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REGULAR GAMMA RINGS

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0. Introduction

Let M and Γ be additive abelian groups. If for all $a,b,c\in M$ and $\alpha,\beta,\gamma\in\Gamma$, the conditions

(1) $a\alpha b\in M$, $\alpha a\beta\in\Gamma$,

(2) $(a+b)\alpha c=a\alpha c+b\alpha c$, $a(\alpha+\beta)b=a\alpha b+a\beta b$, $a\alpha(b+c)=a\alpha b+a\alpha c$,
$(\alpha+\beta)a\gamma=\alpha a\gamma+\beta a\gamma$, $\alpha(a+b)\beta=\alpha a\beta+\alpha b\beta$, $\alpha a(\beta+\gamma)=\alpha a\beta+\alpha a\gamma$,

(3) $(a\alpha b)\beta c=a(\alpha b\beta)c=a\alpha(b\beta c)$, $(\alpha a\beta)b\gamma=\alpha(a\beta b)\gamma=\alpha a(\beta b\gamma)$,

are satisfied, then M is called a *weak gamma ring in the sense of Nobusawa* and denoted by $(\Gamma, M)_{wN}$.

In this note (Γ, M) denotes $(\Gamma, M)_{wN}$, unless otherwise specified.

A gamma ring (Γ, M) is regular if for each $a\in M$ there exists $\delta\in\Gamma$ such that $a\delta a=a$. For a left R-module M, letting $\Gamma=\mathrm{Hom}_R(M, R)$, we have a gamma ring (Γ, M). A left R-module M is called regular, if for any element $m\in M$ there exists $f\in\mathrm{Hom}_R(M, R)$ with $(mf)m=m$, [8]. Thus, the concept of regular gamma rings is a natural generalization of regular modules.

In this note, we study various properties of regular gamma rings. In 1, we obtain a couple of necessary and sufficient conditions that (Γ, M) is regular, and then characterize a commutative regular Nobusawa gamma ring as a subdirect sum of gamma fields (Th. 1.7).

In 2, we define a regular ideal and prove a basic theorem: If $J\subseteq K$ are two ideals in M, then, K is regular if and only if J and K/J are both regular (Th. 2.2). If $\mathcal{R}$ is the class of all regular gamma rings, then this theorem shows that $\mathcal{R}$ is a radical class. Next, we introduce the concept of a weakly nilpotent element, and we obtain that a non-zero subdirectly irreducible regular gamma ring with no non-zero weakly nilpotent elements is a division gamma ring (Th. 2.11).

In 3, we obtain relations among the regularities of the operator rings L, R and a gamma ring (Γ, M) as follows: If (Γ, M) has the left and right unities, then the following conditions are equivalent: (1) L is regular; (2) R is regular; (3) M is regular (Th. 3.2). By this theorem, we have that, when Mod-$R\approx$

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Mod-L, R is regular if and only if L is regular (Corollary 3.5). Furthermore, we show that if (Γ, M) is a semi-prime gamma ring with min-r and min-l conditions, every left (right) L-module and every left (right) R-module are regular. In particular, L, M and R are regular (Th. 3.8).

In 4, we consider the regularity of a Morita context (Q, R, S, T, μ, ν), where μ, ν are surjective. Here, it is not assumed that Q, R have unities nor that S, T are unital. We obtain an extension (Th. 4.1) of Theorem 3.2.

For the definitions of the following basic notions in gamma rings we refer, respectively, to [3] for the right operator ring R, the left operator ring L, a right (left, two-sided) ideal of M, $|a>$, $[N, \Phi]$, where $N\subseteq M$ and $\Phi\subseteq\Gamma$ and to [4] for semiprime ideals, nilpotent elements, the right unity and the left unity.

1. Regular Gamma Rings.

1.1 DEFINITION. A gamma ring (Γ, M) is *regular* if for each $x\in M$ there exists $\delta\in\Gamma$ such that $x\delta x=x$. We abbreviate this as M is regular, when Γ is understood.

1.2 THEOREM. *For a gamma ring (Γ, M) with the left and right unities, the following conditions are equivalent*:

(1) *(Γ, M) is regular.*

(2) *Every principal right ideal of M is generated by an idempotent of the left operator ring L.*

(2′) *Every principal left ideal of M is generated by an idempotent of the right operator ring R.*

(3) *Every finitely generated right ideal of M is generated by an idempotent of the left operator ring L.*

(3′) *Every finitely generated left ideal of M is generated by an idempotent of the right operator ring R.*

PROOF. We note that for any $a\in M$ $|a>=a\Gamma M$, since $|a>=Za+a\Gamma M\subseteq a\Gamma M$. ($Z$ is the set of all integers.)

(1)⇒(2): Suppose that for each $a\in M$ there exists $\delta\in\Gamma$ such that $a\delta a=a$. Then $[a, \delta][a, \delta]=[a, \delta]$ and so $[a, \delta]$ is an idempotent in L. Since $a\Gamma M=a\delta a\Gamma M\subseteq a\delta M$, $a\Gamma M=a\delta M$. Thus, $|a>=a\delta M$.

(2)⇒(3): It suffices to show that for any $a, b\in M$, $|a>+|b>=tM$, where t is an idempotent in L. By (2), $|a>=hM$, $h^2=h\in L$, and $|b>=fM$ where $f^2=f\in L$. Then, since $b-hb\in fM+hM$, $|b-hb>\subseteq fM+hM$, and so $hM+|b-hb>\subseteq hM+fM$. On the other hand, $b=hb+b-hb\in hM+|b-hb>$, whence $fM=|b>\subseteq hM+|b-hb>$. Thus, $hM+fM\subseteq hM+|b-hb>$. Therefore, $hM+fM=hM+$

$|b-hb>$. Again by (2) $|b-hb>=sM$, where $s^2=s\in L$. Then, $hsM=h|b-hb>=0$, and it follows that $hs=hs^2=0$. So if $g=s-sh$, then g is an idempotent and orthogonal to h. Since $sg=g$ and $gs=s$, we see that $gM=sM=|b-hb>$. Therefore, $|a>+|b>=hM+gM$. Since h and g are orthogonal, we have $|a>+|b>=(h+g)M$.

(3)$\Rightarrow$(1): Suppose that for any $x\in M$, $|x>=hM$, where $h^2=h\in L$. Then, $x=hy=h^2y=h(hy)=hx$, where $y\in M$. On the other hand, $hL=[hM, \Gamma]=[|x>, \Gamma]=[Zx+x\Gamma M, \Gamma]\subseteq[x, \Gamma]$, which implies $h=h^2=[x, \delta]$, where $\delta\in\Gamma$. Hence $x=hx=[x, \delta]x=x\delta x$. □

1.3 DEFINITION. A gamma ring (Γ, M) is *right semi-hereditary* if every finitely generated right ideal of M is a projective R-module. A right ideal I in M is called *essential* if for every non-zero right ideal A in M, $I\cap A\neq 0$. Let $\varphi(M)$ be the set of all essential right ideals in M, and $Z_r(M)=\{x\in M|x\Gamma I=0$ for some $I\in\varphi(M)\}$. (Γ, M) is called a *right nonsingular* gamma ring if $Z_r(M)=0$. Similarly, a left semi-hereditary gamma ring and a left nonsingular gamma ring are defined.

1.4 COROLLARY. *Let (Γ, M) be a regular gamma ring. Then*

(1) *All one-sided ideals in M are idempotent.*

(2) *All two-sided ideals in M are semi-prime.*

(3) *The Jacobson radical of M is zero.*

(4) *(Γ, M) with the left and right unities is right and left semi-hereditary.*

(5) *(Γ, M) is right and left nonsingular.*

PROOF. Let J be a right ideal of M. Since M is regular, for each $x\in J$ $x\gamma x=x$ for some $\gamma\in\Gamma$. Consequently, $x=x\gamma x\in J\Gamma J$ and so $J=J\Gamma J$. Thus, we have (1).

Let I be a two-sided ideal of M. If A is a two-sided ideal in M such that $A\Gamma A\subseteq I$, then $A\subseteq I$, because by (1) $A=A\Gamma A$. Hence we have (2).

To show (3), suppose that e is right quasi-regular and $e=e\delta e$. Then, there exists $r\in R$ such that $[\delta, e]\circ r=r+[\delta, e]-[\delta, e]r=0$. It follows $[\delta, e]=[\delta, e]\circ 0=[\delta, e]\circ([\delta, e]\circ r)=([\delta, e]\circ[\delta, e])\circ r=[\delta, e]\circ r=0$. Thus, $e=e\delta e=e[\delta, e]=e0=0$. Recall that $J(M)=\{e\in M|\langle e\rangle$ is right quasi-reqular$\}$. Since $\langle e\rangle=0$, $e=0$ and so $J(M)=0$.

Now we prove (4). By Theorem 1.2.(3), every finitely generated right ideal in M may be written as hM, where $h^2=h\in L$. Let $A=\{x\in M|hx=0\}$. Clearly A is a right ideal in M. For any $x\in M$, $x=hx+(x-hx)$, and $M=hM\oplus A$, because if $a\in hM\cap A$ then $a=ha=0$. Thus, hM is a direct summand of M and so every finitely generated right ideal in M is a projective R-module. Similarly it can be proved that (Γ, M) is left semi-hereditary.

For (5), let J be an essential right ideal in M. Suppose that $a\Gamma J=0$ for some $a\in M$, and that there exists $\delta\in\Gamma$ such that $a\delta a=a$. Then, $a\delta M\cap J=0$, for if $x\in a\delta M\cap J$ then $x=a\delta x=0$. Since J is essential, $a\delta M=0$ and so $a=0$. Similarly we obtain the same result for left ideals. □

Given an ideal I in M, we form a residue class gamma ring $(\Gamma/I^*, M/I)$, where $I^*=\{\gamma\in\Gamma|M\gamma M\subseteq I\}$.

1.5 THEOREM. *A gamma ring (Γ, M) is regular if and only if the following* (1), (2) *and* (3) *hold.*

(1) *M is semi-prime,*

(2) *The union of any chain of semi-prime ideals of M is semi-prime,*

(3) *M/P are regular for all prime ideals P of M.*

PROOF. Let M be regular. Corollary 1.4 (2) shows that all ideals in M are semi-prime, whence (1) and (2) hold. (3) obviously holds, for, $(x+P)(\gamma+P^*)(x+P)=x\gamma x+P=x+P$.

Conversely, assume that (1), (2) and (3) hold. If M is not regular, then there is $a\in M$ such that $a\notin a\Gamma a$. By (2), there is a semi-prime ideal I in M which is maximal among semi-prime ideals such that $a\notin a\Gamma a+I$. Note that $\{0\}$ is a semi-prime ideal of M such that $a\notin a\Gamma a+\{0\}$. M/I is not regular, because otherwise, for any $x\in M$, $(x+I)(\gamma+I^*)(x+I)=x+I$ would imply $x\in x\Gamma x+I$, a contradiction. Hence, by (3) I is not prime. Thus, there are ideals A and B which properly contain I and $A\Gamma B\subseteq I$. Indeed, since $A\nsubseteq I$ and $B\nsubseteq I$, $I\subsetneq A+I$ and $I\subsetneq B+I$. If we set $A+I=A'$ and $B+I=B'$, then $A'\Gamma B'=A\Gamma B+I\subseteq I+I=I$ and $I\subsetneq A'$ and $I\subsetneq B'$. Thus, we can take A, B instead of A', B' from the beginning. Now set $P=\{x\in M|x\Gamma B\subseteq I\}$ and $Q=\{x\in M|P\Gamma x\subseteq I\}$. Since I is semi-prime, P and Q are semi-prime. For, $K\Gamma K\subseteq P\Rightarrow K\Gamma K\Gamma B\subseteq I\Rightarrow K\Gamma B\Gamma K\Gamma B\subseteq K\Gamma K\Gamma B\subseteq I\Rightarrow K\Gamma B\subseteq I\Rightarrow K\subseteq P$, and $U\Gamma U\subseteq Q\Rightarrow P\Gamma U\Gamma U\subseteq I\Rightarrow P\Gamma U\Gamma P\Gamma U\subseteq P\Gamma U\Gamma U\subseteq I\Rightarrow P\Gamma U\subseteq I\Rightarrow U\subseteq Q$.

Since $(P\cap Q)\Gamma(P\cap Q)\subseteq P\Gamma Q\subseteq I$, we have $P\cap Q\subseteq I$. Clearly, $A\subseteq P$ and $B\subseteq Q$, and hence P and Q properly contain I. By the maximality of I, there exist elements γ, $\omega\in\Gamma$ such that $a-a\gamma a\in P$ and $a-a\omega a\in Q$. Then, $a-a(\gamma+\omega-\gamma a\omega)a=a-a\gamma a-(a-a\gamma a)\omega a\in P$. Also $a-a(\gamma+\omega-\gamma a\omega)a=a-a\omega a-a\gamma(a-a\omega a)\in Q$. It follows that $a\in a\Gamma a+P\cap Q\subseteq a\Gamma a+I$, which is a contradiction. Hence, M is regular. □

1.6 COROLLARY. *A gamma ring (Γ, M) is regular if and only if all ideals* *of M are idempotent and M/P are regular for all prime ideals P of M.*

PROOF. If all ideals of M are idempotent, all ideals of M are semi-prime. □

1.7 THEOREM. *A commutative regular Nobusawa gamma ring with more than one element is a subdirect sum of gamma fields.*

PROOF. A regular gamma ring has no non-zero nilpotent elements. For, suppose $(a\gamma)^n a=0$ for any $\gamma\in\Gamma$. Then we have $a=(a\delta)^m a=0$ since there exists $\delta\in\Gamma$ such that $a=a\delta a$. A homomorhpic image of a regular gamma ring is regular, and so it has no non-zero nilpotent elements. Then, the theorem follows immediately from Theorems 3 and 4 in [5]. □

2. Regular Ideals

2.1 DEFINITION. A two-sided ideal J in M is *regular* if for each $x\in J$ there exists $\gamma\in J^*$ such that $x\gamma x=x$, where $J^*=\{\gamma\in\Gamma|M\gamma M\subseteq J\}$.

2.2 THEOREM. *Let $J\subseteq K$ be two-sided ideals in M. Then K is regular if and only if J and K/J are both regular.*

PROOF. Let $J^*=\{\gamma\in\Gamma|M\gamma M\subseteq J\}$ and $K^*=\{\gamma\in\Gamma|M\gamma M\subseteq K\}$. Then (J^*, J), (K^*, K) and $(K^*/J^*, K/J)$ are gamma rings. Suppose that K is regular. For each $k\in K$ there exists $\gamma\in K^*$ such that $k\gamma k=k$. Thus, $(k+J)(\gamma+J^*)(k+J)=k\gamma k+J=k+J$ and so K/J is regular.

Given $x\in J$, we have $x\delta x=x$ for some $\delta\in K^*$, since $J\subseteq K$. Then, $\omega=\delta x\delta\in J^*$, for $M\omega M=M\delta x\delta M\subseteq J\delta M\subseteq J$. Hence, $x\omega x=x\delta x\delta x=x\delta x=x$, and so J is regular.

Conversely, assume that J and K/J are both regular. For a given $a\in K$, $a+J=(a+J)(\gamma+J^*)(a+J)=a\gamma a+J$, where $\gamma\in K^*$ from the regularity of K/J. Hence, $a-a\gamma a\in J$, for some $\gamma\in K^*$. Consequently, $a-a\gamma a=(a-a\gamma a)\omega(a-a\gamma a)$, where $\omega\in J^*$. Then,

$$\begin{aligned}
a&=a-a\gamma a+a\gamma a\\
&=(a-a\gamma a)\omega(a-a\gamma a)+a\gamma a\\
&=a(\omega-\gamma a\omega)(a-a\gamma a)+a\gamma a\\
&=a(\omega-\gamma a\omega-\omega a\gamma+\gamma a\omega a\gamma)a+a\gamma a\\
&=a(\omega-\gamma a\omega-\omega a\gamma+\gamma a\omega a\gamma+\gamma)a\\
&=a\lambda a,\ \text{where } \lambda=\omega-\gamma a\omega-\omega a\gamma+\gamma a\omega a\gamma+\gamma\in K^*,
\end{aligned}$$

because $J^*\subseteq K^*$ and K^* is an ideal in Γ.

Therefore, K is regular. □

2.3 REMARK. Let $\mathcal{R}$ be the class of all regular gamma rings. Theorem 2.2 shows that $\mathcal{R}$ is a radical class, since other two conditions: $\mathcal{R}$ is homomorphically

closed and $\mathscr{R}$ has the inductive property are trivially satisfied.
(See, for instance, [7]) In fact, a radical N for any gamma ring (Γ, M) may be defined by the conditions in Proposition 2.6.

2.4 PROPOSITION. *Any finite subdirect sum of regular Nobusawa gamma rings is regular.*

PROOF. It suffices to show that a subdirect sum of two regular Nobusawa gamma rings is regular. Suppose that M has two ideals J and K such that $J\cap K=0$. Then $J^*\cap K^*=0$, where $J^*=\{\gamma\in\Gamma|M\gamma M\subseteq J\}$ and $K^*=\{\gamma\in\Gamma|M\gamma M\subseteq K\}$. For, if $\gamma\in J^*\cap K^*$, then $M\gamma M\subseteq J\cap K=0$ and $\gamma=0$. Let the gamma rings $(\Gamma/J^*, M/J)$ and $(\Gamma/K^*, M/K)$ be both regular. Consider the homomorhpism

$$(\varphi, \theta): \quad (J^*, J)\to(J^*+K^*/K^*, J+K/K)$$

where

θ is the natural epimorphism: $J\to J+K/K$, $x\theta=x+K$ and Ker $\theta=J\cap K=0$,
φ is the natural epimorphism: $J^*\to J^*+K^*/K^*$, $\alpha\varphi=\alpha+K^*$ and Ker $\varphi=J^*\cap K^*=0$.

Then
$(x\alpha y)\theta=x\alpha y+K=(x+K)(\alpha+K^*)(y+K)=x\theta\alpha\varphi y\theta$, and $(\alpha x\beta)\varphi=\alpha x\beta+K^*=(\alpha+K^*)(x+K)(\beta+K^*)=\alpha\varphi x\theta\beta\varphi$. Hence, (φ, θ) is an isomorphism from (J^*, J) onto $(J^*+K^*/K^*, J+K/K)$. Since $J+K/K$ is an ideal in M/K, $J+K/K$ is regular. Theorem 2.2 shows J is regular. Hence, J and M/J are regular, and again by Theorem 2.2 M is regular. □

2.5 REMARK. A subdirect sum of infinitely many regular Nobusawa gamma rings need not be regular. For example, (Z, Z) is the subdirect sum of infinitely many regular Nobusawa gamma rings $(Z/(p), Z/(p))$, where p runs through all prime numbers.

2.6 PROPOSITION. *For a gamma ring* (Γ, M), *set* $N=\{x\in M|\langle x\rangle$ *is regular*$\}$. *Then,*

(1) *N is a regular ideal in M,*
(2) *N contains all regular ideals of M,*
(3) *M/N has no non-zero regular ideals.*

PROOF. Let $x, y\in N$. Then $\langle y\rangle$ is regular and $\langle x\rangle+\langle y\rangle/\langle y\rangle$ is regular. Hence by Theorem 2.2 $\langle x\rangle+\langle y\rangle$ is regular. For any $a\in\langle x\rangle+\langle y\rangle$, $\langle a\rangle\subseteq\langle x\rangle+\langle y\rangle$. Theorem 2.2 shows $\langle a\rangle$ is regular, and so $a\in N$. Thus, $\langle x\rangle+\langle y\rangle\subseteq N$, whence N is an ideal in M. For any $x\in N$, since $\langle x\rangle$ is regular, there exists

$\delta\in\langle x\rangle^*$, where $\langle x\rangle^*=\{\gamma\in\Gamma|M\gamma M\subseteq\langle x\rangle\}$, such that $x\delta x=x$. Since N is an ideal. $\langle x\rangle\subseteq N$ and then $\langle x\rangle^*\subseteq N^*$. Thus, $\delta\in N^*$ and N is regular. This completes the proof of (1).

To prove (2), let A be any regular ideal in M. For any $a\in A$, $\langle a\rangle\subseteq A$. Thus, by Theorem 2.2, $\langle a\rangle$ is regular and so $a\in N$. Hence $A\subseteq N$.

If A/N is a non-zero regular ideal in M/N, A is regular by Theorem 2.2, and A contains N properly, which contradicts to (2). □

2.7 DEFINITION. An element $a\in M$ is said to be a *weakly nilpotent element* if there exist a non-zero element $\gamma\in\Gamma$ and an integer $n>1$ such that $(a\gamma)^{n-1}a=0$.

2.8 PROPOSITION. *In a gamma ring (Γ, M) with no non-zero weakly nilpotent elements, every idempotent commutes with every element in M.*

PROOF. Let $e\delta e=e$, $\delta\in\Gamma$, and $x\in M$. If $e=0$, $e\delta x=0=x\delta e$. Suppose $e\neq 0$. Then $\delta\neq 0$. Since $(e\delta x-e\delta x\delta e)\delta(e\delta x-e\delta x\delta e)=(e\delta x\delta e-e\delta x\delta e)([\delta, x]-[\delta, x\delta e])=0$ and (Γ, M) has no non-zero weakly nilpotent elements, $e\delta x-e\delta x\delta e=0$ or $e\delta x=ex\delta e$. Similarly, $x\delta e=e\delta x\delta e$, and so $e\delta x=x\delta e$. □

2.9 PROPOSITION. *Let (Γ, M) be a regular gamma ring with no non-zero weakly nilpotent elements. Then*

(1) *Every principal one-sided ideal is generated by an idempotent which commutes with any element in M.*

(2) *Every one-sided ideal is a two-sided ideal.*

PROOF. Let $a=a\delta a$ for some $\delta\in\Gamma$. Then, $|a>=Za+a\Gamma M=a[\delta, Za]+a\Gamma M=a\Gamma M=a\delta a\Gamma M\subseteq a\delta M$, and hence $|a>=a\delta M$. Proposition 2.8 shows that a commutes with any element in M. Thus we have (1).

To prove (2), let A be a right ideal in M. For any $a\in A$, $a\delta M\subseteq A$, where $a\delta a=a$ for some $\delta\in\Gamma$. By Proposition 2.8 $a\delta M=M\delta a$. Since $M\delta a=M\Gamma a$, $M\Gamma a\subseteq A$, and so A is a left ideal. □

2.10 DEFINITION. A gamma ring (Γ, M) is said to be a *division gamma ring* if (Γ, M) has the strong left unity $[e, \delta]$ and the strong right unity $[\delta, e]$, and if for each non-zero element $a\in M$ there exists $b\in M$ such that $a\delta b=b\delta a=e$. A gamma ring (Γ, M) is said to be *subdirectly irreducible* if the intersection of all non-zero ideals of M is not zero.

2.11 THEOREM. *A non-zero subdirectly irreducible regular gamma ring with no non-zero weakly nilpotent elements is a division gamma ring.*

PROOF. Let (Γ, M) be a non-zero subdirectly irreducible regular gamma ring with no non-zero weakly nilpotent elements. For each non-zero element $e\in M$ there exists $\delta\in\Gamma$ such that $e\delta e=e$. Proposition 2.8 shows that for any $x\in M$ $e\delta x=x\delta e$. Let us consider two ideals $e\delta M$ and $A=\{x-e\delta x|x\in M\}$, whose intersection is zero. M is subdirectly irreducible, so $e\delta M=0$ or $A=0$. But $e\delta M\neq 0$, hence $A=0$, and thus $e\delta x=x\delta e=x$. This means that $[e, \delta]$ and $[\delta, e]$ are the strong left and right unities, respectively. Let a be a non-zero element of M. Then, there exists $\omega\in\Gamma$ such that $a\omega a=a$. By the observation made above, $a\omega x=x=x\omega a$ for any $x\in M$ and so $a\omega e=e=e\omega a$, whence $(a\delta e)\omega e=e=e\omega(e\delta a)$ or $a\delta(e\omega e)=e=(e\omega e)\delta a$. Therefore, (Γ, M) is a division gamma ring. □

3. Relations among the regularities of the operator rings and a gamma ring.

Assuming the existence of the left and right unities in a gamma ring (Γ, M), we prove that the left (right) operator ring $L(R)$ is regular if and only M is regular. From this, we can conclude that the regularity may be considered one of Morita invariants.

For a ring A we prepare the following:

3.1 PROPOSITION. *For a ring A with the unity, the following conditions are equivalent*:

(1) *A is regular.*

(2) *Every principal right (left) ideal of A is generated by an idempotent.*

(3) *Every finitely generated right (left) ideal of A is generated by an idempotent.*

The proof is analogous to the proof of Theorem 1.2. □

3.2 THEOREM. *Suppose (Γ, M) has the left and right unities. Then, following conditions are equivalent*:

(1) *L is regular.* (2) *R is regular.* (3) *M is regular.*

PROOF. (2)⇒(3): Suppose that R is regular and let $M\Gamma m$, where $m\in M$, be a principal left ideal of M. We shall show that there exists $e\in R$ such that $e^2=e$ and $M\Gamma m=Me$. Let $1_L=\sum[e_i, \delta_i]$, where $e_i\in M$, $\delta_i\in\Gamma$. Then, $\Gamma=\Gamma\sum[e_i, \delta_i]=\sum \Gamma e_i\delta_i\subseteq\sum R\delta_i$. Clearly, $\sum R\delta_i\subseteq\Gamma$. Hence $\Gamma=\sum R\delta_i$. So, $[\Gamma, m]=\sum Rr_i$, where $r_i=[\delta_i, m]\in R$. Since R is regular by Proposition 3.1 $\sum Rr_i=Re$, with $e\in R$, $e^2=e$. Now, $M\Gamma m=MRe=Me$, as reguired. By Theorem 1.2, M is regular.

(3)$\Rightarrow$(2): Suppose that M is regular, and let Rr be a principal left ideal of R. Let $1_R=\sum[\varepsilon_j, f_j]$, where $\varepsilon_j\in\Gamma$ and $f_j\in M$. Then, $M=M1_R=\sum(M\varepsilon_j)f_j\subseteq\sum Lf_j$. Since $\sum Lf_j\subseteq M$, we have $M=\sum Lf_j$. Then, $Mr=\sum Lm_j$, where $m_j=f_jr\in M$. Since M is regular, by Theorem 1.2 $\sum Lm_j=Me$, with $e\in R$, $e^2=e$. Therefore, $Rr=\Gamma Mr=\Gamma Me=Re$. By Proposition 3.1, R is regular.

(1)$\Longleftrightarrow$(3) is proved analogously. □

3.3 COROLLARY. *Suppose (Γ, M) has the left and right unities, and R and L are the right and left operator rings, respectively. Then, for any positive integers m, n, R_n is regular if and only if L_m is regular, where R_n and L_m denote the total matrix rings of $n\times n$ matrices over R and of $m\times m$ matrices over L, respectively.*

PROOF. Consider the matrix gamma ring $(\Gamma_{n,m}, M_{m,n})$ over (Γ, M). Then $R_n=[\Gamma_{n,m}, M_{m,n}]$ and $L_m=[M_{m,n}, \Gamma_{n,m}]$ are the right and left operator rings of $(\Gamma_{n,m}, M_{m,n})$, respectively. □

3.4 REMARK. In Corollary 3.3, put $m=1$, then R_n is regular if and only if L is regular. Also we know L is regular if and only if R is regular. Hence, we have R_n is regular if and only if R is regular. Likewise, R_n is regular if and only if $M_{m,n}$ is regular, and R is regular if and only if M is regular. Hence, M is regular if and only if $M_{m,n}$ is regular.

Now, let R and R' be ordinary rings with the unities. Suppose the categories Mod-R and Mod-R' are equivalent, written Mod-$R\approx$Mod-R'. Then, there exist bimodules ${}_{R'}P_R$, ${}_RP'_{R'}$ and a Morita context $(R, R', P, P', \tau, \mu)$ for which τ and μ are surjective, so Morita I holds (see [2, p. 178]). Thus, (P', P) forms a gamma ring having the right operator ring R and the left operator ring R'. Thus, Theorem 3.2 shows the following:

3.5 COROLLARY. *If R and R' are rings with the unities and* Mod-$R\approx$Mod-R', then R is regular if and only if R' is regular.

By this corollary, the regularity may be considered as one of Morita invariants.

3.6 DEFINITION. A left R-module M is called *regular* if, given any element $m\in M$, there exists $f\in\mathrm{Hom}_R(M, R)$ with $(mf)m=m$.

Chung and Luh [1] proved the following:

3.7 THEOREM. *Let R be a ring with unity. For unital left R-modules, the following conditions are equivalent:*

(1) *R is a seme-simple artinian ring.*

(2) *Every R-module is regular.*

(3) *Every simple R-module is regular.*

Using Theorem 3.7 we have

3.8 THEOREM. *Let (Γ, M) be a semi-prime gamma ring with min-r and min-l conditions. Let L and R be the left and right operator rings respectively. Then, every left (right) L-module and every left (right) R-module are regular. In particular, L, M and R are regular.*

PROOF. First we note that by Corollaries 3.6 and 3.7 in [4] M has the left unity 1_L and the right unity 1_R. Here, $1_L=\sum_i[e_i, \delta_i]$, where $[e_1, \delta_1], \cdots, [e_n, \delta_n]$ are mutually orthogonal primitive idempotents. Similarly for 1_R. Thus,

$L=\oplus_i[e_i, \delta_i]L=\oplus_i L[e_i, \delta_i]$, where $[e_i, \delta_i]L$ and $L[e_i, \delta_i]$ are right and left minimal ideals respectively. Hence, L is left and right artinian. So, we have

$L=\oplus_{i,j}[e_i, \delta_i]L[e_j, \delta_j]$, where $[e_i, \delta_i]L[e_j, \delta_j]$ are division rings. Thus, L is a semi-simple artinian ring. By Theorem 3.7, every left (right) L-module is regular. In particular, L is regular as s left (right) L-module. Since L has the unity 1_L, $L=\mathrm{End}({}_LL)$ $(\mathrm{End}(L_L))$, and so L is regular as a ring, because for any $h\in L$ there exists $h'\in \mathrm{End}({}_LL)=L$ such that $hh'h=h$. Now by Theorem 3.2 M is regular. Similarly, every left (right) R-module is regular, and in particular R is regular. □

4. Regularity of Morita pairs.

Let (Q, R, S, T, μ, ν) be a Morita context, where Q and R are rings, S and T are bimodules such that $S={}_QS_R$ and $T={}_RT_Q$, and μ and ν are mappings such that $\mu: S\otimes_R T\rightarrow Q$ and $\nu: T\otimes_Q S\rightarrow R$. For $s, s'\in S$, and $t, t'\in T$, denote

$$st=\mu(s\otimes t)\in Q,\ ts=\nu(t\otimes s)\in R,$$
$$sts'=(st)s'\in S,\ tst'=(ts)t'\in T.$$

Due to the associative laws in a Morita context, the conditions (1), (2) and (3) of **0** are satisfied, and we obtain a gamma ring (T, S).

Conversely, if (Γ, M) is a gamma ring with the left and the right operator rings L and R, we obtain a Morita context $(L, R, M, \Gamma, \mu, \nu)$. However, note that Q and R of a Morita context are not the operator rings of a gamma ring (T, S), because S (or T) is not necessarily a faithful module.

For a Morita context, we let $ST=\{\sum s_it_i\}$, $TS=\{\sum t_is_i\}$. For the case $Q=ST$ and $R=TS$ we say that Q and R are related through a Morita context, or simply (Q, R) is a *Morita pair*, [6]. Let (L, R) be a Morita pair, where $L=ST$ and

$R=TS$. Define $L_0=\{h\in L|Th=0\}$, $R_0=\{r\in R|rT=0\}$, and $S_0=\{s\in S|TsT=0\}$. L_0 and R_0 are ideals of L and of R, respectively, and S_0 is an L-R-submodule of S. It is easy to see that $S_0T\subseteq L_0$ and $TS_0\subseteq R_0$. When S is a finitely generated left L- module, we simply say that ${}_LS$ is finitely generated. The same convention is used for S_R, ${}_RT$ and T_L. With the notations above, we have the following theorem:

4.1 THEOREM. *Suppose that ${}_LS$, S_R, ${}_RT$ and T_L are all finitely generated. Then, the following conditions are equivalent.*

(1) *L/L_0 is a regular ring.*

(2) *R/R_0 is a regular ring.*

(3) *For any element $s\in S$, there exists an element $t\in T$ such that $sts\equiv s$ mod S_0.*

PROOF. The proof consists of the following four steps.

Step 1. Suppose that T_L is finitely generated. Then (1) implies (3).

Proof of Step 1. Suppose that (1) holds. Since T_L is finitely generated, we have $T=\sum t_iL$, $(t_i\in T)$. For any element $s\in S$, $sT=\sum st_iL$. Here st_iL are principal right ideals of L, and since L/L_0 is regular, there exists $e\in L$ such that $e^2\equiv e$ mod L_0 and $\sum st_iL\equiv eL$ mod L_0. So, $sT\equiv eL$ mod L_0. Then, there exists an element $t_0\in T$ such that $st_0\equiv e$ mod L_0. On the other hand, for any $t\in T$, $st\equiv eh$ mod L_0 with some $h\in L$. Therefore, $est\equiv e^2h\equiv eh\equiv st$ mod L_0, $(es-s)t\equiv 0$ mod L_0, and hence $(st_0s-s)t\in L_0$. This implies that $T(st_0s-s)t=0$ for any t. We have shown that $st_0s-s\in S_0$. So, (3) holds.

Step 2. Suppose that ${}_LS$ is finitely generated. Then, (3) implies (2).

Proof of Step 2. Suppose that (3) holds. Since ${}_LS$ is finitely generated, $S=\sum Lu_i(u_i\in S)$. For any element $r\in R$, $Sr=\sum Lu_ir=\sum Ls_i$, where $s_i=u_ir\in S$. By (3), there exist t_i such that $s_it_is_i\equiv s_i$ mod S_0. Let $e_i=t_is_i\in R$. Then, $e_i^2=t_is_it_is_i\equiv t_is_i$ mod R_0, as $TS_0\subseteq R_0$. Hence, $e_i^2\equiv e_i$ mod R_0. Clearly, $Re_i=Rt_is_i=TSt_is_i\subseteq TLs_i$. On the other hand, $TLs_i\equiv TLs_it_is_i$ mod R_0, and $TLs_it_is_i=TLs_ie_i\subseteq Re_i$. So, $TLs_i\equiv Re_i$ mod R_0. Hence, $Rr\equiv\sum Re_i$ mod R_0. By a well known argument in ring theory, we have that $\sum Re_i\equiv Re$ mod R_0 with $e^2\equiv e$ mod R_0. Thus, every principal left ideal of R/R_0 is generated by an idempotent and hence R/R_0 is regular. Thus, (3) holds.

Step 3. Suppose that ${}_RT$ is finitely generated. Then, (2) implies (3).

Proof of Step 3. The proof is similar to the proof of the step 1, using R in place of L, and changing the order of multiplication. Namely, let $T=\sum Rt_i$ and $Ts=\sum Rt_is$. We can show that there exists $e\in R$ such that $e^2\equiv e$ mod R_0 and

$Ts \equiv Re$ mod R_0. Then, $e \equiv t_0 s$ mod R_0 with some t_0. We can also show that $t(st_0 s - s) \equiv 0$ mod R_0, and hence $st_0 s \equiv s$ mod S_0.

Step 4. Suppose that S_R is finitely generated. Then (3) implies (1).

Proof of Step 4. The proof is similar to the proof of Step 2. □

4.2 COROLLARY. *Suppose that $_LS$ and T_L are finitely generated. Assume, further, that $rR=0$ implies $r=0$. Then, R is regular if L is regular.*

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