ d)=\sum_{\substack{a<n\leqq x\\n\equiv a(q)\\(n,a)=1}}\Big(\sum_{\substack{d\mid n\\(d,q)=1}}\lambda_d\Big)-\frac{\phi(a)}{a}\frac{x-a}{q}\Big(\sum_{(d,aq)=1}\frac{\lambda_d}{d}\Big).$$

By (3.1),

$$\begin{aligned}&\sum_{\substack{(q,a)=1\\Q<q\leqq 2Q}}\Big|\sum_{(d,q)=1}\lambda_d r(\mathcal{A},\ d)\Big|^2\\&\ll\sum_{\substack{(q,a)=1\\Q<q\leqq 2Q}}\Big|\sum_{\substack{a<n\leqq x\\n\equiv a(q)\\(n,a)=1}}\Big(\sum_{\substack{d\mid n\\(d,q)=1}}\lambda_d\Big)-\Big(\sum_{(d,aq)=1}\frac{\lambda_d}{d}\Big)\sum_{\substack{a<n\leqq x\\n\equiv a(q)\\(n,a)=1}}1\Big|^2+\tau(a)^2Q\\&=W-2V+U+\tau(a)^2Q,\qquad\text{say.}\end{aligned} \tag{3.2}$$

Firstly we consider W.

$$W=\sum_{\substack{(q\ a)=1\\ Q<q\leq 2Q}}\sum_{\substack{a<n_1\leq x\\ n_1\equiv a(q)\\ (n_1,a)=1}}\Big(\sum_{\substack{d_1|n_1\\ (d_1,q)=1}}\lambda_{d_1}\Big)\sum_{\substack{a<n_2\leq x\\ n_2\equiv a(q)\\ (n_2,a)=1}}\Big(\sum_{\substack{d_2|n_2\\ (d_2,q)=1}}\lambda_{d_2}\Big)$$

$$=\sum_{\substack{(q,a)=1\\ Q<q\leq 2Q}}\sum\sum_{(d_1d_2,aq)=1}\lambda_{d_1}\lambda_{d_2}\sum_{\substack{a<n_1\leq x\\ n_1\equiv a(q)\\ n_1\equiv 0(d_1)\\ (n_1,a)=1\\ n_1\neq n_2}}1\sum_{\substack{a<n_2\leq x\\ n_2\equiv a(q)\\ n_2\equiv 0(d_2)\\ (n_2,a)=1}}1+O\Big(\sum_q\sum_{\substack{a<n\leq x\\ n\equiv a(q)}}\tau(n)^2\Big)$$

We express the congruent condition $n_1\equiv a(q)$, $n_2\equiv a(q)$ as $n_1=a+ql_1$, $n_2=a+ql_2$. Then, the condition for l_1 and l_2 is

$$l_1\neq l_2\leqq\frac{x-a}{q},\quad \begin{cases} a+ql_1\equiv 0\ (d_1)\\ a+ql_2\equiv 0\ (d_2)\end{cases},\quad (l_1l_2,\ a)=1\,.$$

Since $(l_1l_2,\ a)=1$, $(l_1,\ d_1)=(l_2,\ d_2)=1$. Changing the order of summation, we have

$$W=\sum\sum_{(d_1d_2,a)=1}\lambda_{d_1}\lambda_{d_2}\sum\sum_{\substack{l_1\neq l_2\leq\frac{x-a}{Q}\\ (l_1l_2,a)=1\\ (l_1,d_1)=1\\ (l_2,d_2=1)}}\sum_{\substack{Q<q\leq\min\left(2Q,\frac{x-a}{l_1},\frac{x-a}{l_2}\right)\\ q\equiv -al_1(d_1)\\ q\equiv -al_2(d_2)\\ (q,ad_1d_2)=1}}1+O\Big(\sum_{a<n\leq x}\tau(n)^2\tau(n-a)\Big)$$

We write $L=(x-a)/Q$ and $Q'=Q'(l_1,\ l_2)=\min(2Q,\ (x-a)/l_1,\ (x-a)/l_2)$. The congruences are soluble iff $l_1\equiv l_2$ $((d_1,\ d_2))$ and expressed as one congruence

$$q\equiv b\ ([d_1,\ d_2]),\quad \begin{cases} b\equiv -a\bar{l}_1(d_1)\\ b\equiv -a\bar{l}_2(d_2^*),\quad d_2^*=d_2/(d_1,\ d_2).\end{cases} \tag{3.3}$$

This congruence then absorbs the condition $(q,\ d_1d_2)=1$. Hence

$$W=\sum\sum_{(d_1d_2,a)=1}\lambda_{d_1}\lambda_{d_2}\sum\sum_{\substack{l_1\neq l_2\leq L\\ l_1\equiv l_2((d_1,d_2))\\ (l_1l_2,a)=1\\ (l_1,d_1)=(l_2,d_2)=1}}\sum_{\substack{Q<q\leq Q'\\ q\equiv b([d_1,d_2])\\ (q,a)=1}}1+O\Big(\sum_{a<n\leq x}\tau_4(n)\tau(n-a)\Big)$$

It is expected, by (3.1), that the innermost sum is approximately equal to

$$\frac{\phi(a)}{a}\frac{Q'-Q}{[d_1,\ d_2]}\,.$$

Thus,

$$W=W_0+R+O(x(\log x)^4) \tag{3.4}$$

where

$$W_0=\sum\sum_{(d_1d_2,a)=1}\frac{\lambda_{d_1}\lambda_{d_2}}{[d_1,\ d_2]}\frac{\phi(a)}{a}\sum\sum_{\substack{l_1\neq l_2\leq L\\ l_1\equiv l_2((d_1,d_2))\\ (l_1,ad_1)=(l_2,ad_2)=1}}\Big(\min\Big(2Q,\ \frac{x-a}{l_1},\ \frac{x-a}{l_2}\Big)-Q\Big) \tag{3.5}$$

and

$$R=\sum\sum_{(d_1d_2,a)=1}\lambda_{d_1}\lambda_{d_2}\sum\sum_{\substack{l_1\neq l_2\leq L\\ l_1\equiv l_2((d_1,d_2))\\ (l_1,ad_1)=(l_2,ad_2)=1}}\Big(\sum_{\substack{Q<q\leq Q\\ q\equiv b([d_1,d_2])\\ (q,a)=1}}1-\frac{\phi(a)}{a}\frac{Q'-Q}{[d_1,\ d_2]}\Big). \tag{3.6}$$

In section 5 we shall estimate R. We proceed to consider W_0. We perform the summation over l_1 and l_2. Write

$$\sum_{l_1\neq l_2}=\sum_{l_1<l_2}+\sum_{l_1>l_2}.$$

Then,

$$\sum_{l_1<l_2}=\sum_{\substack{l_2\leq L\\(l_2,ad_2)=1}}\left(\min\left(2Q,\frac{x-a}{l_2}\right)-Q\right)\sum_{\substack{l_1<l_2\\l_1\equiv l_2((d_1,d_2))\\(l_1,ad_1l=1}}1.$$

By (3.1), the inner sum is equal to

$$\sum_{\substack{l_1<l_2\\l_1\equiv l_2((d_1,d_2))\\(l_1,ad_1^*)=1}}1=\frac{\phi(ad_1^*)}{ad_1^*}\frac{l_2}{(d_1,d_2)}+O(\tau(ad_1^*))$$

$$=\frac{1}{\phi((d_1,d_2))}\sum_{\substack{l_1<l_2\\(l_1,ad_1)=1}}1+O(\tau(ad_1)),$$

whence

$$\sum_{l_1<l_2}=\frac{1}{\phi((d_1,d_2))}\sum_{\substack{l_2\leq L\\(l_2,ad_2)=1}}\left(\min\left(2Q,\frac{x-a}{l_1}\right)-Q\right)\sum_{\substack{l_1<l_2\\(l_1,ad_1)=1}}1+O(x(ad_1)x) \tag{3.7}$$

Similarly,

$$\sum_{l_1>l_2}=\frac{1}{\phi((d_1,d_2))}\sum_{\substack{l_1\leq L\\(l_1,ad_1)=1}}\left(\min\left(2Q,\frac{x-a}{l_1}\right)-Q\right)\sum_{\substack{l_2<l_1\\(l_2,ad_2)=1}}1+O(\tau(ad_2)x\Big). \tag{3.8}$$

In conjunction with (3.5), (3.7) and (3.8) we have

$$W_0=\sum_{(d_1d_2,a)=1}\sum\frac{\lambda_{d_1}\lambda_{d_2}}{d_1d_2}\frac{\phi(a)}{a}\frac{(d_1,d_2)}{\phi((d_1,d_2))}\sum_{\substack{l_1\neq l_2\leq L\\(l_1,ad_1)=(l_2,ad_2)=1}}\sum\left(\min\left(2Q,\frac{x-a}{l_1},\frac{x-a}{l_2}\right)-Q\right)$$

$$+O(\tau(a)x(\log x)^3)$$

$$=W_1+O(\tau(a)x(\log x)^3),\qquad \text{say.}$$

Combining this with (3.4) we get

$$W=W_1+R+O(\tau(a)x(\log x)^4). \tag{3.9}$$

We turn to V.

$$V=\sum_{\substack{(q,a)=1\\Q<q\leq 2Q}}\sum_{\substack{a<n_1\leq x\\n_1\equiv a(q)\\(n_1,a)=1}}\Big(\sum_{\substack{d_1|n_1\\(d_1,q)=1}}\lambda_{d_1}\Big)\Big(\sum_{(d_2,aq)=1}\frac{\lambda_{d_2}}{d_2}\Big)\sum_{\substack{a<n_2\leq x\\n_2\equiv a(q)\\(n_2,a)=1}}1$$

$$=\sum_{\substack{(q,a)=1\\Q<q\leq 2Q}}\sum_{(d_1d_2,aq)=1}\sum\lambda_{d_1}\frac{\lambda_{d_2}}{d_2}\sum_{\substack{a<n_1\leq x\\n_1\equiv a(q)\\n_1\equiv 0(d_1)\\(n_1,a)=1}}1\sum_{\substack{a<n_2\leq x\\n_2\equiv a(q)\\(n_2,a)=1\\n_1\neq n_2}}1+O((\log x)\sum_{a<n\leq x}\tau(n)\tau(n-a)).$$

As before we write $n_1=a+ql_1$ and $n_2=a+ql_2$. Then the condition for l_1 and l_2 is

$$l_1\neq l_2\leq\frac{x-a}{q}.\qquad a+ql_1\equiv 0(d_1),\qquad \begin{cases}(l_1l_2,\ a)=1\\ (l_1,\ d_1)=1\end{cases}.$$

Changing the order of summation we have

$$V=\sum_{(d_1d_2,\,a)=1}\sum\lambda_{d_1}\frac{\lambda_{d_2}}{d_2}\sum_{\substack{l_1\neq l_2\leq L\\ (l_1l_2,\,a)=1\\ (l_1,\,d_1)=1}}\ \sum_{\substack{Q<q\leq Q'\\ q\equiv -a\bar{l}_1(d_1)\\ (q,\,ad_1d_2)=1}}1+O(x(\log x)^3).$$

By (3.1) the innermost sum is equal to

$$\sum_{\substack{Q<q\leq Q'\\ q\equiv -a\bar{l}_1(d_1)\\ (q,\,ad_2^*)=1}}1=\frac{\phi(ad_2^*)}{ad_2^*}\frac{(Q'-Q)}{d_1}+O(\tau(ad_2^*)),$$

whence

$$V=\sum_{(d_1d_2,\,a)=1}\sum\frac{\lambda_{d_1}\lambda_{d_2}}{d_1d_2}\frac{\phi(ad_2^*)}{ad_2^*}\sum_{\substack{l_1\neq l_2\leq L\\ (l_1l_2,\,a)=1\\ (l_1,\,d_1)=1}}\sum(Q'-Q)$$

$$+O\Big(\tau(a)\Big(\frac{x}{Q}\Big)^2D(\log x)^2+x(\log x)^3\Big) \tag{3.10}$$

Next we carry out the summation over l_1 and l_2.

$$\frac{\phi(d_2^*)}{d_2^*}\sum_{\substack{l_1\neq l_2\leq L\\ (l_1l_2,\,a)=1\\ (l_1,\,d_1)=1}}\sum\Big(\min\Big(2Q,\ \frac{x-a}{l_1},\ \frac{x-a}{l_2}\Big)-Q\Big)$$

$$=\sum_{\substack{l_2\leq L\\ (l_2,\,a)=1}}\Big(\min\Big(2Q,\ \frac{x-a}{l_2}\Big)-Q\Big)\frac{\phi(d_2^*)}{d_2^*}\sum_{\substack{l_1<l_2\\ (l_1,\,ad_1)=1}}1$$

$$+\sum_{\substack{l_1\leq L\\ (l_1,\,ad_1)=1}}\Big(\min\Big(2Q,\ \frac{x-a}{l_1}\Big)-Q\Big)\frac{\phi(d_2^*)}{d_2^*}\sum_{\substack{l_2<l_1\\ (l_1,\,a)=1}}1. \tag{3.11}$$

(3.1) yields

$$\frac{\phi(d_2^*)}{d_2^*}\sum_{\substack{l_1<l_2\\ (l_1,\,ad_1)=1}}1=\frac{\phi([d_1,\ d_2])}{[d_1,\ d_2]}\sum_{\substack{l_1<l_2\\ (l_1,\,a)=1}}1+O(\tau(ad_1))$$

and

$$\frac{\phi(d_2^*)}{d_2^*}\sum_{\substack{l_2<l_1\\ (l_2,\,a)=1}}1=\frac{(d_1,\ d_2)}{\phi((d_1,\ d_2))}\sum_{\substack{l_2<l_1\\ (l_2,\,ad_1)=1}}1+O(\tau(ad_1)\log x).$$

Thus (3.11) is equal to

$$\frac{\phi([d_1,\ d_2])}{[d_1,\ d_2]}\sum_{\substack{l_1<l_2\leq L\\ (l_1l_2,\,a)=1}}\sum(Q'-Q)+\frac{(d_1,\ d_2)}{\phi((d_1,\ d_2))}\sum_{\substack{l_2<l_1\leq L\\ (l_1,\,ad_1)=1\\ (l_2,\,ad_2)=1}}\sum(Q'-Q)+O(\tau(ad_1)\log x)$$

Combining this with (3.10) we have

$$V=\sum_{(d_1d_2,a)=1}\frac{\lambda_{d_1}\lambda_{d_2}}{d_1d_2}\frac{\phi(a)}{a}\cdot\Big(\frac{\phi([d_1,d_2])}{[d_1,d_2]}\sum_{\substack{l_1<l_2\leq L\\(l_1l_2,a)=1}}\sum(Q'-Q)$$

$$+\frac{(d_1,d_2)}{\phi((d_1,d_2))}\sum_{\substack{l_2<l_1\leq L\\(l_1,ad_1)=1\\(l_2,ad_2)=1}}\sum(Q'-Q)\Big)$$

$$+O\Big(\tau(a)x(\log x)^4+\tau(a)\Big(\frac{x}{Q}\Big)^2D(\log x)^2\Big) \tag{3.12}$$

Interchanging the role of (d_1, l_1) with that of (d_2, l_2), we may obtain the corresponding expression to (3.12). Hence

$$2V=U_1+W_1+O\Big(\tau(a)x(\log x)^4+\tau(a)\Big(\frac{x}{Q}\Big)^2D(\log x)^2\Big) \tag{3.13}$$

where

$$U_1=\sum_{(d_1d_2,a)=1}\sum\frac{\lambda_{d_1}\lambda_{d_2}}{d_1d_2}\frac{\phi(a[d_1,d_2])}{a[d_1,d_2]}\sum_{\substack{l_1\neq l_2\leq L\\(l_1l_2,a)=1}}\sum(Q'-Q).$$

Finaly we consider U. By the same argument as above, we have

$$U=\sum_{\substack{(q,a)=1\\Q<q\leq 2Q}}\Big(\sum_{(d_1,aq)=1}\frac{\lambda_{d_1}}{d_1}\Big)\Big(\sum_{(d_2,aq)=1}\frac{\lambda_{d_2}}{d_2}\Big)\sum_{\substack{a<n_1\leq x\\n_1\equiv a\,(q)\\(n_1,a)=1}}1\sum_{\substack{a<n_2\leq x\\n_2\equiv a\,(q)\\(n_2,a)=1}}1$$

$$=\sum_{\substack{(q,a)=1\\Q<q\leq 2Q}}\Big(\sum_{(d_1,aq)=1}\frac{\lambda_{d_1}}{d_1}\Big)\Big(\sum_{(d_2,aq)=1}\frac{\lambda_{d_2}}{d_2}\Big)\sum_{\substack{l_1\neq l_2\leq(x-a)/q\\(l_1l_2,a)=1}}\sum 1+O\Big((\log x)^2\sum_{a<n\leq x}\tau(n-a)\Big)$$

$$=\sum_{(d_1d_2,a)=1}\sum\frac{\lambda_{d_1}\lambda_{d_2}}{d_1d_2}\sum_{\substack{l_1\neq l_2\leq L\\(l_1l_2,a)=1}}\sum\sum_{\substack{Q<q\leq Q'\\(q,a[d_1,d_2])=1}}1+O(x(\log x)^3)$$

$$=\sum_{(d_1d_2,a)=1}\sum\frac{\lambda_{d_1}\lambda_{d_2}}{d_1d_2}\sum_{\substack{l_1\neq l_2\leq L\\(l_1l_2,a)=1}}\sum\Big(\frac{\phi(a[d_1,d_2])}{a[d_1,d_2]}(Q'-Q)+O(\tau(a[d_1,d_2]))\Big)$$

$$+O(x(\log x)^3)$$

$$=U_1+O\Big(\tau(a)\Big(\frac{x}{Q}\Big)^2(\log x)^4+x(\log x)^3\Big) \tag{3.14}$$

In conjunction with (3.2), (3.9), (3.13) and (3.14) we get

$$\sum_{\substack{Q<q\leq 2Q\\(q,a)=1}}\Big|\sum_{(d,q)=1}\lambda_d r(\mathcal{A},d)\Big|^2\ll W_1+R-(U_1+W_1)+U_1+\tau(a)x(\log x)^4$$

$$+\tau(a)\Big(\frac{x}{Q}\Big)^2D(\log x)^2+\tau(a)^2Q$$

$$\leq R+\tau(a)x(\log x)^4+\tau(a)^2Q+\Big(\frac{x}{Q}\Big)^2Dx^{\varepsilon}. \tag{3.15}$$

4. Auxiliary results.

In the next section we use the following lemmas, see [5].

LEMMA 2. *Let* $\psi(t)=[t]-t+1/2$. *For* $H>2$, *we have*

$$\psi(t)=\sum_{0<|h|\leq H}\frac{e(ht)}{2\pi ih}+O\left(\min\left(1,\frac{1}{H\|t\|}\right)\right)$$

where $e(x)=e^{2\pi ix}$ *and* $\|x\|=\min_{n\in Z}|x-n|$. *Moreover,*

$$\min\left(1,\frac{1}{H\|t\|}\right)=\sum_{h\in Z}C_{h}e(ht),$$

with

$$|C_{h}|\ll\min\left(\frac{\log H}{H},\frac{H}{h^{2}}\right).$$

LEMMA 3. *We have, for any* $\varepsilon>0$,

$$\sum_{\substack{m\sim M\\ m\equiv x\ (\mathrm{mod}\ y)\\ (m,cd)=1}}e\left(l\frac{\bar{m}}{d}\right)\ll\tau(c)(l,d)^{1/2}d^{1/2+\varepsilon}\left(1+\frac{M}{d}\right).$$

5. Proof of Lemma 1, continued.

In this section we shall estimate R by appealing to lemmas 3 and 4. We begin with treating the condition $(q,a)=1$ by Moebius function.

$$R=\sum_{\substack{l_1\neq l_2\leq L\\ (l_1l_2,a)=1}}\sum\ \sum_{\substack{(d_1,d_2)\mid l_1-l_2\\ (d_1,al_1)=1\\ (d_2,al_2)=1}}\sum\lambda_{d_1}\lambda_{d_2}\sum_{e\mid a}\mu(e)\left(\sum_{\substack{Q<q\leq Q'\\ q\equiv b([d_1,d_2])\\ q\equiv O(e)}}1-\frac{Q'-Q}{e[d_1,d_2]}\right)$$

$$=\sum_{e\mid a}\mu(a)\sum_{\substack{l_1\neq l_2\leq L\\ (l_1l_2,a)=1}}\sum\ \sum_{\substack{(d_1,d_2)\mid l_1-l_2\\ (d_1,al_1)=1\\ (d_2,al_2)=1}}\sum\lambda_{d_1}\lambda_{d_2}$$

$$\times\left\{\psi\left(\frac{Q'}{e[d_1,d_2]}-\frac{b\bar{e}}{[d_1,d_2]}\right)-\psi\left(\frac{Q}{e[d_1,d_2]}-\frac{b\bar{e}}{[d_1,d_2]}\right)\right\}. \tag{5.1}$$

We consider the argument in ψ-function. By the definition (3.3) of b (mod $[d_1,d_2]$) and the relalation, for $(m,n)=1$,

$$\frac{\bar{m}}{n}+\frac{\bar{n}}{m}\equiv\frac{1}{mn}\qquad(\mathrm{mod}\ 1), \tag{5.2}$$

we have

$$-\frac{b\bar{e}}{[d_1, d_2]}\equiv-\frac{b\overline{d_2{}^*e}}{d_1}-\frac{b\overline{d_1e}}{d_2{}^*}$$

$$\equiv\frac{a\overline{l_1d_2{}^*e}}{d_1}+\frac{a\overline{l_2d_1e}}{d_2{}^*}$$

$$\equiv\frac{a}{e}\left(\frac{\overline{d_2{}^*l_1}}{d_1}+\frac{\overline{d_1l_2}}{d_2{}^*}\right)$$

$$\equiv\frac{a}{e}\left(\frac{1}{d_1d_2{}^*l_1}-\frac{\bar{d}_1}{d_2{}^*l_1}+\frac{\overline{d_1l_2}}{d_2{}^*}\right)\quad(\mathrm{mod}\ 1). \tag{5.3}$$

We write $l_1{}^*=l_1/(l_1, l_2)$, $l_2{}^*=l_2/(l_1, l_2)$. When

$$(l_1, l_2)=p_1{}^{a_1}\cdots p_r{}^{a_r}$$

$$l_1{}^*=p_1{}^{b_1}\cdots p_r{}^{b_r}p_{r+1}{}^{a_{r+1}}\cdots p_s{}^{a_s}$$

$$p_i\neq p_j\ (i\neq j),\quad a_i\geqq 1,\quad b_i\geqq 0,$$

we define

$$l_1{}^{**}=p_{r+1}{}^{a_{r+1}}\cdots p_s{}^{a_s}$$

$$l_1{}^{\#}=p_1{}^{a_1+b_1}\cdots p_r{}^{a_r+b_r}.$$

Then it follows $(l_1{}^{**}, l_1{}^{\#})=1$ and $(d_1d_2{}^*, l_1{}^{\#})=1$. By (5.2),

$$\frac{\bar{d}_1}{d_2{}^*l_1}\equiv\frac{\bar{d}_1}{d_2{}^*l_1{}^{**}\cdot l_1{}^{\#}}\equiv\frac{\overline{d_1l_1{}^{\#}}}{d_2{}^*l_1{}^{**}}+\frac{\overline{d_1d_2{}^*l_1{}^{**}}}{l_1{}^{\#}}\quad(\mathrm{mod}\ 1).$$

Moreover,

$$-\frac{\overline{d_1l_1{}^{\#}}}{d_2{}^*l_1{}^{**}}+\frac{\overline{d_1l_2}}{d_2{}^*}\equiv\frac{k}{d_2{}^*l_1{}^{**}}\quad(\mathrm{mod}\ 1)$$

where $k=-\overline{d_1l_1{}^{\#}}+\widehat{d_1l_2}l_1{}^{**}$ with $\widehat{d_1l_2}d_1l_2\equiv 1\ (\mathrm{mod}\ d_2{}^*)$. We then have, with a certain integer n,

$$d_1l_1{}^{\#}l_2{}^*k=-\overline{d_1l_1{}^{\#}}d_1l_1{}^{\#}l_2{}^*+\widehat{d_1l_2}d_1l_2{}^*l_1{}^{\#}l_1{}^{**}$$

$$\equiv-l_2{}^*+\widehat{d_1l_2}d_1l_2l_1{}^*$$

$$\equiv-l_2{}^*+(1+nd_2{}^*)l_1{}^*$$

$$\equiv l_1{}^*-l_2{}^*\quad(\mathrm{mod}\ d_2{}^*l_1{}^{**}).$$

Since $(d_1l_1{}^{\#}l_2{}^*, d_2{}^*l_1{}^{**})=1$,

$$k\equiv(l_1{}^*-l_2{}^*)\overline{d_1l_1{}^{\#}l_2{}^*}\quad(\mathrm{mod}\ d_2{}^*l_1{}^{**}).$$

The condition $(d_1, d_2)|l_1-l_2$ implies $(d_1, d_2)|l_1{}^*-l_2{}^*$ since $((d_1, d_2), l_1l_2)=1$, whence

$$k\equiv\frac{(l_1{}^*-l_2{}^*)}{(d_1, d_2)}\overline{d_1{}^*l_1{}^{\#}l_2{}^*}\quad(\mathrm{mod}\ d_2{}^*l_1{}^{**}),$$

or

$$-\frac{\overline{d_1 l_1^{\#}}}{d_2^* l_1^{**}}+\frac{\overline{d_1 l_2}}{d_2^*}\equiv\frac{l_1^*-l_2^*}{(d_1,\ d_2)}\frac{\overline{d_1^* l_1^{\#} l_2^*}}{d_2^* l_1^{**}} \pmod 1 . \tag{5.5}$$

In conjunction with (5.3), (5.4) and (5.5) we finaly obtain

$$\begin{aligned}-\frac{b\bar{e}}{[d_1,\ d_2]}&\equiv\frac{a}{e}\Big(\frac{1}{d_1 d_2^* l_1}-\frac{\bar{d}_1}{d_2^* l_1}+\frac{\overline{d_1 l_2}}{d_2^*}\Big)\\ &\equiv\frac{a}{e}\Big(\frac{1}{d_1 d_2^* l_1}-\frac{\overline{d_1 l_1^{\#}}}{d_2 l_1^{**}}-\frac{\overline{d_1 d_2^* l_1^{**}}}{l_1^{\#}}+\frac{\overline{d_1 l_2}}{d_2^*}\Big)\\ &\equiv\frac{a}{e}\Big(\frac{1}{d_1 d_2^* l_1}-\frac{\overline{d_1 d_2^* l_1^{**}}}{l_1^{\#}}+\frac{l_1^*-l_2^*}{(d_1,\ d_2)}\frac{\overline{d_1^* l_1^{\#} l_2^*}}{d_2^* l_1^{**}}\Big) \pmod 1\\ &=\frac{a/e}{d_1 d_2^* l_1}+\theta\Big(\frac{d_1^*}{d_2^*}\Big),\quad \text{say.}\end{aligned} \tag{5.6}$$

We write

$$a=ef,\qquad (d_1,\ d_2)=\delta,\qquad d_j^*=\nu_j,\qquad \frac{l_1^*-l_2^*}{\delta}=l$$

$$\frac{fQ'}{a\delta}+\frac{f}{\delta l_1}=\xi_2,\qquad \frac{fQ}{a\delta}+\frac{f}{\delta l_1}=\xi_1 . \tag{5.7}$$

By (5.6) and (5.7), (5.1) is reduced to the following:

$$R=\sum_{f\mid a}\mu\Big(\frac{a}{f}\Big)\sum_{\substack{l_1\neq l_2\leq L\\ (l_1 l_2,\ a)=1}}\sum\ \sum_{\substack{\delta\mid l_1-l_2\\ (\delta,\ a l_1 l_2)=1}} R_1(f,\ l_1,\ l_2,\ \delta)$$

where

$$\begin{aligned}R_1&=R_1(f,\ l_1,\ l_2,\ \delta)\\ &=\sum_{\substack{(\nu_1,\nu_2)=1\\ (\nu_1,\ a l_1)=1\\ (\nu_2,\ a l_2)=1}}\lambda_{\delta\nu_1}\lambda_{\delta\nu_2}\Big\{\psi\Big(\frac{\xi_2}{\nu_1\nu_2}+\theta\Big(\frac{\nu_1}{\nu_2}\Big)\Big)-\psi\Big(\frac{\xi_1}{\nu_1\nu_2}+\theta\Big(\frac{\nu_1}{\nu_2}\Big)\Big)\Big\}\end{aligned} \tag{5.8}$$

with

$$\begin{aligned}\theta\Big(\frac{\nu_1}{\nu_2}\Big)&=\theta\Big(\frac{\nu_1}{\nu_2}:f,\ l_1,\ l_2,\ \delta\Big)\\ &=-f\frac{\overline{\delta\nu_1\nu_2 l_1^{**}}}{l_1^{\#}}+fl\frac{\overline{\nu_1 l_1^{\#} l_2^*}}{\nu_2 l_1^{**}} .\end{aligned} \tag{5.9}$$

and ξ_j's are defined by (5.7).

Next we decompose $(\lambda_{\delta\nu})$. Since $\mu^2(\delta\nu)=1$, we may write

$$\lambda_{\delta\nu}=\sum_{k\leq(\log D)^2}\ \sum_{\substack{bd=\delta\\ bm\leq M}}\ \sum_{\substack{mn=\nu\\ dn\leq N\\ \mu^2(n)=1}} a_{bm}(k,\ M,\ N)b_{dn}(k,\ M,\ N).$$

Thus,

$$R_1 \ll (\log D)^4 \tau(\delta)^2 \sum_{\substack{M_1 \\ M_1, M_2 \le M \\ M_1, N_2 \le M}} \sum_{N_1} \sum_{M_2} \sum_{N_2} \sup |R_2|$$

where M_1, N_1, M_2, N_2's run through prowers of 2;

$$R_2 = R_2(M_1, N_1, M_2, N_2; \alpha_1, \beta_1, \alpha_2, \beta_2)$$

$$= \sum_{\substack{m_1 \sim M_1 \\ n_1 \sim N_1 \\ (m_1 n_1, m_2 n_2)=1 \\ (m_1 n_1, a l_1) = (n_2 n_2, a l_2) = 1 \\ \mu^2(n_1 n_2)=1}} \sum \sum_{\substack{m_2 \sim M_2 \\ n_2 \sim N_2}} \sum \alpha_1(m_1)\beta_1(n_1)\alpha_2(m_2)\beta_2(n_2) \left\{ \psi\left(\frac{\xi_2}{m_1 n_1 m_2 n_2} + \theta\left(\frac{m_1 n_1}{m_2 n_2}\right)\right) \right.$$

$$\left. - \psi\left(\frac{\xi_1}{m_1 n_1 m_2 n_2} + \theta\left(\frac{m_1 n_1}{m_2 n_2}\right)\right) \right\}; \qquad (5.10)$$

and the supremum is taken over all sequences (α_1), (β_1), (α_2), (β_2) such that $|\alpha_1|$, $|\beta_1|$, $|\alpha_2|$, $|\beta_2| \le 1$. Moreover

$$R_1 \ll (\log D)^8 \tau(\delta)^2 \{ \sup_{\substack{x_1, x_2, \beta_1, \beta_2 \\ M_1 N_1 M_2 N_2 > Q x^{-2\varepsilon}}} |R_2| + Q x^{-2\varepsilon} \}. \qquad (5.11)$$

We shall show

$$\sup |R_2| \ll Q x^{-3\varepsilon/2}$$

from which Lemma 1 follows. Actually, by (5.9), we then have

$$R \ll \sum_{f | a} \sum_{l_1 \ne l_2 \le L} \sum_{\delta | l_1 - l_2} (\log D)^8 \tau(\delta)^2 Q x^{-3\varepsilon/2}$$

$$\ll \left(\frac{x}{Q}\right)^2 x^{\varepsilon} Q x^{-3\varepsilon/2}$$

$$\ll \frac{x^{2-3\varepsilon/2}}{Q}.$$

Combining this with (3.15) we get Lemma 1.

Now we estimate R_2 defined by (5.10). By Lemma 2 we have

$$R_2 = R_3 + R_4$$

where

$$R_3 = \sum_{\substack{m_1 \\ (m_1 n_1, m_2 n_2)=1 \\ (m_1 n_1, a l_1) = (n_2 n_2, a l_2) = 1 \\ \mu^2(n_1 n_2)=1}} \sum_{n_1} \sum_{m_2} \sum_{n_2} \frac{\alpha_1(m_1)\beta_1(n_1)\alpha_2(m_2)\beta_2(n_2)}{m_1 n_1 m_2 n_2} \sum_{0 < |h| \le H} e\left(h\theta\left(\frac{m_1 n_1}{m_2 n_2}\right)\right) \int_{\xi_1}^{\xi_2} e\left(\frac{ht}{m_1 n_1 m_2 n_2}\right) dt, \qquad (5.12)$$

$$R_4 \ll \sum_{j=1,2} \sum_{\substack{m_1 \\ (n_1 n_1, n_2 n_2)=1 \\ (m_1 n_1, a l_1)=1 \\ (m_2 n_2, a l_2)=1}} \sum_{n_1} \sum_{m_2} \sum_{n_2} \min\left(1, \frac{1}{H \left\| \left(\frac{\xi_j}{m_1 n_1 m_2 n_2} + \theta\left(\frac{m_1 n_1}{m_2 n_2}\right)\right) \right\|}\right).$$

Firstly we consider R_4. By Lemma 2

$$R_4 \ll x^{\varepsilon} \sum_{j=1,2} \sum_{k\sim M_1N_1} \sum_{\substack{r\sim M_2N_2 \\ (k,r)=1 \\ (k,al_1)=(r,al_2)=1}} \min\left(1, \frac{1}{H\left\|\left(\frac{\xi_j}{kr}+\theta\left(\frac{k}{r}\right)\right)\right\|}\right)$$

$$\ll x^{\varepsilon} \sum_{j=1,2} \sum_{h\in Z} |C_h| \left| \sum_k \sum_r e\left(\frac{h\xi_j}{kr}\right) e\left(h\theta\left(\frac{k}{r}\right)\right)\right|$$

$$= x^{\varepsilon} \sum_{j=1,2} \sum_{h\in Z} |C_h|\,|S(h)|, \qquad \text{say.} \tag{5,13}$$

We proceed to estimate $S(h)$. Trivially,

$$S(h) \ll M_1N_1M_2N_2\,. \tag{5.14}$$

For $h\neq 0$, we get, by partial summation and the definition (5.9) of θ,

$$S(h) \ll \sum_{\substack{r\sim M_2N_2 \\ (r,al_2)=1}} \left(1+\frac{h\xi_j}{M_1N_1r}\right) \left| \sum_{\substack{k\sim M_1N_1 \\ (k,ral_1)=1}} e\left(h\theta\left(\frac{k}{r}\right)\right)\right|$$

$$\ll \left(1+\frac{hQ}{\delta M_1N_1M_2N_2}\right) \sum_{\substack{r\sim M_2N_2 \\ (r,al_2)=1}} \left| \sum_{b=1}^{l_1^{\#}}{}^{*} e\left(-hf\frac{\delta \overline{krl_1^{**}}}{l_1^{\#}}\right) \sum_{\substack{k\sim M_1N_1 \\ k\equiv b(l_1^{\#}) \\ (k,ral_1^{**})=1}} e\left(hfl\frac{\overline{kl_1^{\#}l_2^{*}}}{rl_1^{**}}\right)\right|$$

Lemma 3 yields

$$S(h) \ll \left(1+\frac{hQ}{M_1N_1M_2N_2}\right) \sum_{\substack{r\sim M_2N_2 \\ (r,al_2)=1}} \sum_{b=1}^{l_1^{\#}}{}^{*} \left| \sum_{\substack{k\sim M_2N_2 \\ k\equiv b(l_1^{\#}) \\ (k,arl_1^{**})=1}} e\left(hfl\frac{\overline{kl_1^{\#}l_2^{*}}}{rl_1^{**}}\right)\right|$$

$$\ll x^{\varepsilon/2}\left(1+\frac{hQ}{M_1N_1M_2N_2}\right) \sum_{(r,al_2)=1} l_1^{\#}\tau(a)(hfl,\, rl_1^{**})^{1/2}(rl_1^{**})^{1/2}\left(1+\frac{M_1N_1}{rl_1^{**}}\right)$$

$$\ll \tau(a)x^{\varepsilon/2}\left(1+\frac{hQ}{M_1N_1M_2N_2}\right) l_1 \sum_r (hl,\, r)^{1/2}\left(r^{1/2}+\frac{M_1N_1}{r^{1/2}}\right)$$

$$\ll \tau(a)x^{\varepsilon/2}\left(1+\frac{hQ}{M_1N_1M_2N_2}\right) l_1 \left(\sum_r \frac{(hl,\, r)}{r}\right)^{1/2} \left\{\left(\sum_r r^2\right)^{1/2} + M_1N_1\left(\sum_r 1\right)^{1/2}\right\}$$

$$\ll \tau(a)x^{\varepsilon}\left(1+\frac{hQ}{M_1N_1M_2N_2}\right) l_1\tau(h)\{(M_2N_2)^{3/2}+M_1N_1(M_2N_2)^{1/2}\}, \tag{5.15}$$

since

$$\sum_{m\sim M} \frac{(A,\, m)}{m} \ll \tau(A). \tag{5.16}$$

In conjunction with (5.13), (5.14), (5.15) and Lemma 2, we obtain

$$R_4 \ll x^{\varepsilon} M_1N_1M_2N_2 \Big(|C_0| + \sum_{|h|>HM_2N_2} |C_h|\Big)$$

$$+\tau(a)x^{2\varepsilon}l_1\{(M_2N_2)^{3/2}+M_1N_1(M_2N_2)^{1/2}\} \sum_{0<|h|\leq HM_2N_2} |C_h|\tau(h)\left(1+\frac{hQ}{M_1N_1M_2N_2}\right)$$

$$\ll x^{\varepsilon}M_1N_1M_2N_2\Big(\frac{\log H}{H}+\sum_{h>HM_2N_2}\frac{H}{h^2}\Big)+\tau(a)x^{2\varepsilon}l_1\{(M_2N_2)^{3/2}+M_1N_1(M_2N_2)^{1/2}\}.$$

$$\cdot\sum_{0<h\leq H}\frac{\log H}{H}\tau(h)\Big(1+\frac{HQ}{M_1N_1M_2N_2}\Big)+\sum_{H<h\leq HN_2M_2}H\tau(h)\Big(\frac{1}{h^2}+\frac{Q}{hM_1N_1M_2N_2}\Big)\Big\}$$

$$\ll x^{\varepsilon}M_1N_1M_2N_2\Big(\frac{\log H}{H}+\frac{1}{M_2N_2}\Big)$$

$$+\tau(a)x^{2\varepsilon}l_1\{(M_2N_2)^{3/2}+M_1N_1(M_2N_2)^{1/2}\}(\log H)^2\Big(1+\frac{HQ}{M_1N_1M_2N_2}\Big).$$

Now, we choose

$$H=\frac{M_1N_1M_2N_2X^{4\varepsilon}}{Q},$$

then $H>2$ since $M_1N_1M_2N_2>Qx^{-2\varepsilon}$ in (5.11). Thus,

$$R_4\ll x^{\varepsilon}(Qx^{-3\varepsilon}+M_1N_1)+\tau(a)x^{7\varepsilon}l_1\{(M_2N_2)^{3/2}+M_1N_1(M_2N_2)^{1/2}\}$$

or

$$\sup|R_4|\ll Qx^{-2\varepsilon}+\tau(a)x^{7\varepsilon}\Big(\frac{x}{Q}\Big)(MN)^{3/2}$$

$$\ll Qx^{-2\varepsilon}+x^{8\varepsilon}\Big(\frac{x}{Q}\Big)(Q^{3/4}x^{-1/5-4\varepsilon})^{3/2}$$

$$\ll Qx^{-2\varepsilon}+Qx^{2\varepsilon}\Big(\frac{x^{1/5}}{Q^{1/4}}\Big)^{7/2}$$

$$\ll Qx^{-3\varepsilon/2}. \tag{5.17}$$

We turn to R_3 defined by (5.12).

$$R_3=\frac{1}{N_1N_2}\sum_{\substack{m_1\\(n_1,m_2)=1\\(m_1,al_1)=1\\(m_2,al_2)=1}}\sum_{m_2}\alpha_1(m_1)\alpha_2(m_2)\int_{\xi_1/m_1m_2}^{\xi_2/m_1m_2}\sum_{0<|h|\leq H}\sum_{\substack{n_1\\(m_1n_1,m_2n_2)=1\\(n_1,al_1)=(n_2,al_2)=1\\\mu^2(n_1n_2)=1}}\sum_{n_2}\beta_1(n_1)\beta_2(n_2)$$

$$\cdot\frac{N_1N_2}{n_1n_2}e\Big(\frac{ht}{n_1n_2}\Big)e\Big(h\theta\Big(\frac{m_1n_1}{m_2n_2}\Big)\Big)dt$$

$$\ll\frac{1}{N_1N_2}\sum_{\substack{m_1\\(m_1,m_2)=1\\(m_1,l_1)=1\\(m_2,fl_2)=1}}\sum_{m_2}\int_0^{4Q/\delta M_1M_2}\Big|\sum_{0<h\leq H}\sum_{\substack{n_1\\(m_11m,m_2n_2)=1\\(n_1,al_1)=(n_2,al_2)=1}}\sum_{n_2}\beta_1(n_1)\beta_2(n_2)$$

$$\cdot\frac{N_1N_2}{n_1n_2}e\Big(\frac{ht}{n_1n_2}\Big)\cdot e\Big(h\theta\Big(\frac{m_1n_1}{m_2n_2}\Big)\Big)\Big|dt$$

$$\ll\frac{Q}{\delta M_1N_1M_2N_2}\sup\sum_{m_1}\sum_{m_2}\Big|\sum_h\sum_{n_1}\sum_{n_2}c(h,n_1,n_2)e\Big(h\theta\Big(\frac{m_{11}n}{m_2n_2}\Big)\Big)\Big| \tag{5.18}$$

where the suprenum is taken over all sequences $|c(h, n_1, n_2)| \leqq 1$.

In the next section we shall prove the following:

LEMMA 4. *For any sequence* (c) *with* $|c|<1$, *we have*

$$S=\sum_{\substack{m_1\sim M_1 \\ (m_1,m_2)=1 \\ (m_1,l_1)=1 \\ (m_2,fl_2)=1}}\sum_{m_2\sim M_2}\left|\sum_{0<h\leqq H}\sum_{\substack{n_1\sim N_1 \\ (m_1n_1,m_2n_2)=1 \\ (n_1,al_1)=(n_2,al_2)=1 \\ \mu^2(n_1n_2)=1}}\sum_{n_2\sim N_2} c(h, n_1, n_2)e\left(h\theta\left(\frac{m_1n_1}{m_2n_2}\right)\right)\right|^2$$

$$\ll x^{\varepsilon}HM_1N_1M_2N_2+x^{\varepsilon}l_1H^2(M_1+M_2)M_2^{1/2}N_1^2N_2^3 .$$

We apply Lemma 4 to (5.18).

$$R_3\ll\frac{Q}{\delta M_1N_1M_2N_2}\sup(M_1M_2)^{1/2}(S)^{1/2}$$

$$\ll\frac{x^{4\varepsilon}}{H}\{x^{\varepsilon}H(M_1M_2)^2N_1N_2+x^{\varepsilon}l_1H^2(M_1+M_2)M_1M_2^{3/2}N_1^2N_2^3\}^{1/2}$$

$$\ll x^{9\varepsilon/2}\{Qx^{-4\varepsilon}M_1M_2+l_1(M_1+M_2)M_1M_2^{3/2}N_1^2N_2^3\}^{1/2}$$

or

$$\sup|R_3|\ll x^{9\varepsilon/2}\{Q^{1/2}x^{-2\varepsilon}M+l_1^{1/2}M^{7/4}N^{5/2}\}$$

$$\ll x^{9\varepsilon/2}\left\{Qx^{-6\varepsilon}+\left(\frac{x}{Q}\right)^{1/2}(Q^{1/2}x^{-4\varepsilon})^{7/4}\left(\frac{Q^{1/4}}{x^{1/5}}\right)^{5/2}\right\}$$

$$\ll Qx^{-3\varepsilon/2} .$$

Combining this with (5.17) we get

$$\sup|R_2|\leqq\sup|R_3|+\sup|R_4|\ll Qx^{-3\varepsilon/2} ,$$

as required.

6. Proof of Lemma 4.

It remains to establish Lemma 4. By expanding the square and changing the order of summation, we have

$$S=\sum_{0<h_1,h_2\leqq H}\sum\sum_{\substack{n_j(j=1,3,2,4) \\ (n_1,n_2)=(n_3,n_4)=1 \\ (n_1n_3,al_1)=(n_2n_4,al_2)=1 \\ \mu^2(n_1n_2)=\mu^2(n_3n_4)=1}} c(h_1, n_1, n_2)c((h_2, n_3, n_4)\sum_{\substack{m_1 \\ (m_1n_1,n_2n_2)=1 \\ (n_1n_3,n_2n_4)=1 \\ (m_1,l_1)=(m_2,fl_2)=1}}\sum_{m_2} e\left(h_1\theta\left(\frac{m_1n_1}{m_2n_2}\right)-h_2\theta\left(\frac{m_1n_3}{m_2n_4}\right)\right)$$

$$\leqq\sum_{h_1}\sum_{h_2}\sum_{n_j}\sum_{m_2}\sum_{b=1}^{l_1^{\#}}{}^{*}\left|\sum_{\substack{m_1\sim M_1 \\ m_1\equiv b(l_1^{\#}) \\ (m_1,m_2n_2n_4l_1^{**})=1}} e\left(fl\left(h_1\frac{\overline{m_1n_1l_1^{\#}l_2^{*}}}{m_2n_2l_1^{**}}-h_2\frac{\overline{m_1n_3l_1^{\#}l_2^{*}}}{m_2n_4l_1^{**}}\right)\right)\right|$$

We proceed to treat the argument in the above exponential sums. We have

$$h_1\frac{\overline{m_1n_1l_1^{\#}l_2^{*}}}{m_2n_2l_1^{**}}-h_2\frac{\overline{m_1n_3l_1^{\#}l_2^{*}}}{m_2n_4l_1^{**}}=\frac{g}{m_2\frac{n_2}{(n_2,\ n_3)}\frac{n_4}{(n_1,\ n_4)}l_1^{**}}$$

where

$$g=h_1\frac{1}{(n_2,\ n_3)(n_1,\ n_4)}\frac{n_4}{}(m_1n_1l_1{}^{\#}l_2{}^*)'-h_2\frac{1}{(n_1,\ n_4)}\frac{n_2}{(n_2,\ n_3)}(m_2n_3l_1{}^{\#}l_2{}^*)''$$

with $A'A\equiv 1\ (\mathrm{mod}\ m_2n_2l_1{}^{**})$ and $A''A\equiv 1\ (\mathrm{mod}\ m_2n_4l_1{}^{**})$. We then find, with certain integers k_1 and k_2, that

$$\begin{aligned}
&m_1n_1n_3l_1{}^{\#}l_2{}^*g\\
&=h_1\frac{n_3}{(n_2,\ n_3)}\frac{n_4}{(n_1,\ n_4)}(1+k_1m_2n_2l_1{}^{**})-h_2\frac{n_1}{(n_1,\ n_4)}\frac{n_2}{(n_2,\ n_3)}(1+k_2m_2n_4l_1{}^{**})\\
&\equiv h_1\frac{n_3}{(n_2,\ n_3)}\frac{n_4}{(n_1,\ n_4)}-h_2\frac{n_1}{(n_1,\ n_4)}\frac{n_2}{(n_2,\ n_3)}\quad\left(\mathrm{mod}\ m_2\frac{n_2}{(n_2,\ n_3)}\frac{n_4}{(n_1,\ n_4)}l_1{}^{**}\right)\\
&=r,\qquad\text{say.}
\end{aligned}$$

Since $(m_1n_1,\ m_2n_2)=(m_1n_3,\ m_2n_4)=1$ and $\mu^2(n_1n_2)=\mu^2(n_3n_4)=1$, we have

$$\left(m_1n_1n_3l_1{}^{\#}l_2{}^*,\ m_2\frac{n_2}{(n_2,\ n_3)}\frac{n_4}{(n_1,\ n_4)}l_1{}^{**}\right)=1\,,$$

whence

$$g\equiv\overline{m_1n_1n_3l_1{}^{\#}l_2{}^*}r\qquad\left(\mathrm{mod}\ m_2\frac{n_2}{(n_2,\ n_3)}\frac{n_4}{(n_1,\ n_4)}l_1{}^{**}\right).$$

Now Lemma 3 yields

$$\begin{aligned}
S&\leqq\sum_{h_1}\sum_{h_2}\sum_{\substack{n_j(j=1,2,3,4)\\ h_1n_3n_4-h_2n_1n_2=r(n_2,n_3)(n_1,n_4)\\ m_2n_2n_4=d(n_2,n_3)(n_1,n_4)\\ (d,f)=1}}\sum_{m_2}\sum_{b}\left|\sum_{\substack{m_1\sim M_1\\ m_1\equiv b(l_1\#)\\ (m_1,\,dl_1{}^{**}(n_2,n_3)(n_1,n_4))=1}}e\left(flr\frac{\overline{m_1n_1n_3l_1{}^{\#}l_2{}^*}}{dl_1{}^{**}}\right)\right|\\
&\ll\sum_{h_1}\sum_{h_2}\sum_{\substack{n_j\\ h_1n_3n_4=h_2n_1n_2}}M_1M_2+x^{\varepsilon/2}\sum_{h_1}\sum_{h_2}\sum_{n_j}\sum_{\substack{m_2\\ h_1n_3n_4-h_2n_1n_2=r(n_2,n_3)(n_1,n_4)\neq 0\\ m_2n_2n_4=d(n_2,n_3)(n_1,n_4)\\ (d,f)=1}}l_1{}^{\#}(flr,\ d)^{1/2}(dl_1{}^{**})^{1/2}\left(1+\frac{M_1}{d}\right)\\
&\ll M_1M_2\sum_{s\leqq 4HN_1N_2}\tau_3(s)^2+x^{\varepsilon/2}l_1\sum_{h_1,h_2}\left(\sum_{n_j}\sum_{m_2}\frac{(kr,\ d)}{d}\right)^{1/2}\left\{\left(\sum_{n_j}\sum_{m_2}d^2\right)^{1/2}+M_1\left(\sum_{n_j}\sum_{m_2}1\right)^{1/2}\right\}\\
&\ll x^{\varepsilon}HM_1N_1M_2N_2+x^{\varepsilon/2}l_1\sum_{h_1,h_2}\left(\sum_{n_j}\sum_{m_2}\frac{(kr,\ d)}{d}\right)^{1/2}\\
&\qquad\qquad\{((M_2N_2{}^2)^3N_1{}^2)^{1/2}+M_1(M_2N_2{}^2N_1{}^2)^{1/2}\}.
\end{aligned}$$

Here we easily see

$$\Sigma=\sum_{n_j}\sum_{m_2}\frac{(kr,\ d)}{d}\ll x^{\varepsilon}N_1{}^2.$$

In fact, if we write

$$\delta_1=(n_1,\ n_2),\qquad \delta_2=(n_2,\ n_3)$$

$$n_1=\delta_1n_1',\qquad n_2=\delta_2n_2',\qquad n_3=\delta_2n_3',\qquad n_4=\delta_1n_4',$$

and

$$\gamma=(n_2', n_4'), \qquad n_2'=\gamma n_2, \qquad n_4'=\gamma n_4,$$

then we have

$$\Sigma=\sum_{n_1}\sum_{n_3}\sum_{\delta_1|n_1}\sum_{\substack{\delta_2|n_3 \\ k(h_1n_3'n_4-h_2n_1'n_2)\neq 0}}\sum_{m_2}\sum_{\gamma}\sum_{n_2}\sum_{\substack{n_4 \\ (n_2,n_4)=1}}\frac{(k(h_1n_3'n_4-h_2n_1'n_2),\ m_2\gamma n_2n_4)}{m_2\gamma n_2n_4}$$

$$\leqq\sum_{n_1}\sum_{n_3}\sum_{\delta_1}\sum_{\delta_2}\sum_{n_2}\sum_{n_4}\sum_{m_2}\sum_{\gamma}\frac{(B,\ m_2)}{m_2}\frac{(B,\ \gamma)}{\gamma}\frac{(B,\ n_2)}{n_2}\frac{(B,\ n_4)}{n_4}$$

where $B=k(h_1n_3'n_4-h_2n_1'n_2)\neq 0$. Since $(n_2, n_4)=1$,

$$(B,\ n_2)=(kh_1n_3'n_4,\ n_2)=(kh_1n_3',\ n_2)$$

and

$$(B,\ n_4)=(kh_2n_1',\ n_4).$$

By (5.16),

$$\Sigma\ll(x^{\varepsilon/5})^2\sum_{n_1}\sum_{n_3}\sum_{\delta_1|n_1}\sum_{\delta_2|n_3}\sum_{n_2}\frac{(kh_1n_3',\ n_2)}{n_2}\sum_{n_4}\frac{(kh_2n_1',\ n_4)}{n_4}$$

$$\ll(x^{\varepsilon/5})^4\sum_{n_1}\sum_{n_3}\tau(n_1)\tau(n_3)$$

$$\ll x^{\varepsilon}N_1^2.$$

Therefore we get

$$S\ll x^{\varepsilon}HM_1N_1M_2N_2+x^{\varepsilon}l_1H^2\{M_2^{3/2}N_1^2N_2^3+M_1M_2^{1/2}N_1^2N_2\}$$

$$\ll x^{\varepsilon}HM_1N_1M_2N_2+x^{\varepsilon}l_1H^2(M_1+M_2)M_2^{1/2}N_1^2N_2^3,$$

as required.

This conpletes the proof of our Theorem.

References

[1] Fouvry, E., Autour du théorème de Bombieri-Vinogradov, Acta Math. **152** (1984), 219-244.

[2] Fouvry, E., Théorème de Brun-Titchmarsh; application au théorème Fermat, Invent. Math. **79** (1985), 383-407.

[3] Fouvry, E., Autour du théorème de Bombieri-Vinnogradov. II, Ann. scient. Éc. Norm. Sup. **20** (1987), 617-640.

[4] Hooley, C., On the Brun-Titchmarsh theorem. II, Proc. London Math. Soc. (3), **30** (1975), 114-128.

[5] Hooley, C., Applications of sieve methods to the theory of numbers, Cambridge 1976.

[6] Iwaniec, H., A new form of the error term in the linear sieve, Acta Arith. **37** (1980), 307-321.

[7] Iwaniec, H., On the Brun-Titchmarsh theorem, J. Math. Soc. Japan **34** (1982), 95-123.

[8] Mikawa, H., Almost-primes in arithmetic progressions and short intervals, Tsukuba J. Math. **13** (1989), 387-401.
[9] Motohashi, Y., On almost-primes in arithmetic progressions, II, Proc. Japan Acad. **51** (1975), 545-547.
[10] Richert, H.E., Selberg's sieve with weight, Mathematika **16** (1969), 545-547.

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