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ON PROPER HELICES AND EXTRINSIC SPHERES IN PSEUDO-RIEMANNIAN GEOMETRY

By

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Abstract. In this paper, we define the notion of a proper helix of order d in a pseudo-Riemannian manifold and investigate those curves in a totally umbilical pseudo-Riemannian submanifold.

Introduction.

In Riemannian geometry, properties of regular curves are well discribed by the Frenet formula. In [8], K. Sakamoto called a regular curve which has constant curvatures of osculating order d a *helix of order d*. Note that a helix of order one (resp. two) is a geodesic (resp. circle). The research of geodesics, circles and helices (of order three) in Riemannian submanifold theory, has been done by K. Nomizu and K. Yano ([5]), H. Nakagawa ([2]), K. Sakamoto ([7]) and other geometricians. Furthermore, K. Sakamoto also has investigated helices of general order in the theory (cf. [8]). For regular curves in a pseudo-Riemannian manifold, we can not necessarily define a formula corresponding to the Frenet formula. Especially, we call a regular curve with a formula corresponding to the Frenet formula a *proper curve*. Furthermore, we call a proper curve which has constant curvatures of osculating order d a *proper helix of order d*. N. Abe, Y. Nakanishi and S. Yamaguchi defined general circles and helices (of order three) in a pseudo-Riemannian manifold. They investigated those curves in a pseudo-Riemannian submanifold (cf. [1], [3], [4]). We shall investigate proper helices of general order in a totally umbilical pseudo-Riemannian submanifold.

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§1. Notations and Basic Equations.

In this paper, the differentiability of all geometric objects will be C^∞. Let M be a pseudo-Riemannian submanifold in pseudo-Riemannian manifold $\overline{M}$ isometrically immersed by f and denote by g (resp. $\overline{g}$) the pseudo-Riemannian metric of M (resp. $\overline{M}$). For all local formulas and calculations, we may assume f as an imbedding and thus we shall often identify $p\in M$ with $f(p)\in\overline{M}$. The tangent space T_pM at p is identified with a subspace $f_*(T_pM)$ of the tangent space $T_p\overline{M}$. We put $\|X\|:=\sqrt{|\overline{g}(X,X)|}$ for $X\in T_p\overline{M}$. We denote the tangent bundle of M by TM and the normal bundle by $T^\perp M$. Let $\overline{\nabla}$ and ∇ be the Levi-Civita connections of $\overline{M}$ and M, respectively. Then the Gauss formula is given by

$$\overline{\nabla}_XY=\nabla_XY+B(X,Y), \tag{1.1}$$

where X and Y are tangent vector fields of M and B is the second fundamental form of M. The Weingarten formula is given by

$$\overline{\nabla}_X\xi=-A_\xi X+\nabla^\perp_X\xi, \tag{1.2}$$

where X (resp. ξ) is a tangent (resp. normal) vector field of M and A (resp. $\nabla^\perp$) is the shape operator (resp. the normal connection) of M. Clearly A is related to B as

$$g(A_\xi X,Y)=\overline{g}(B(X,Y),\xi).$$

The mean curvature vector field H of M is defined by

$$H:=\frac{1}{n}\sum_{i=1}^{n}g(e_i,e_i)B(e_i,e_i),$$

where n = dim M and $\{e_1,\cdots,e_n\}$ is an orthonormal frame of M. If the second fundamental form B satisfies

$$B(X,Y)=g(X,Y)H$$

for every tangent vector fields X, Y of M, then M is called a totally umbilical submanifold. The mean curvature vector field H is said to be parallel if $\nabla^\perp_XH=0$ for every tangent vector field X of M. A totally umbilical submanifold with the parallel mean curvature vector field is called an *extrinsic sphere*. If the second fundamental form B vanishes identically, then M is called a totally geodesic submanifold of $\overline{M}$.

Next we shall define the notion of a proper helix of order d in a pseudo-Riemannian manifold N. Let $\sigma:I\to N$ be a non-null curve in N parametrized by the arclength s, where I is an open interval of the real line $\boldsymbol{R}$. We denote the tangent vector field of σ by v_0. We assume that σ satisfies the following Frenet formula:

$$\begin{cases} \nabla_{v_0} v_0 = \lambda_1 v_1 \\ \nabla_{v_0} v_1 + \varepsilon_0\varepsilon_1\lambda_1 v_0 = \lambda_2 v_2 \\ \nabla_{v_0} v_2 + \varepsilon_1\varepsilon_2\lambda_2 v_1 = \lambda_3 v_3 \\ \quad\vdots \\ \nabla_{v_0} v_{d-2} + \varepsilon_{d-3}\varepsilon_{d-2}\lambda_{d-2} v_{d-3} = \lambda_{d-1} v_{d-1} \\ \nabla_{v_0} v_{d-1} + \varepsilon_{d-2}\varepsilon_{d-1}\lambda_{d-1} v_{d-2} = 0, \end{cases}$$

where

$$\begin{cases} \lambda_1 := \|\nabla_{v_0} v_0\| > 0, \\ \lambda_i := \|\nabla_{v_0} v_{i-1} + \varepsilon_{i-2}\varepsilon_{i-1}\lambda_{i-1} v_{i-2}\| > 0, \quad (2 \le i \le d-1) \\ \varepsilon_j := g(v_j, v_j)(=\pm 1) \quad (0 \le j \le d-1) \quad \text{on } I. \end{cases}$$

We call such a curve a *proper curve of order d*, λ_i the i-th curvature and $v_0, \ldots, v_{d-1}$ the Frenet frame field. Furthermore, if $\lambda_i (1 \le i \le d-1)$ are constant along σ, then we call this curve a *proper helix of order d*.

§2. Proper helices in a totally umbilical pseudo-Riemannian submanifold.

Let M be a totally umbilical pseudo-Riemannian submanifold in a pseudo-Riemannian manifold $\overline{M}$ isometrically immersed by f and σ a proper helix of order d in M. We denote a curve $f \circ \sigma$ in $\overline{M}$ by $\bar{\sigma}$. Assume that $\bar{\sigma}$ is a proper helix of order $\bar{d}$. Let $\lambda_1, \cdots, \lambda_{d-1}$ (resp. $\bar{\lambda}_1, \cdots, \bar{\lambda}_{\bar{d}-1}$) be the curvatures of σ (resp. $\bar{\sigma}$) and $v_0, \cdots, v_{d-1}$ (resp. $\bar{v}_0, \cdots, \bar{v}_{\bar{d}-1}$) the Frenet frame field of σ (resp. $\bar{\sigma}$). For convenience, let $\lambda_i = 0, v_i = 0, \bar{\lambda}_j = 0$ and $\bar{v}_j = 0 (i \ge d, j \ge \bar{d})$. Set $\varepsilon_i := g(v_i, v_i)$ and $\bar{\varepsilon}_i := \bar{g}(\bar{v}_i, \bar{v}_i)(i \ge 0)$. We define $\nabla^{\perp\,(i)}_{v_0} H (i \ge 0)$ by $\nabla^{\perp\,(0)}_{v_0} H := H$ and $\nabla^{\perp\,(i)}_{v_0} H := \nabla^{\perp}_{v_0} (\nabla^{\perp\,(i-1)}_{v_0} H)(i \ge 1)$. Also, we define $\beta_{i,j}$ and $\bar{\beta}_{i,j} (i \ge j \ge 1, i+j: \text{even})$ by

$$(2.1) \qquad \begin{cases} \beta_{1,1} = \lambda_1, \quad \bar{\beta}_{1,1} = \bar{\lambda}_1 \\ \beta_{i,i} = \lambda_i \beta_{i-1,i-1}, \quad \bar{\beta}_{i,i} = \bar{\lambda}_i \bar{\beta}_{i-1,i-1} \quad (i \ge 2) \\ \beta_{2i+1,1} = -\varepsilon_1\varepsilon_2\lambda_2\beta_{2i,2}, \quad \bar{\beta}_{2i+1,1} = -\bar{\varepsilon}_1\bar{\varepsilon}_2\bar{\lambda}_2\bar{\beta}_{2i,2} \quad (i \ge 1) \\ \beta_{i,j} = -\varepsilon_j\varepsilon_{j+1}\lambda_{j+1}\beta_{i-1,j+1} + \lambda_j\beta_{i-1,j-1} \quad (i > j \ge 2) \\ \bar{\beta}_{i,j} = -\bar{\varepsilon}_j\bar{\varepsilon}_{j+1}\bar{\lambda}_{j+1}\bar{\beta}_{i-1,j+1} + \bar{\lambda}_j\bar{\beta}_{i-1,j-1} \quad (i > j \ge 2). \end{cases}$$

LEMMA 2.1. *The vector fields* $v_i (i \geq 0)$ *and* $\bar{v}_j (j \geq 0)$ *along* σ *are related as follows:*

(F_0) $\quad \bar{v}_0 = v_0,$

(F_{2i-1}) $\quad \sum_{j=1}^{i} \bar{\beta}_{2i-1,2j-1} \bar{v}_{2j-1} = \sum_{j=1}^{i} \beta_{2i-1,2j-1} v_{2j-1} + \varepsilon_0 \nabla_{v_0}^{\perp(2i-2)} H \qquad (i \geq 1),$

(F_{2i}) $\quad \sum_{j=1}^{i} \bar{\beta}_{2i,2j} \bar{v}_{2j} = \sum_{j=1}^{i} \beta_{2i,2j} v_{2j} + \varepsilon_0 \nabla_{v_0}^{\perp(2i-1)} H \qquad (i \geq 1).$

PROOF. By using (1.1), the Frenet formulas and the assumption that M is totally umbilic, we get

$$\bar{\lambda}_1 \bar{v}_1 = \bar{\nabla}_{v_0} v_0 = \nabla_{v_0} v_0 + \varepsilon_0 H = \lambda_1 v_1 + \varepsilon_0 H.$$

Thus we obtain (F_1). Operating $\bar{\nabla}_{v_0}$ to (F_1), we get

$$\bar{\beta}_{1,1}(-\varepsilon_0 \bar{\varepsilon}_1 \bar{\lambda}_1 v_0 + \bar{\lambda}_2 \bar{v}_2) = \beta_{1,1}(-\varepsilon_0 \varepsilon_1 \lambda_1 v_0 + \lambda_2 \bar{v}_2) - \varepsilon_0 \bar{g}(H, H) v_0 + \varepsilon_0 \nabla_{v_0}^{\perp} H,$$

where we use (1.1), (1.2), the Frenet formulas and the assumption that M is totally umbilic. By noticing $\{v_0\}^{\perp}$-component of this equality, we see that

$$\bar{\lambda}_2 \bar{\beta}_{1,1} \bar{v}_2 = \lambda_2 \beta_{1,1} v_2 + \varepsilon_0 \nabla_{v_0}^{\perp} H,$$

which implies (F_2) by (2.1). Assume that (F_{2k}) holds. Operating $\bar{\nabla}_{v_0}$ to (F_{2k}), we have

$$\sum_{j=1}^{k} \bar{\beta}_{2k,2j} \bar{\nabla}_{v_0} \bar{v}_{2j} = \sum_{j=1}^{k} \beta_{2k,2j} \nabla_{v_0} v_{2j} - \varepsilon_0 \bar{g}(\nabla_{v_0}^{\perp(2k-1)} H, H) v_0 + \varepsilon_0 \nabla_{v_0}^{\perp(2k)} H,$$

where we use (1.1), (1.2) and the assumption that M is totally umbilic. Furthermore, by using the Frenet formulas and (2.1), we have

$$\sum_{j=1}^{k+1} \bar{\beta}_{2k+1,2j-1} \bar{v}_{2j-1} = \sum_{j=1}^{k+1} \bar{\beta}_{2k+1,2j-1} v_{2j-1} - \varepsilon_0 \bar{g}(\nabla_{v_0}^{\perp(2k-1)} H, H) v_0 + \varepsilon_0 \nabla_{v_0}^{\perp(2k)} H.$$

Therefore, by noticing Span$\{v_0\}^{\perp}$-component of this equality, we obtain (F_{2k+1}). Similarly, by operating $\bar{\nabla}_{v_0}$ to (F_{2k+1}) and using the Frenet formulas and (2.1), we also have

$$-\varepsilon_0 \bar{\varepsilon}_1 \bar{\lambda}_1 \bar{\beta}_{2k+1,1} v_0 + \sum_{j=1}^{k+1} \bar{\beta}_{2k+2,2j} \bar{v}_{2j} = -\varepsilon_0 \{\varepsilon_1 \lambda_1 \beta_{2k+1,1} + \bar{g}(\nabla_{v_0}^{\perp(2k)} H, H)\} v_0 + \sum_{j=1}^{k+1} \beta_{2k+2,2j} v_{2j} + \nabla_{v_0}^{\perp(2k+1)} H.$$

Thus, by noticing Span $\{v_0\}^{\perp}$-component of this equality, we also have (F_{2k+2}). Therefore, by the induction, we see that (F_i) holds for every $i \geq 0$. □

Now we define column vectors $\boldsymbol{b}_i (i \geq 3)$ and matrices $\boldsymbol{B}_i (i \geq 1)$ by

$$\boldsymbol{b}_{2j-1} := \begin{pmatrix} \beta_{2j-1,1} \\ \beta_{2j-1,3} \\ \vdots \\ \beta_{2j-1,2j-3} \end{pmatrix}, \quad \boldsymbol{b}_{2j} := \begin{pmatrix} \beta_{2j,2} \\ \beta_{2j,4} \\ \vdots \\ \beta_{2j,2j-2} \end{pmatrix} \qquad (j \geq 2)$$

and

$$\boldsymbol{B}_{2j-1} := \begin{pmatrix} \beta_{1,1} & 0 & \cdots & 0 \\ \beta_{3,1} & \beta_{3,3} & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ \beta_{2j-1,1} & \beta_{2j-1,3} & \cdots & \beta_{2j-1,2j-1} \end{pmatrix},$$

$$\boldsymbol{B}_{2j} := \begin{pmatrix} \beta_{2,2} & 0 & \cdots & 0 \\ \beta_{4,2} & \beta_{4,4} & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ \beta_{2j,2} & \beta_{2j,4} & \cdots & \beta_{2j,2j} \end{pmatrix} \qquad (j \geq 1).$$

Also, we define formal column vectors $\boldsymbol{V}_i (i \geq 1)$ and $\boldsymbol{H}_i (i \geq 0)$ whose components are vector fields along σ by

$$\boldsymbol{V}_{2j-1} := \begin{pmatrix} v_1 \\ v_3 \\ \vdots \\ v_{2j-1} \end{pmatrix}, \quad \boldsymbol{V}_{2j} := \begin{pmatrix} v_2 \\ v_4 \\ \vdots \\ v_{2j} \end{pmatrix} \qquad (j \geq 1)$$

and

$$\boldsymbol{H}_{2j} := \begin{pmatrix} H \\ \nabla^{\perp (2)}_{v_0} H \\ \vdots \\ \nabla^{\perp (2j)}_{v_0} H \end{pmatrix}, \quad \boldsymbol{H}_{2j+1} := \begin{pmatrix} \nabla^{\perp}_{v_0} H \\ \nabla^{\perp (3)}_{v_0} H \\ \vdots \\ \nabla^{\perp (2j+1)}_{v_0} H \end{pmatrix} \qquad (j \geq 0).$$

Similarly, we define $\bar{\boldsymbol{b}}_i (i \geq 3), \bar{\boldsymbol{B}}_i (i \geq 1)$ and $\bar{\boldsymbol{V}}_i (i \geq 1)$ in terms of $\bar{\beta}_{i,j}$ and $\bar{v}_i$ instead of $\beta_{i,j}$ and v_i. Note that $\boldsymbol{B}_i (i \leq d-1) \bar{\boldsymbol{B}}_i (i \leq \bar{d}-1)$ are nonsingular by (2.1). By using these notations, (F_i) is expresses as follows:

(2.2) $$ {}^t\bar{b}_i\bar{V}_{i-2}+\bar{\beta}_{i,i}\bar{v}_i={}^t b_iV_{i-2}+\beta_{i,i}v_i+\varepsilon_0\nabla^{\perp\,(i-1)}_{v_0}H.\ (i\geq 3). $$

Moreover, the systems $(F_1),(F_3),\cdots,(F_{2i-1})(i\geq 1)$ and $(F_2),(F_4),\cdots,(F_{2i})(i\geq 1)$ are expressed as

$$ \bar{B}_{2i-1}\bar{V}_{2i-1}=B_{2i-1}V_{2i-1}+\varepsilon_0H_{2i-2}, $$
$$ \bar{B}_{2i}\bar{V}_{2i}=B_{2i}V_{2i}+\varepsilon_0H_{2i-1}, $$

respectively. Thus we have

(2.3) $$ \bar{B}_i\bar{V}_i=B_iV_i+\varepsilon_0H_{i-1},\qquad (i\geq 1). $$

From (2.2) and (2.3), we have

$$ (MF_i)\quad -\bar{\beta}_{i,i}\bar{v}_i+\beta_{i,i}v_i=({}^t\bar{b}_i\bar{B}_{i-2}^{-1}B_{i-2}-{}^tb_i)V_{i-2} $$
$$ +\varepsilon_0({}^t\bar{b}_i\bar{B}_{i-2}^{-1}H_{i-3}-\nabla^{\perp\,(i-1)}_{v_0}H)\qquad (3\leq i\leq \bar{d}+1). $$

LEMMA 2.2. *The inequality $d\leq\bar{d}\leq d+r$ holds, where r is the codimension of M in $\bar{M}$.*

PROOF. Suppose $d>\bar{d}$. Then we have $v_{\bar{d}}\neq 0$ and $\bar{v}_{\bar{d}}=0$. Hence, it follows from $(MF_{\bar{d}})$ that

$$ \beta_{\bar{d},\bar{d}}v_{\bar{d}}=({}^t\bar{b}_{\bar{d}}\bar{B}_{\bar{d}-2}^{-1}B_{\bar{d}-2}-{}^tb_{\bar{d}})V_{\bar{d}-2} $$
$$ +\varepsilon_0({}^t\bar{b}_{\bar{d}}\bar{B}_{\bar{d}-2}^{-1}H_{\bar{d}-3}-\nabla^{\perp\,(\bar{d}-1)}_{v_0}H). $$

Since $v_{\bar{d}}$ is linearly independent of $v_i(i\leq\bar{d}-2)$ and $\nabla^{\perp\,(i)}_{v_0}H(i\leq\bar{d}-1)$, we have $\beta_{\bar{d},\bar{d}}v_{\bar{d}}=0$. From (2.1) and $d>\bar{d},\beta_{\bar{d},\bar{d}}=\lambda_1\lambda_2\cdots\lambda_{\bar{d}}\neq 0$ is deduced. Therefore, we have $v_{\bar{d}}=0$. This contradicts $d>\bar{d}$. Thus we have $d\leqq\bar{d}$. The remaining part is trivial. □

LEMMA 2.3. (i) *If $\bar{d}=d(\geq 3)$, then $\nabla^{\perp\,(d-1)}_{v_0}H={}^tb_dB_{d-2}{}^{-1}H_{d-3}$ holds.*
(ii) *If $\bar{d}=d+1(d\geq 2)$, then $\nabla^{\perp\,(d)}_{v_0}H={}^tb_{d+1}B_{d-1}{}^{-1}H_{d-2}$ holds.*

PROOF. (i) By the assumption, $v_d=0$ and $\bar{v}_d=0$ holds. Substituting these to (MF_d), we have

$$ ({}^t\bar{b}_d\bar{B}_{d-2}^{-1}B_{d-2}-{}^tb_d)V_{d-2}+\varepsilon_0({}^t\bar{b}_d\bar{B}_{d-2}^{-1}H_{d-3}-\nabla^{\perp\,(d-1)}_{v_0}H)=0. $$

By noticing the tangential component and the normal component of this equality, we have

$$ {}^t\bar{b}_d\bar{B}_{d-2}^{-1} = {}^t b_d B_{d-2}{}^{-1} $$

and

$$ \nabla_{v_0}^{\perp\,(d-1)} H = {}^t\bar{b}_d\bar{B}_{d-2}^{-1}H_{d-3}. $$

These imply

$$ \nabla_{v_0}^{\perp\,(d-1)} H = {}^t b_d B_{d-2}{}^{-1}H_{d-3}. $$

(ii) By the assumption, $v_{d+1}=0$ and $\bar{v}_{d+1}=0$ holds. Substituting these to (MF_{d+1}), we have

$$ ({}^t\bar{b}_{d+1}\bar{B}_{d-1}^{-1}B_{d-1} - {}^t b_{d+1})V_{d-1} + \varepsilon_0({}^t\bar{b}_{d+1}\bar{B}_{d-1}^{-1}H_{d-2} - \nabla_{v_0}^{\perp\,(d)}H) = 0. $$

By noticing the tangential component and the normal component of this equality, we have

$$ {}^t\bar{b}_{d+1}\bar{B}_{d-1}^{-1} = {}^t b_{d+1}B_{d-1}{}^{-1} $$

and

$$ \nabla_{v_0}^{\perp\,(d)} H = {}^t\bar{b}_{d+1}\bar{B}_{d-1}^{-1}H_{d-2}. $$

These imply

$$ \nabla_{v_0}^{\perp\,(d)} H = {}^t b_{d+1}B_{d-1}{}^{-1}H_{d-2}. \qquad \square $$

Since ${}^t b_{2i+1}B_{2i-1}{}^{-1}(1\le 2i-1\le d-1)$ is the solution of the equation $(x_1,\cdots,x_i)B_{2i-1} = {}^t b_{2i+1}$, by Cramér formula, we have

$$ {}^t b_{2i+1}B_{2i-1}{}^{-1} = \frac{1}{|B_{2i-1}|}(P_{2i+1,1}(\lambda_1,\cdots,\lambda_{2i}),\cdots,P_{2i+1,i}(\lambda_1,\cdots,\lambda_{2i})), \tag{2.4} $$

where $P_{2i+1,j}(\lambda_1,\cdots,\lambda_{2i})(1\le j\le i)$ is the determinant replaced the j-th row of $|B_{2i-1}|$ by ${}^t b_{2i+1}$. Similarly, we have

$$ {}^t b_{2i}B_{2i-2}{}^{-1} = \frac{1}{|B_{2i-2}|}(P_{2i,1}(\lambda_1,\cdots,\lambda_{2i-1}),\cdots,P_{2i,i-1}(\lambda_1,\cdots,\lambda_{2i-1})) \quad (2\le 2i-2\le d-1), \tag{2.5} $$

where $P_{2i,j}(\lambda_1,\cdots,\lambda_{2i-1})(1\le j\le i-1)$ is the determinant replaced the j-th row of $|B_{2i-2}|$ by ${}^t b_{2i}$. Then we have the following lemma.

LEMMA 2.4. (i) *The polynomial* $P_{2i+1,j}(\lambda_1,\cdots,\lambda_{2i})(1\le j\le i)$ *is a homogeneous* *polynomial of degree* $(i^2+2i-2j+2)$ *and* $P_{2i,j}(\lambda_1,\cdots,\lambda_{2i-1})(1\le j\le i-1)$ *is a homogeneous polynomial of degree* (i^2+i-2j).

(ii) *The polynomial* $P_{2i+1,1}(\lambda_1,\cdots,\lambda_{2i})$ *is expressed as follows:*

$$P_{2i+1,1}(\lambda_1,\cdots,\lambda_{2i})=-\varepsilon_1\varepsilon_2\cdots\varepsilon_{2i}\lambda_2\lambda_4\cdots\lambda_{2i}|\boldsymbol{B}_{2i}|.$$

PROOF. (i) By (2.1), we see that $\beta_{i,j}$ is a homogeneous polynomial of degree i with variables $\lambda_1,\cdots,\lambda_i$. Hence the conclusion is directly deduced from the definitions of $P_{2i+1,j}(\lambda_1,\cdots,\lambda_{2i})$ and $P_{2i,j}(\lambda_1,\cdots,\lambda_{2i-1})$.

(ii) Define $\hat{\beta}_{j,k}(j>k\ge 1, j+k:\text{even})$ by

$$\hat{\beta}_{j,k}=\begin{cases}0 & (j>k=1)\\ \lambda_k\beta_{j-1,k-1} & (j>k>1).\end{cases}$$

Then, from (2.1), we have

$$(b_{j,k})\quad \beta_{j,k}=-\varepsilon_k\varepsilon_{k+1}\lambda_{k+1}\beta_{j-1,k+1}+\hat{\beta}_{j,k}\quad (j>k>2).$$

Also, we define a matrix C_j of type $(2,j)$ and a matrix D_j of type $(j,2)(j\ge 1)$ by

$$C_j:=\begin{pmatrix}\beta_{2j+3,1} & \beta_{2j+3,3} & \ldots & \beta_{2j+3,2j-1}\\ \beta_{2j+5,1} & \beta_{2j+5,3} & \ldots & \beta_{2j+5,2j-1}\end{pmatrix}$$

and

$$D_j:=\begin{pmatrix}0 & 0\\ \vdots & \vdots\\ 0 & 0\\ \beta_{2j+1,2j+1} & 0\end{pmatrix}.$$

Furthermore, we define matrices A_j and $\hat{A}_j(j\ge 1)$ by

$$A_1:=(\beta_{3,1}),\quad A_2:=\begin{pmatrix}\beta_{3,1} & \beta_{3,3}\\ \beta_{5,1} & \beta_{5,3}\end{pmatrix},$$

$$A_j:=\begin{pmatrix}A_{j-2} & D_{j-2}\\ C_{j-2} & \begin{pmatrix}\beta_{2j-1,2j-3} & \beta_{2j-1,2j-1}\\ \beta_{2j+1,2j-3} & \beta_{2j+1,2j-1}\end{pmatrix}\end{pmatrix}\quad (j\ge 3)$$

and

$$\hat{A}_1:=(\hat{\beta}_{3,1}),\quad \hat{A}_2:=\begin{pmatrix}\beta_{3,1} & \beta_{3,3}\\ \hat{\beta}_{5,1} & \beta_{5,3}\end{pmatrix},$$

$$\hat{A}_j:=\begin{pmatrix}A_{j-2} & D_{j-2}\\ C_{j-2} & \begin{pmatrix}\beta_{2j-1,2j-3} & \beta_{2j-1,2j-1}\\ \hat{\beta}_{2j+1,2j-3} & \beta_{2j+1,2j-1}\end{pmatrix}\end{pmatrix}\quad (j\ge 3).$$

From the definition of $P_{2j+1,1}(\lambda_1,\cdots,\lambda_{2j})$, we have

$$P_{2j+1,1}(\lambda_1,\cdots,\lambda_{2j})=(-1)^{j-1}|A_j|$$
$$=(-1)^{j-1}\begin{vmatrix} A_{j-2} & D_{j-2} \\ C_{j-2} & \begin{pmatrix} \beta_{2j-1,2j-3} & \beta_{2j-1,2j-1} \\ \beta_{2j+1,2j-3} & \beta_{2j+1,2j-1} \end{pmatrix} \end{vmatrix}.$$

Substituting $(b_{2j+1,2j-1})$ to this equality and using the linearity of the determinant for the final column, we have

$$P_{2j+1,1}(\lambda_1,\cdots,\lambda_{2j})$$
$$(2.6)\qquad =(-1)^{j-1}\left\{\begin{vmatrix} A_{j-2} & D_{j-2} \\ C_{j-2} & \begin{pmatrix} \beta_{2j-1,2j-3} & 0 \\ \beta_{2j+1,2j-3} & -\varepsilon_{2j-1}\varepsilon_{2j}\lambda_{2j}\beta_{2j,2j} \end{pmatrix} \end{vmatrix}+|\hat{A}_j|\right\}$$
$$=(-1)^{j-1}\left\{-\varepsilon_{2j-1}\varepsilon_{2j}\lambda_{2j}\beta_{2j,2j}|A_{j-1}|+|\hat{A}_j|\right\}$$
$$=\varepsilon_{2j-1}\varepsilon_{2j}\lambda_{2j}\beta_{2j,2j}P_{2j-1,1}(\lambda_1,\cdots,\lambda_{2j-2})+(-1)^{j-1}|\hat{A}_j| \quad (j\geq 2).$$

Next we shall show $|\hat{A}_j|=0 (j\geq 1)$. Clearly we have $|\hat{A}_j|=|\hat{\beta}_{3,1}|=0$. Assume that $|\hat{A}_j|=0$ for every $j\leq k$. Substituting $(b_{2k+1,2k-1}),(b_{2k+3,2k-1}),\beta_{2k+1,2k+1}=\lambda_{2k+1}\beta_{2k,2k}$ and $\hat{\beta}_{2k+3,2k+1}=\lambda_{2k+1}\beta_{2k+2,2k}$ to

$$|\hat{A}_{k+1}|=\begin{vmatrix} A_{k-1} & D_{k-1} \\ C_{k-1} & \begin{pmatrix} \beta_{2k+1,2k-1} & \beta_{2k+1,2k+1} \\ \beta_{2k+3,2k-1} & \beta_{2k+3,2k+1} \end{pmatrix} \end{vmatrix}$$

and adding $\dfrac{\varepsilon_{2k-1}\varepsilon_{2k}\lambda_{2k}}{\lambda_{2k+1}}$ multiple of the final column to the k-th column, we obtain

$$|\hat{A}_{k+1}|=\begin{vmatrix} A_{k-1} & D_{k-1} \\ C_{k-1} & \begin{pmatrix} \hat{\beta}_{2k+1,2k-1} & \beta_{2k+1,2k+1} \\ \hat{\beta}_{2k+3,2k-1} & \hat{\beta}_{2k+3,2k+1} \end{pmatrix} \end{vmatrix}.$$

Expanding this determinant with respect to the final column and using the assumption of the induction, we obtain

$$|\hat{A}_{k+1}|=-\beta_{2k+1,2k+1}\begin{vmatrix} A_{k-2} & D_{k-2} \\ \begin{pmatrix} \beta_{2k-1,1} & \cdots & \beta_{2k-1,2k-5} \\ \beta_{2k+3,1} & \cdots & \beta_{2k+3,2k-5} \end{pmatrix} & \begin{pmatrix} \beta_{2k-1,2k-3} & \beta_{2k-1,2k-1} \\ \beta_{2k+3,2k-3} & \hat{\beta}_{2k+3,2k-1} \end{pmatrix} \end{vmatrix}$$
$$+\hat{\beta}_{2k+3,2k+1}|\hat{A}_k|$$

$$= -\beta_{2k+1,2k+1}\left|\begin{matrix} \overset{A_{k-2}}{\begin{pmatrix} \beta_{2k-1,1} & \cdots & \beta_{2k-1,2k-5} \\ \beta_{2k+3,1} & \cdots & \beta_{2k+3,2k-5} \end{pmatrix}} & \overset{D_{k-2}}{\begin{pmatrix} \beta_{2k-1,2k-3} & \beta_{2k-1,2k-1} \\ \beta_{2k+3,2k-3} & \hat{\beta}_{2k+3,2k-1} \end{pmatrix}} \end{matrix}\right|.$$

By repeating the same process, we can obtain

$$\begin{aligned}
|\hat{A}_{k+1}| &= (-1)^{k-2}\beta_{7,7}\beta_{9,9}\cdots\beta_{2k+1,2k+1}\left|\begin{matrix} \overset{A_1}{\begin{pmatrix} \beta_{5,1} \\ \beta_{2k+3,1} \end{pmatrix}} & \overset{D_1}{\begin{pmatrix} \beta_{5,3} & \beta_{5,5} \\ \beta_{2k+3,3} & \hat{\beta}_{2k+3,5} \end{pmatrix}} \end{matrix}\right| \\
&= (-1)^{k-2}\beta_{7,7}\beta_{9,9}\cdots\beta_{2k+1,2k+1}\begin{vmatrix} \beta_{3,1} & \beta_{3,3} & 0 \\ \beta_{5,1} & \beta_{5,3} & \beta_{5,5} \\ \beta_{2k+3,1} & \beta_{2k+3,3} & \hat{\beta}_{2k+3,5} \end{vmatrix} \\
&= (-1)^{k-1}\beta_{5,5}\beta_{7,7}\cdots\beta_{2k+1,2k+1}\begin{vmatrix} \beta_{3,1} & \beta_{3,3} \\ \beta_{2k+3,1} & \hat{\beta}_{2k+3,3} \end{vmatrix} \\
&= (-1)^{k}\beta_{3,3}\beta_{5,5}\cdots\beta_{2k+1,2k+1}|\hat{\beta}_{2k+3,1}| \\
&= 0.
\end{aligned}$$

Thus, by the induction, we can conclude $|\hat{A}_j| = 0$ every $j \geq 1$. Substituting $|\hat{A}_j| = 0$ to (2.6), we have

$$P_{2j+1,1}(\lambda_1,\cdots,\lambda_{2j}) = \varepsilon_{2j-1}\varepsilon_{2j}\lambda_{2j}\beta_{2j,2j}P_{2j-1,1}(\lambda_1,\cdots,\lambda_{2j-2}) \quad (j \geq 2).$$

After all we can obtain

$$\begin{aligned}
&P_{2i+1,1}(\lambda_1,\cdots,\lambda_{2i}) \\
&= \varepsilon_3\varepsilon_4\cdots\varepsilon_{2i}\lambda_4\lambda_6\cdots\lambda_{2i}\beta_{4,4}\beta_{6,6}\cdots\beta_{2i,2i}P_{3,1}(\lambda_1,\lambda_2) \\
&= -\varepsilon_1\varepsilon_2\cdots\varepsilon_{2i}\lambda_2\lambda_4\cdots\lambda_{2i}\beta_{2,2}\beta_{4,4}\cdots\beta_{2i,2i} \\
&= -\varepsilon_1\varepsilon_2\cdots\varepsilon_{2i}\lambda_2\lambda_4\cdots\lambda_{2i}|B_{2i}|.
\end{aligned}$$
□

Also, we have the following lemma.

LEMMA 2.5. *(i) The normal vector field* $\nabla^{\perp\,(2i)}_{v_0}H (i \geq 1)$ *along* σ *is written as*

$$(H_{2i})\quad \nabla^{\perp\,(2i)}_{v_0}H = \sum_{j=1}^{i-1} Q_{2i,2j-1}(\lambda_1,\cdots,\lambda_{2i-2})\nabla^{\perp}_{v_{2j-1}}H + \lambda_1\lambda_2\cdots\lambda_{2i-1}\nabla^{\perp}_{v_{2j-1}}H + N_{2i}(\lambda_1,\cdots,\lambda_{2i-2}),$$

where $Q_{2i,2j-1}(\lambda_1,\cdots,\lambda_{2i-1})(1 \leq j \leq i-1)$ *is a homogeneous polynomial of degree*

$(2i-1)$ *and* $N_{2i}(\lambda_1,\cdots,\lambda_{2i-2})$ *is a normal vector field-valued polynomial of degree at most* $(2i-2)$,

(ii) The normal vector field $\nabla^{\perp\,(2i+1)}_{v_0}H(i\geq 1)$ *along* σ *is written as*

$$(H_{2i+1})\quad \nabla^{\perp\,(2i+1)}_{v_0}H=\sum_{j=0}^{i-1}Q_{2i+1,2j}(\lambda_1,\cdots,\lambda_{2i-1})\nabla^{\perp}_{v_{2j}}H+\lambda_1\lambda_2\cdots\lambda_{2i}\nabla^{\perp}_{v_{2i}}H$$
$$+N_{2i+1}(\lambda_1,\cdots,\lambda_{2i-1}),$$

where $Q_{2i+1,2j}(\lambda_1,\cdots,\lambda_{2i})(0\leq j\leq i-1)$ *is a homogeneous polynomial of degree* $2i$ *and* $N_{2i+1}(\lambda_1,\cdots,\lambda_{2i-1})$ *is a normal vector field-valued polynomial of degree at most* $(2i-1)$.

PROOF. Define a normal bundle-valued $(0,j)$-tensor field T_j on M by $T_1:=\nabla^{\perp}H$ and $T_k(X_1,\cdots,X_k):=(\overline{\nabla}_{X_1}T_{k-1})(X_2,\cdots,X_k)(k\geq 2)$ for $X_1,\cdots,X_k\in TM$, where $\overline{\nabla}$ is the connection induced from ∇ and $\nabla^{\perp}$. We shall show (H_3). By using the definition of T_j and the Frenet formula, $\nabla^{\perp\,(3)}_{v_0}H$ is rewritten in terms of T_j as follows:

$$\begin{aligned}\nabla^{\perp\,(3)}_{v_0}H&=\nabla^{\perp\,(2)}_{v_0}(T_1(v_0))=\nabla^{\perp}_{v_0}(T_2(v_0,v_0)+\lambda_1T_1(v_1))\\&=T_3(v_0,v_0,v_0)+\lambda_1T_2(v_1,v_0)+2\lambda_1T_2(v_0,v_1)\\&\quad-\varepsilon_0\varepsilon_1\lambda_1{}^2\nabla^{\perp}_{v_0}H+\lambda_1\lambda_2\nabla^{\perp}_{v_0}H\\&=Q_{3,0}(\lambda_1)\nabla^{\perp}_{v_0}H+\lambda_1\lambda_2\nabla^{\perp}_{v_0}H+N_3(\lambda_1),\end{aligned}$$

where we set $Q_{3,0}(\lambda_1):=-\varepsilon_0\varepsilon_1\lambda_1{}^2$ and $N_3(\lambda_1):=T_3(v_0,v_0,v_0)+\lambda_1T_2(v_1,v_0)+2\lambda_1T_2(v_0,v_1)$. Thus (H_3) is shown. Similarly, $(H_i)(i\geq 4)$ is also shown. □

By using these lemmas, we can prove the following theorem.

THEOREM 2.6. *Let* M *be a totally umbilical pseudo-Riemannian submanifold in* $\overline{M}$ *isometrically immersed by* f. *Assume that for every proper helix* σ *of order* d *in* M, $\overline{\sigma}(:=f\circ\sigma)$ *is a proper helix of order* d *in* $\overline{M}$, *where* d *is a positive integer. Then*

(i) *if* d *is odd, then* M *is totally geodesic,*

(ii) *if* d *is even, then* M *is an extrinsic sphere.*

PROOF. Assume that $d\geq 3$. Fix $p\in M$. For any orthonormal system $X_0,X_1,\cdots,X_{d-1}$ of T_pM and any positive numbers $\lambda_1,\cdots,\lambda_{d-1}$, there exists a proper helix σ of order d through p with the curvatures $\lambda_1,\cdots,\lambda_{d-1}$ whose Frenet frame field $v_0,v_1,\cdots,v_{d-1}$ coincide with $X_0,X_1,\cdots,X_{d-1}$ at p. Since $\overline{\sigma}(:=f\circ\sigma)$ is a proper helix of order d in M, by Lemma 2.3, we have

$$\nabla_{v_0}^{\perp\,(d-1)} H = {}^t b_d B_{d-2}{}^{-1} H_{d-3}. \tag{2.7}$$

(i) Let $d = 2i+1$. It follows from (2.4) and Lemma 2.5 that

$$\begin{aligned}\nabla_{v_0}^{\perp\,(d-1)} H &= \nabla_{v_0}^{\perp\,(2i)} H \\ &= \sum_{k=1}^{i-1} Q_{2i,2k-1}(\lambda_1,\cdots,\lambda_{2i-2}) \nabla_{v_{2k-1}}^{\perp} H + \lambda_1\lambda_2\cdots\lambda_{2i-1}\nabla_{v_{2i-1}}^{\perp} H \\ &\qquad + N_{2i}(\lambda_1,\cdots,\lambda_{2i-2})\end{aligned}$$

and

$$\begin{aligned}{}^t b_d B_{d-2}{}^{-1} H_{d-3} &= \frac{1}{|B_{d-2}|}\sum_{j=0}^{i-1} P_{d,j+1}(\lambda_1,\cdots,\lambda_{d-1})\nabla_{v_0}^{\perp\,(2j)} H \\ = \frac{1}{|B_{d-2}|}&\Bigg\{P_{d,1}(\lambda_1,\cdots,\lambda_{d-1})H + \sum_{j=1}^{i-1} P_{d,j+1}(\lambda_1,\cdots,\lambda_{d-1}) \\ &\left\{\sum_{k=1}^{j-1} Q_{2j,2k-1}(\lambda_1,\cdots,\lambda_{2j-2})\nabla_{v_{2k-1}}^{\perp} H + \lambda_1\lambda_2\cdots\lambda_{2j-1}\nabla_{v_{2j-1}}^{\perp} H + N_{2j}(\lambda_1,\cdots,\lambda_{2j-2})\right\}\Bigg\}.\end{aligned}$$

Substituting these equalities to (2.7) and noticing the point p, we have

$$\begin{aligned}&|B_{d-2}|\left\{\sum_{k=1}^{i-1} Q_{2i,2k-1}(\lambda_1,\cdots,\lambda_{2i-2})\nabla_{X_{2k-1}}^{\perp} H + \lambda_1\lambda_2\cdots\lambda_{2i-1}\nabla_{X_{2i-1}}^{\perp} H + N_{2i}(\lambda_1,\cdots,\lambda_{2i-2})\right\} \\ &= P_{d,1}(\lambda_1,\cdots,\lambda_{d-1})H + \sum_{j=1}^{i-1} P_{d,j+1}(\lambda_1,\cdots,\lambda_{d-1}) \\ &\left\{\sum_{k=1}^{j-1} Q_{2j,2k-1}(\lambda_1,\cdots,\lambda_{2j-2})\nabla_{X_{2k-1}}^{\perp} H + \lambda_1\lambda_2\cdots\lambda_{2j-1}\nabla_{X_{2j-1}}^{\perp} H + N_{2j}(\lambda_1,\cdots,\lambda_{2j-2})\right\}\end{aligned} \tag{2.8}$$

Since the degrees of $|B_{d-2}|, Q_{2j,2k-1}(\lambda_1,\cdots,\lambda_{2j-1})(j>k\ge 1)$ and $P_{d,j}(\lambda_1,\cdots,\lambda_{d-1})(j\ge 1)$ are i^2, $(2j-1)$, and $(i^2+2i-2j+2)$, respectively, the left-hand side of (2.8) is a polynomial of degree (i^2+2i-1), the first term $P_{d,1}(\lambda_1,\cdots,\lambda_{d-1})H$ of the right-hand side is of degree (i^2+2i) and other terms of the right-hand side are of degree at most (i^2+2i-1). Hence, since (2.8) holds for every positive numbers $\lambda_1,\cdots,\lambda_{2i-1}$, we obtain $P_{d,1}(\lambda_1,\cdots,\lambda_{d-1})H=0$. From Lemma 2.4-(ii), $P_{d,1}(\lambda_1,\cdots,\lambda_{d-1})\neq 0$ holds. Therefore, we see that $H=0$ at p. By the arbitrarity of $p\in M$, we see that $H\equiv 0$, that is, M is totally geodesic. In case of $d = 1$, it is directly deduced from Lemma 2.1 that so is M.

(ii) Let $d = 2i$. It follows from (2.5), (2.7) and Lemma 2.5 that

$$|\boldsymbol{B}_{d-2}|\left\{\sum_{k=0}^{i-2} Q_{2i-1,2k}(\lambda_1,\cdots,\lambda_{2i-3})\nabla^{\perp}_{X_{2k}}H+\lambda_1\lambda_2\cdots\lambda_{2i-2}\nabla^{\perp}_{X_{2i-2}}H+N_{2i-1}(\lambda_1,\cdots,\lambda_{2i-3})\right\}$$

$$(2.9)\quad = P_{d,1}(\lambda_1,\cdots,\lambda_{d-1})\nabla^{\perp}_{X_0}H+\sum_{j=1}^{i-2} P_{d,j+1}(\lambda_1,\cdots,\lambda_{d-1})$$

$$\left\{\sum_{k=0}^{j-1} Q_{2j+1,2k}(\lambda_1,\cdots,\lambda_{2j-1})\nabla^{\perp}_{X_{2k}}H+\lambda_1\lambda_2\cdots\lambda_{2j}\nabla^{\perp}_{X_{2j}}H+N_{2j+1}(\lambda_1,\cdots,\lambda_{2j-1})\right\}.$$

Since the degrees of $|\boldsymbol{B}_{d-2}|, Q_{2j+1,2k-1}(\lambda_1,\cdots,\lambda_{2j-1})(j>k\geq 0)$ and $P_{d,j}(\lambda_1,\cdots,\lambda_{d-1})$ $(j\geq 1)$ are (i^2-i), $2j$ and (i^2+i-2j), respectively, both sides of (2.9) are polynomials of degree (i^2+i-2). Hence, since (2.9) holds for every positive numbers $\lambda_1,\cdots,\lambda_{2i-2}$, terms of degree (i^2+i-2) of the both sides are mutually equal, that is,

$$|\boldsymbol{B}_{d-2}|\left\{\sum_{k=0}^{i-2} Q_{2i-1,2k}(\lambda_1,\cdots,\lambda_{2i-3})\nabla^{\perp}_{X_{2k}}H+\lambda_1\lambda_2\cdots\lambda_{2i-2}\nabla^{\perp}_{X_{2i-2}}H\right\}$$

$$= P_{d,1}(\lambda_1,\cdots,\lambda_{d-1})\nabla^{\perp}_{X_0}H+\sum_{j=1}^{i-2} P_{d,j+1}(\lambda_1,\cdots,\lambda_{d-1})$$

$$\left\{\sum_{k=0}^{j-1} Q_{2j+1,2k}(\lambda_1,\cdots,\lambda_{2j-1})\nabla^{\perp}_{X_{2k}}H+\lambda_1\lambda_2\cdots\lambda_{2j}\nabla^{\perp}_{X_{2j}}H\right\}.$$

Furthermore, since this equality holds for every orthonormal system $X_0,X_2,\cdots,X_{2i-2}$ of T_pM, we see that $|\boldsymbol{B}_{d-2}|\lambda_1\lambda_2\cdots\lambda_{2i-2}\nabla^{\perp}_{X_{2i-2}}H=0$, that is $\nabla^{\perp}_{X_{2i-2}}H=0$. By the arbitrarity of X_{2i-2}, we see that $\nabla^{\perp}H=0$ at p. Furthermore, from the arbitrarily of $p\in M, \nabla^{\perp}H\equiv 0$ is deduced. Thus M is an extrinsic sphere. In case of d = 2, it is directly deduced from Lemma 2.1 that so is M. □

In the case where M and $\overline{M}$ are Riemannian manifolds, this theorem is written as follows.

COROLLARY 2.7. *Let M be a totally umbilical submanifold in a Riemannian manifold $\overline{M}$ isometrically immersed by f. Assume that for every helix σ of order d in M, $\bar{\sigma}(:=f\circ\sigma)$ is a helix of order d in $\overline{M}$, where d is a positive integer. Then*

(i) *if d is odd, then M is totally geodesic,*
(ii) *if d is even, then M is an extrinsic sphere.*

Also, we can prove the following theorem.

THEOREM 2.8. *Let M be a totally umbilical pseudo-Riemannian submanifold in M isometrically immersed by f. Assume that for every proper helix σ of order d in M, $\bar{\sigma}(:=f\circ\sigma)$ is a proper helix of order d + 1 in $\overline{M}$, where d is a positive* *integer. Then d is odd and M is an extrinsic sphere.*

PROOF. Assume that $d \geq 2$. Fix $p \in M$. For any orthonormal system $X_0, X_1, \cdots, X_{d-1}$ of T_pM and any positive numbers $\lambda_1, \cdots, \lambda_{d-1}$, there exists a proper helix σ of order d through p with the curvatures $\lambda_1, \cdots, \lambda_{d-1}$ whose Frenet frame field $v_0, v_1, \cdots v_{d-1}$ coincide with $X_0, X_1, \cdots, X_{d-1}$ at p. Since $\bar{\sigma}(:= f \circ \sigma)$ is a proper helix of order $d+1$ in M, by Lemma 2.3, we have

$$\nabla^{\perp (d)}_{v_0} H = {}^t b_{d+1} B_{d-1}{}^{-1} H_{d-2}. \tag{2.10}$$

Suppose that d is even. Let $d = 2i$. It follows from (2.4), (2.10) and Lemma 2.5 that

$$\begin{aligned} &|B_{d-1}|\left\{\sum_{k=1}^{i-1} Q_{2i,2k-1}(\lambda_1, \cdots, \lambda_{2i-2}) \nabla^{\perp}_{X_{2k-1}} H + \lambda_1 \lambda_2 \cdots \lambda_{2i-1} \nabla^{\perp}_{X_{2i-1}} H + N_{2i}(\lambda_1, \cdots, \lambda_{2i-2})\right\} \\ &= P_{d+1,1}(\lambda_1, \cdots, \lambda_d) H + \sum_{j=1}^{i-1} P_{d+1,j+1}(\lambda_1, \cdots, \lambda_d) \\ &\left\{\sum_{k=1}^{j-1} Q_{2j,2k-1}(\lambda_1, \cdots, \lambda_{2j-2}) \nabla^{\perp}_{X_{2k-1}} H + \lambda_1 \lambda_2 \cdots \lambda_{2j-1} \nabla^{\perp}_{X_{2j-1}} H + N_{2j}(\lambda_1, \cdots, \lambda_{2j-2})\right\}. \end{aligned} \tag{2.11}$$

Since (2.11) holds for every positive numbers $\lambda_1, \cdots, \lambda_{2i-1}$, by noticing the term of the highest degree, we have $P_{d+1,1}(\lambda_1, \cdots, \lambda_d) H = 0$. From Lemma 2.4-(ii), $P_{d+1,1}(\lambda_1, \cdots, \lambda_d) \neq 0$ holds. Therefore, we obtain $H = 0$ at p. By the arbitrarity of $p \in M$, we see that $H \equiv 0$, that is, M is totally geodesic. This implies $\bar{d} = d$. Thus a contradiction results. Therefore, d is odd. Let $d = 2i + 1$. It follows from (2.5), (2.10) and Lemma 2.5 that

$$\begin{aligned} &|B_{d-1}|\left\{\sum_{k=0}^{i-1} Q_{2i+1,2k}(\lambda_1, \cdots, \lambda_{2i-1}) \nabla^{\perp}_{X_{2k}} H + \lambda_1 \lambda_2 \cdots \lambda_{2i} \nabla^{\perp}_{X_{2i}} H + N_{2i+1}(\lambda_1, \cdots, \lambda_{2i-1})\right\} \\ &= P_{d+1,1}(\lambda_1, \cdots, \lambda_d) \nabla^{\perp}_{X_0} H + \sum_{j=1}^{i-1} P_{d+1,j+1}(\lambda_1, \cdots, \lambda_d) \\ &\left\{\sum_{k=0}^{j-1} Q_{2j+1,2k}(\lambda_1, \cdots, \lambda_{2j-1}) \nabla^{\perp}_{X_{2k}} H + \lambda_1 \lambda_2 \cdots \lambda_{2j} \nabla^{\perp}_{X_{2j}} H + N_{2j+1}(\lambda_1, \cdots, \lambda_{2j-1})\right\}. \end{aligned} \tag{2.12}$$

Since (2.12) holds for every positive numbers $\lambda_1, \cdots, \lambda_{2i-1}$, by noticing terms of the highest degree, we have

$$\begin{aligned} &|B_{d-1}|\left\{\sum_{k=0}^{i-1} Q_{2i+1,2k}(\lambda_1, \cdots, \lambda_{2i-1}) \nabla^{\perp}_{X_{2k}} H + \lambda_1 \lambda_2 \cdots \lambda_{2i} \nabla^{\perp}_{X_{2i}} H\right\} \\ &= P_{d+1,1}(\lambda_1, \cdots, \lambda_d) \nabla^{\perp}_{X_0} H + \sum_{j=1}^{i-1} P_{d+1,j+1}(\lambda_1, \cdots, \lambda_d) \\ &\left\{\sum_{k=0}^{j-1} Q_{2j+1,2k}(\lambda_1, \cdots, \lambda_{2j-1}) \nabla^{\perp}_{X_{2k}} H + \lambda_1 \lambda_2 \cdots \lambda_{2j} \nabla^{\perp}_{X_{2j}} H\right\}. \end{aligned}$$

Furthermore, since this equality holds for every orthonormal system $X_0, X_2, \ldots, X_{2i}$ of T_pM, we see that $|B_{d-1}|\lambda_1\lambda_2\cdots\lambda_{2i}\nabla^{\perp}_{X_{2i}}H = 0$, that is, $\nabla^{\perp}_{X_{2i}}H = 0$. By the arbitrarity of X_{2i}, we see that $\nabla^{\perp}H = 0$ at p. Furthermore, from the arbitrarity of $p \in M, \nabla^{\perp}H \equiv 0$ is deduced. Thus M is an extrinsic sphere. In case of $d = 1$, it is directly deduced from Lemma 2.1 that so is M. □

In the case where M and $\overline{M}$ are Riemannian manifolds, this theorem is written as follows.

COROLLARY 2.9. *Let M be a totally umbilical submanifold in a Riemannian manifold M isometrically immersed by f. Assume that for every helix σ of order d in M, $\bar{\sigma}(:= f \circ \sigma)$ is a helix of order $d + 1$ in $\overline{M}$, where d is a positive integer. Then d is odd and M is an extrinsic sphere.*

§3. Proper helices in an extrinsic sphere.

Let M be an extrinsic sphere in a pseudo-Riemannian manifold $\overline{M}$ isometrically immersed by f and σ a proper helix of order d in M. We put $\bar{\sigma} := f \circ \sigma$. Assume that $\bar{\sigma}$ is a proper curve of order $\bar{d}$. Let $\lambda_1, \cdots, \lambda_{d-1}$(resp. $\bar{\lambda}_1, \cdots, \bar{\lambda}_{\bar{d}-1}$) be the curvatures of σ(resp. $\bar{\sigma}$), $v_0, \cdots, v_{d-1}$(resp. $\bar{v}_0, \ldots, \bar{v}_{\bar{d}-1}$)the Frenet frame field of σ(resp. $\bar{\sigma}$). For convenience, let $\lambda_i = 0, v_i = 0$, $\bar{\lambda}_j = 0$ and $\bar{v}_j = 0\,(i \geq d, j \geq \bar{d})$. Set $\varepsilon_i := g(v_i, v_i)$and $\bar{\varepsilon}_i := \bar{g}(\bar{v}_i, \bar{v}_i)(i \geq 0)$. Also, we define $\beta_{i,j}$ and $\bar{\beta}_{i,j}(i \geq j \geq 1, i + j : \text{even})$ as (2.1).

LEMMA 3.1. *The curve $\bar{\sigma}$ is a proper helix in $\overline{M}$ and the vector fields $v_i(i \geq 0)$ and $\bar{v}_j(j \geq 0)$ along σ are related as follows:*

(F_0') $\quad \bar{v}_0 = v_0,$

(F_1') $\quad \bar{\beta}_{1,1}\bar{v}_1 = \beta_{1,1}v_1 + \varepsilon_0 H,$

(F_{2i-1}') $\quad \sum_{j=1}^{i} \bar{\beta}_{2i-1,2j-1}\bar{v}_{2j-1} = \sum_{j=1}^{i} \beta_{2i-1,2j-1}v_{2j-1} \quad (i \geq 2),$

(F_{2i}') $\quad \sum_{j=1}^{i} \bar{\beta}_{2i,2j}\bar{v}_{2j} = \sum_{j=1}^{i} \beta_{2i,2j}v_{2j} \quad (i \geq 1).$

PROOF. From (1.1), the Frenet formulas and the assumption that M is totally umbilic, we get

$$\bar{\lambda}_1\bar{v}_1 = \overline{\nabla}_{v_0} v_0 = \nabla_{v_0} v_0 + \varepsilon_0 H = \lambda_1 v_1 + \varepsilon_0 H.$$

Thus we obtain (F_1'). Furthermore, from this equality, we get

$$\bar{\varepsilon}_1\bar{\lambda}_1^2=\varepsilon_1\lambda_1{}^2+\bar{g}(H,H).$$

Since M is an extrinsic sphere, $\bar{g}(H,H)$ is constant. Therefore, $\bar{\lambda}_1$ is constant. Operating $\overline{\nabla}_{v_0}$ to (F_1'), we get

$$\bar{\beta}_{1,1}(-\varepsilon_0\bar{\varepsilon}_1\bar{\lambda}_1 v_0+\bar{\lambda}_2\bar{v}_2)=\beta_{1,1}(-\varepsilon_0\varepsilon_1\lambda_1 v_0+\lambda_2 v_2)-\varepsilon_0\bar{g}(H,H)v_0,$$

where we use (1.1), (1.2), the Frenet formulas and the assumption that M is an extrinsic sphere. By noticing Span$\{v_0\}^{\perp}$-component of this equality, we see that

$$\bar{\lambda}_2\bar{\beta}_{1,1}\bar{v}_2=\lambda_2\beta_{1,1}v_2,$$

which implies (F_2') by (2.1). Furthermore, from this equality, we get

$$\bar{\varepsilon}_2\bar{\lambda}_2^2\bar{\beta}_{1,1}^2=\varepsilon_2\lambda_2^2\beta_{1,1}^2,$$

which implies that $\bar{\lambda}_2$ is constant. Assume that (F_{2k}') holds and $\bar{\lambda}_i(1\le i\le 2k)$ are constant. Since $\beta_{2k,2j}(resp.\,\bar{\beta}_{2k,2j})(1\le i\le k)$ are polynomials with variables $\lambda_1,\cdots,\lambda_{2k}$(resp. $\bar{\lambda}_1,\cdots,\bar{\lambda}_{2k}$), these are constant along σ. Hence, operating $\overline{\nabla}_{v_0}$ to (F_{2k}'), we have

$$\sum_{j=1}^{k}\bar{\beta}_{2k,2j}\overline{\nabla}_{v_0}\bar{v}_{2j}=\sum_{j=1}^{k}\beta_{2k,2j}\nabla_{v_0}v_{2j},$$

where we use (1.1) and the assumption that M is an extrinsic sphere. Applying the Frenet formulas and (2.1) to this equality, we obtain (F_{2k+1}'). Furthermore, from (F_{2k+1}'), we get

$$\sum_{j=1}^{k+1}\bar{\varepsilon}_{2j-1}\bar{\beta}_{2k+1,2j-1}^2=\sum_{j=1}^{k+1}\varepsilon_{2j-1}\beta_{2k+1,2j-1}^2,$$

that is,

$$\bar{\varepsilon}_{2k+1}\bar{\beta}_{2k+1,2k+1}^2=\sum_{j=1}^{k+1}\varepsilon_{2j-1}\beta_{2k+1,2j-1}^2-\sum_{j=1}^{k}\bar{\varepsilon}_{2j-1}\bar{\beta}_{2k+1,2j-1}^2. \tag{3.1}$$

Since $\beta_{2k+1,2j-1}(1\le j\le k+1)$ are polynomials with variables $\lambda_1,\cdots,\lambda_{2k+1}$ and $\bar{\beta}_{2k+1,2j-1}(1\le i\le k)$ are polynomials with variables $\bar{\lambda}_1,\cdots,\bar{\lambda}_{2k}$, these are constant along σ, that is, the right-hand side of (3.1) is constant along σ. Also, the left-hand side of (3.1) is equal to $\bar{\varepsilon}_{2k+1}\bar{\lambda}_1^2\bar{\lambda}_2^2\cdots\bar{\lambda}_{2k+1}^2$. Therefore, we see that $\bar{\lambda}_{2k+1}$ is constant. Since $\beta_{2k+1,2j-1}$ (resp. $\bar{\beta}_{2k+1,2j-1}$)$(1\le j\le k+1)$ are polynomials with variables $\lambda_1,\cdots,\lambda_{2k+1}$(resp. $\bar{\lambda}_1,\cdots,\bar{\lambda}_{2k+1}$), these are constant along σ. Hence, operating $\overline{\nabla}_{v_0}$ to (F_{2k+1}'), we have

$$\sum_{j=1}^{k+1}\overline{\beta}_{2k+1,2j-1}\overline{\nabla}_{v_0}\bar{v}_{2j-1}=\sum_{j=1}^{k+1}\beta_{2k+1,2j-1}\nabla_{v_0}v_{2j-1},$$

where we use (1.1) and the assumption that M is an extrinsic sphere. Applying the Frenet formulas and (2.1) to this equality, we obtain (F'_{2k+2}). Furthermore, from (F'_{2k+2}), we get

$$\sum_{j=1}^{k+1}\bar{\varepsilon}_{2j}\overline{\beta}^2_{2k+2,2j}=\sum_{j=1}^{k+1}\varepsilon_{2j}\beta^2_{2k+2,2j},$$

that is,

$$\bar{\varepsilon}_{2k+2}\overline{\beta}^2_{2k+2,2k+2}=\sum_{j=1}^{k+1}\varepsilon_{2j}\beta^2_{2k+2,2j}-\sum_{j=1}^{k}\bar{\varepsilon}_{2j}\overline{\beta}^2_{2k+2,2j}. \tag{3.2}$$

Since $\beta_{2k+2,2j}(1\le j\le k+1)$ are polynomials with variables $\lambda_1,\cdots,\lambda_{2k+2}$ and $\overline{\beta}_{2k+2,2j}(1\le i\le k)$ are polynomials with variables $\overline{\lambda}_1,\cdots,\overline{\lambda}_{2k+1}$, these are constant along σ, that is, the right-hand side of (3.2) is constant along σ. Also, the left-hand side of (3.2) is equal to $\bar{\varepsilon}_{2k+2}\overline{\lambda}_1^2\overline{\lambda}_2^2\cdots\overline{\lambda}^2_{2k+2}$. Therefore, we see that $\overline{\lambda}_{2k+2}$ is constant. Thus, by the induction, we see that $(F_i')(i\ge 0)$ hold and $\lambda_j(j\ge 1)$ are constant (i.e., $\bar{\sigma}$ is a proper helix). □

By using this lemma, we can prove the following theorem.

THEOREM 3.2. *Let M be an extrinsic sphere in a pseudo-Riemannian manifold $\overline{M}$ isometrically immersed by f and σ a proper helix of order d in M such that $\bar{\sigma}(:=f\circ\sigma)$ is a proper curve in $\overline{M}$, where d is a positive integer. Then*

(i) *if d is odd, then $\bar{\sigma}$ is a proper helix of order d or $d+1$,*

(ii) *if d is even, then $\bar{\sigma}$ is a proper helix of order d.*

PROOF. Let $v_i(0\le i\le d-1)$(resp. $v_i(0\le i\le \bar{d}-1)$) the Frenet frame field of σ(resp. $\bar{\sigma}$) and, for convenience, $v_i=0(i\ge d)$ and $\bar{v}_i=0(i\ge \bar{d})$. According to Lemma 3.1, $\bar{\sigma}$ is a proper helix, $\bar{v}_{2i}\in$ Span $\{v_0,v_2,\cdots,v_{2i}\}(i\ge 0)$ and $\bar{v}_{2i+1}\in$ Span $\{v_1,v_3,\cdots,v_{2i+1},H\}(i\ge 0)$. The conclusion is directly deduced from these facts. □

In the case where M and $\overline{M}$ are Riemannian manifolds, this theorem is written as follows.

COROLLARY 3.3. *Let M be an extrinsic sphere in a Riemannian manifold $\overline{M}$ isometrically immersed by f and σ a helix of order d in M, where d is a positive integer. Then*

(i) *if d is odd, then* $f \circ \sigma$ *is a helix of order* d *or* $d+1$,

(ii) *if d is even, then* $f \circ \sigma$ *is a helix of order* d.

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