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COALGEBRA ACTIONS ON AZUMAYA ALGEBRAS

By

Akira MASUOKA

Introduction.

The notion of measuring actions of coalgebras on an algebra unifies the notions of algebra automorphisms, of derivations and of higher derivations. In this paper we examine such actions of a k-coalgebra C on an Azumaya k-algebra A, where k is a commutative ring. In (2.4) we show a 1-1 correspondence between the set of measurings $C\to\operatorname{End} A$ and the set of certain right C^*-submodules of $C^*\otimes A$. Using this result, we show a Noether-Skolem type theorem (3.1): For example, *if k is a field, then any measuring $C\to\operatorname{End} A$ is inner for arbitrary C and A.*

Throughout the paper we fix a commutative ring k with 1. A linear map, an algebra, a coalgebra, $\otimes$, Hom and End mean a k-linear map, a k-algebra, a k-algebra, a k-coalgebra, $\otimes_k$, Hom_k and End_k, respectively. We fix an algebra A and a coalgebra C. C^* denotes $\operatorname{Hom}(C, k)$, the dual algebra of C [**9**, Prop. 1.1.1, p. 9].

1. Preliminaries.

Let Δ, ε be the structure maps of C and write

$$\Delta(c)=\sum_{(c)} c_{(1)}\otimes c_{(2)} \qquad \text{for } c\in C.$$

The k-module $\operatorname{Hom}(C, A)$ is an algebra with the $*$-product [**9**, p. 69]. $\operatorname{Hom}(C, A)^{\times}$ denotes the group of units in $\operatorname{Hom}(C, A)$.

1.1. DEFINITION. A linear map $f: C\to\operatorname{End} A$ is called a *measuring*, if $a\mapsto(c\mapsto f(c)(a))$, $A\to\operatorname{Hom}(C, A)$ is an algebra map, or equivalently if

$$f(c)(1)=\varepsilon(c)1,$$

$$f(c)(ab)=\sum_{(c)} f(c_{(1)})(a)f(c_{(2)})(b)$$

for $c\in C$, $a, b\in A$ [**9**, Def. p. 138]. We denote by

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$$\mathrm{Meas}(C, \mathrm{End}\, A)$$

the set of measurings $C \to \mathrm{End}\, A$.

For any $u \in \mathrm{Hom}(C, A)^{\times}$, the linear map $inn\, u : C \to \mathrm{End}\, A$ determined by

$$inn\, u(c)(a) = \sum_{(c)} u(c_{(1)}) a u^{-1}(c_{(2)}) \qquad c \in C,\ a \in A \tag{1.2}$$

is a measuring. Thus we have a map

$$inn : \mathrm{Hom}(C, A)^{\times} \longrightarrow \mathrm{Meas}(C, \mathrm{End}\, A). \tag{1.3}$$

1.4. DEFINITION (cf. [2, Def. 1.2, p. 674]). We write

$$\mathrm{Inn}(C, \mathrm{End}\, A) = \text{the image of } inn$$

and call an element of this set an *inner measuring*.

2. A 1-1 correspondence.

Throughout this section, let A be an Azumaya algebra [6, p. 95]. Thus A is a progenerator k-module and

$$A \otimes A \simeq \mathrm{End}\, A \qquad \text{via } a \otimes b \mapsto (x \mapsto axb). \tag{2.1}$$

Let D be an arbitrary algebra. $\mathrm{Alg}(A, D \otimes A)$ denotes the set of algebra maps $A \to D \otimes A$.

2.2. DEFINITION. $\mathbf{I}(D \otimes A)$ denotes the set of right D-submodules I of $D \otimes A$ such that

$$\kappa : I \otimes A \longrightarrow D \otimes A, \qquad \kappa(x \otimes a) = x(1 \otimes a)$$

is an isomorphism.

2.3. PROPOSITION. *Let A, D be as above.*

(1) *Let $f \in \mathrm{Alg}(A, D \otimes A)$ and define*

$$I_f = \{x \in D \otimes A \mid f(a)x = x(1 \otimes a) \quad \text{for all } a \in A\}.$$

Then $I_f \in \mathbf{I}(D \otimes A)$.

(2) *Let $I \in \mathbf{I}(D \otimes A)$ and suppose $\kappa^{-1}(1 \otimes 1) = \Sigma_i x_i \otimes a_i$. Define $f_I \in \mathrm{Hom}(A, D \otimes A)$ by*

$$f_I(a) = \Sigma_i x_i (1 \otimes a a_i), \qquad a \in A.$$

Then f_I is an algebra map.

(3) *$f \mapsto I_f$ and $I \mapsto f_I$ establish a 1-1 correspondence between $\mathrm{Alg}(A, D \otimes A)$ and $\mathbf{I}(D \otimes A)$.*

PROOF. We modify the proof of [6, Prop. 1.2, p. 107].

Let ${}_f(D\otimes A)$ denote the k-module $D\otimes A$ with the twisted A-bimodule structure represented by

$$A\otimes A \xrightarrow{f\otimes 1} D\otimes A\otimes A \xrightarrow{1\otimes(2.1)} D\otimes \mathrm{End}\, A \subset \mathrm{End}\,(D\otimes A).$$

Then I_f is identified with the A-centralizer of ${}_f(D\otimes A)$. This, together with [**6**, Cor. 5.3, p. 95], implies $I_f\in\mathbf{I}(D\otimes A)$.

f_I coincides with the composition of algebra maps

$$A \longrightarrow \mathrm{End}_{-D\otimes A}(I\otimes A) \xrightarrow{\simeq} \mathrm{End}_{-D\otimes A}(D\otimes A)=D\otimes A\,,$$

where the first map is $a\mapsto(x\otimes b\mapsto x\otimes ab)$ and the second is $g\mapsto\kappa\circ g\circ\kappa^{-1}$. This is a unique algebra map making $\kappa: I\otimes A\simeq {}_{f_I}(D\otimes A)$ into an A-bimodule isomorphism, so we have

$$f=f_{I_f}, \qquad I=I_{f_I}. \qquad\qquad \text{Q.E.D.}$$

2.4. Theorem. *Let A be an Azumaya algebra, let C be a coalgebra and let $D=C^*$.*

(1) *There is a 1-1 correspondence between* Meas$(C, \mathrm{End}\, A)$ *and* $\mathbf{I}(D\otimes A)$, *which is given by $f\mapsto I_f$, $I\mapsto f_I$ in* (2.3) *through the natural identification*

(2.5) $$\mathrm{Meas}(C, \mathrm{End}\, A)=\mathrm{Alg}(A, D\otimes A).$$

(2) *If $f\mapsto I$ in* (1), *then f is inner if and only if $I\simeq D$ as right D-modules.*

Proof. (1) By definition (1.1) we have Meas$(C, \mathrm{End}\, A)=\mathrm{Alg}(A, \mathrm{Hom}(C, A))$ by adjointness. Since A is a finitely generated projective k-module, we have $D\otimes A=\mathrm{Hom}(C, A)$. Thus we have (2.5). Then part (1) follows from (2.3) immediately.

(2) We have the correspondences

$$\begin{aligned} inn\, u &\longleftrightarrow (a\mapsto u(1\otimes a)u^{-1}) && \text{in (2.5)} \\ &\longleftrightarrow uD && \text{in (2.3)(3)} \end{aligned}$$

for $h\in(D\otimes A)^{\times}$. If $h: D\to I$, $I\in\mathbf{I}(D\otimes A)$, is a right D-module isomorphism with $u=h(1)$ (so $I=uD$), then $u\in(D\otimes A)^{\times}$, since we have the right $D\otimes A$-module isomorphism

$$D\otimes A=D\otimes_D(D\otimes A) \xrightarrow[h\otimes 1]{\simeq} I\otimes_D(D\otimes A) \xrightarrow[\kappa]{\simeq} D\otimes A$$

sending $1\otimes 1$ to u. Thus part (2) follows. Q.E.D.

2.6. Fact. *Let A, C, D be as in* (2.4). *Suppose C is cocommutative. Then*:

(1) Meas$(C, \mathrm{End}\, A)$ *forms a group with respect to the $*$-product.*

(2) $f \mapsto I_f$ *in* (2.3) *induces an exact sequence of groups*

$$1 \longrightarrow \mathrm{Inn}(C, \mathrm{End}\, A) \longrightarrow \mathrm{Meas}(C, \mathrm{End}\, A) \xrightarrow{\phi} \mathrm{Pic}(D)$$

and

$$\mathrm{Im}\, \phi = \{I \in \mathrm{Pic}(D) \mid I \otimes A \simeq D \otimes A \text{ as right or left } D \otimes A\text{-modules}\},$$

where Pic(D) *is the Picard group of* D.

PROOF. As is easily verified, if C is cocommutative (so D is commutative), then Meas$(C$, End $A)$ is a sub-monoid of Hom$(C$, End $A)$ and the natural bijection

$$\mathrm{Meas}(C, \mathrm{End}\, A) = \mathrm{Alg}(A, D \otimes A) \simeq \mathrm{End}_{D\text{-Alg}}(D \otimes A)$$

is a monoid isomorphism. Moreover since $D \otimes A$ is an Azumaya D-algebra, the assertions follow from [**6**, Cor. 5.4, p. 95 and Prop. 1.2, p. 107]. Q. E. D.

3. A Noether-Skolem theorem.

3.1. THEOREM. *Let* C *be a coalgebra and let* $D = C^*$. *Then any measuring* $C \rightarrow$End A *is inner for an arbitrary Azumaya algebra* A, *if either*

(a) *C is cocommutative and the Picard group* Pic(D) *of D is trivial,*

(b) *k, the base ring, is artinian and C is a finitely generated k-module, or*

(c) *k is a field (and C is arbitrary).*

PROOF in case (a). This follows from (2.6).

PROOF in case (b). By (2.4) we have only to show each $I \in \mathbf{I}(D \otimes A)$ is isomorphic to D as a right D-module. Multiplying a primitive idempotent, we may assume k is local artinian. Then A is a free k-module of finite rank, say n. We have

$$I^n \simeq I \otimes A \simeq D \otimes A \simeq D^n$$

as right D-modules, where $(\ )^n$ means the direct sum of n copies of $(\ )$. Since D is right artinian, we can apply the Krull-Schmidt theorem to have $I \simeq D$.

Q. E. D.

More generally, the conclusion of (3.1) holds true, if k is the direct product $\prod k_i$ of finitely many commutative rings k_i such that all finitely generated projective k_i-modules are free and if each Dk_i is contained in the class $\mathfrak{R}$ defined as follows. Let $\mathfrak{R}$ be the class of rings R with 1 satisfying: *A right R-module M is isomorphic to R, if there exists $n \geqq 1$ such that $M^n \simeq R^n$ as right R-modules.* All right artinian rings are contained in $\mathfrak{R}$.

3.2. LEMMA. (1) *If* $R/\mathrm{Rad}\,R\in\mathfrak{R}$, *then* $R\in\mathfrak{R}$, *where* Rad R *is the Jacobson radical of* R.

(2) $\mathfrak{R}$ *is closed under possibly infinite direct products.*

PROOF. (1) This follows from [**1**, (2.12) Prop., p. 90].

(2) Let $R=\prod R_\lambda$. Suppose $M^n\simeq R^n$. Then $M\simeq\prod MR_\lambda$, since so is $M^n=R^n$. Suppose $R_\lambda\in\mathfrak{R}$ for all λ. Then $MR_\lambda\simeq R_\lambda$, since $M^n\simeq R^n$ implies $(MR_\lambda)^n\simeq R_\lambda^n$. Thus we have

$$M\simeq\prod MR_\lambda\simeq\prod R_\lambda=R$$

as right R-modules. Hence $R\in\mathfrak{R}$. Q. E. D.

PROOF in case (c). By (3.2)(1), it is enough to show $D/\mathrm{Rad}\,D\in\mathfrak{R}$. By [**5**, 2.1.5. Prop. (a), p. 224], $D/\mathrm{Rad}\,D\simeq C_0^*$, where C_0 is the coradical [**9**, Def., p. 181] of C. Since C_0^* is a direct product of finite dimensional (simple) algebras [**5**, p. 223], $D/\mathrm{Rad}\,D=C_0^*\in\mathfrak{R}$ by (3.2)(2). Q. E. D.

3.3. REMARKS. (1) Sweedler [8, Thm. 9.5, p. 236] extended the classical results of Noether-Skolem and of Jacobson to Hopf algebra actions. His result cannot be covered by ours, unless $D=B$ in the notation of [8].

(2) Blattner and Montgomery [**3**, Thm. 2.15] prove a Noether-Skolem theorem for Hopf-Galois extensions, generalizing [7, Thm. 6]. Their result follows immediately from (3.1)(c), since, in their notation, an action of H on B trivial on Z gives rise to a Z-linear measuring $Z\otimes H\to\mathrm{End}_Z B$.

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Institute of Mathematics
University of Tsukuba
Ibaraki, 305 Japan

Current address:
Department of Mathematics
Nippon Institute of Tachnology
4-1 Gakuen-Dai, Miyashiro-Machi
Minami-Saitama-Gun, Saitama 345
JAPAN