TSUKUBA J. MATH.
Vol. 11 No. 2 (1987). 265–272

NON-NEGATIVELY CURVED C-TOTALLY REAL SUBMANIFOLDS IN A SASAKIAN MANIFORD

By

Masumi KAMEDA

Dedicated to Professor Y. Tashiro on his 60th birthday

§0. Introduction.

Several authors have investigated minimal totally real submanifolds in a complex space form and obtained many interesting results. Recently F. Urbano [6] and Y. Ohnita [4] have studied pinching problems on their curvatures and stated some theorems.

On the other hand, in a $(2n+1)$-dimensional Sasakian space form of constant ϕ-sectional curvature $c(>-3)$, if a submanifold M is perpendicular to the structure vector field, then M is said to be *C-totally real*. For such a submanifold M, it is well-known that if the mean curvature vector field of M is parallel, then M is minimal. S. Yamaguchi, M. Kon and T. Ikawa [8] obtained that if the squared length of the second fundamental form of M is less than $n(n+1)(c+3)/4(2n-1)$, then M is totally geodesic. Furthermore, D. E. Blair and K. Ogiue [2] proved that if the sectional curvature of M is a greater than $(n-2)(c+3)/4(2n-1)$, then M is totally geodesic.

In this paper, we consider a curvature-invariant C-totally real submanifold M in a Sasakian manifold with η-parallel mean curvature vector field. Then M is not necessary minimal. Making use of methods of [3] and [4], we prove that if the sectional curvature of M is positive, then M is totally geodesic.

In Sec. 1, we recall the differential operators on the unit sphere bundle of a Riemannian manifold. Sec. 2 is devoted to stating about fundamental formulas on a C-totally real submanifold in a Sasakian manifold. In Sec. 3, we prove Theorems and Corollaries. Throughout this paper all manifolds are always C^∞, oriented, connected and complete. The author wishes to thank Professor S. Yamaguchi for his help.

§1. A differential operator defined by A. Gray.

Let M be an n-dimensional Riemannian manifold and $\Gamma(M)$ the Lie algebra

Received August 8, 1986.

of vector fields on M. Denote by $\langle\ ,\ \rangle$, ∇ and $R_{XY}:=[\nabla_X, \nabla_Y]-\nabla_{[X,Y]}$ $(X, Y\in \Gamma(M))$ the metric tensor of M, the Riemannian connection on M and the curvature tensor of M, respectively. The Ricci tensor ρ of M is given by

$$\rho_{XY}:=\sum_{\alpha=1}^n\langle R_{e_\alpha X}Y, e_\alpha\rangle \quad \text{for } X, Y\in \Gamma(M), \tag{1.1}$$

where $\{e_1, \cdots, e_n\}$ is an arbitrary local orthonormal frame field. For $m\in M$ we denote by M_m the tangent space to M at m. Then we write R_{wxyz} in place of $\langle R_{wx}y, z\rangle$ for $w, x, y, z\in M_m$ and shall sometimes use such expressions as $R_{x\alpha y\beta}$ instead of $R_{xe_\alpha(m)ye_\beta(m)}$.

Now we define the unit sphere bundle $S(M)$ of M by

$$S(M)=\{(m, x): m\in M, x\in M_m, \langle x, x\rangle=1\}.$$

For any unit vector x in a fibre S_m we take an orthonormal basis $\{e_1, \cdots, e_n\}$ of M_m such that $x=e_1$. Denote by $(y_2, \cdots, y_n)$ the corresponding system of normal coordinates defined on a neighborhood of x in S_m.

LEMMA A [3]. *Let $F: S_m\longrightarrow \boldsymbol{R}$ be a function. Then we have*

$$\frac{\partial^{\alpha_2+\cdots+\alpha_n}F}{\partial y_2^{\alpha_2}\cdots\partial y_n^{\alpha_n}}(m, x)=\frac{\partial^{\alpha_2+\cdots+\alpha_n}}{\partial u_2^{\alpha_2}\cdots\partial u_n^{\alpha_n}}F((\cos r)x+\left(\frac{\sin r}{r}\right)\sum_{\gamma=2}^n u_\gamma e_\gamma)(0),$$

where we have set $r^2=\sum_{\gamma=2}^n u_\gamma^2$.

Next we lift the frame $\{e_1, \cdots, e_n\}$ to an orthonormal basis $\{f_1, \cdots, f_n; g_2, \cdots, g_n\}$ of the tangent space $S(M)_{(m,x)}$, where we require that $f_1, \cdots, f_n$ are horizontal and $g_2, \cdots, g_n$ are vertical. Denote by $(x_1, \cdots, x_n; y_2, \cdots, y_n)$ the corresponding normal coordinate system on a neighborhood of (m, x) in $S(M)$. We define a second-order linear differential operator $L(\lambda, \mu)$ by

$$L(\lambda, \mu)_{(m,x)}:=[\sum_{\alpha=1}^n\frac{\partial^2}{\partial x_\alpha^2}-\lambda\sum_{\alpha,\beta=2}^n p_{\alpha\beta}\frac{\partial^2}{\partial y_\alpha\partial y_\beta}+\mu\sum_{\alpha=2}^n q_\alpha\frac{\partial}{\partial y_\alpha}\ ]_{(m,x)},$$

where $p_{\alpha\beta}(m, x):=R_{\alpha x\beta x}$, $q_\alpha(m, x):=\rho_{\alpha x}$ and λ, μ are constants to be chosen later. This definition is independent of the choice of normal coordinates at (m, x). Hence $L(\lambda, \mu)_{(m,x)}$ is well-defined. Here we note that the sign of the second term in the right hand side is minus because of the definition on curvature tensor.

For a compact Riemannian manifold M, we define an inner product $(\ ,\ )$ on the space of functions by $(f, g):=\int_M fg*1$. Then the differential operator $L(\lambda, \mu)$ is self-adjoint with respect to $(\ ,\ )$ provided that $\lambda=-\mu$ (*cf.* [3]).

If f is a real-valued function on $S(M)$, we denote by $\mathrm{grad}^v f$ and $\mathrm{grad}^h f$ the vertical and horizontal components of grad f respectively.

LEMMA B [3]. *In a compact Riemannian manifold M, we have*

$$\int_{S(M)}[f L(\lambda, -\lambda)(f)(m, x)+|\mathrm{grad}^h f|^2(m, x)+\lambda K_{x(\mathrm{grad}^v f)(x)}] *1=0,$$

where the letter K indicates the sectional curvature of M.

§ 2. Fundamental formulas.

Let M be a submanifold of a Riemannian manifold N. We denote by the same $\langle\ ,\ \rangle$ the Riemannian metrics of M and N, and by $\bar{\nabla}$ (resp. ∇) the Riemannian connection of N (resp. M) respectively. In the sequel the letters W, X, Y and Z (resp. V) will always denote any vector fields tangent (resp. normal) to M. Then the Gauss and Weingarten formulas are respectively given by

$$\bar{\nabla}_X Y=\nabla_X Y+B(X, Y), \tag{2.1}$$

$$\bar{\nabla}_X V=-A_V X+D_X V, \tag{2.2}$$

where B (resp. A) and D are the second fundamental form (resp. shape operator) and the normal connection of M respectively. Then first and second covariant derivatives of B are respectively defined by

$$(\tilde{\nabla}_X B)(Y, Z)=D_X B(Y, Z)-B(\nabla_X Y, Z)-B(Y, \nabla_X Z), \tag{2.3}$$

$$\begin{aligned}(\tilde{\nabla}^2_{WX} B)(Y, Z)&=D_W(\tilde{\nabla}_X B)(Y, Z)-(\tilde{\nabla}_{\nabla_W X} B)(Y, Z)\\&\quad-(\tilde{\nabla}_X B)(\nabla_W Y, Z)-(\tilde{\nabla}_X B)(Y, \nabla_W Z)\end{aligned} \tag{2.4}$$

Denoting by $\bar{R}$ the Riemannian curvature tensor of N and putting as $(\bar{R}_{WX} Y)^n$ the normal part of $\bar{R}_{WX} Y$, we have the equation of Codazzi:

$$(\bar{R}_{WX} Y)^n=(\tilde{\nabla}_W B)(X, Y)-(\tilde{\nabla}_X B)(W, Y). \tag{2.5}$$

If $(\bar{R}_{WX} Y)^n$ vanishes identically, then we call such a submanifold M *curvature-invariant.*

From (2.4), the formula of Ricci with respect to the second covariant derivative of B is given by

$$\begin{aligned}&(\tilde{\nabla}^2_{WX} B)(Y, Z)-(\tilde{\nabla}^2_{XW} B)(Y, Z)\\&=R^D_{WX} B(X, Z)-B(R_{WX} Y, Z)-B(Y, R_{WX} Z),\end{aligned} \tag{2.6}$$

where $R^D_{WX} := [D_W, D_X] + D_{[W,X]}$ indicates the normal curvature tensor of M.

From now on let M be an n-dimensional C-totally real submanifold in a $(2n+1)$-dimensional Sasakian manifold N with structure (ϕ, ξ, η). Then it is shown that ([7], [8], [9], [11])

$$\langle B(Y, Z), \xi\rangle = 0, \tag{2.7}$$

$$D_X \phi Y = -\langle X, Y\rangle \xi + \phi \nabla_X Y, \tag{2.8}$$

$$\langle R^D_{WX}\phi Y, \phi Z\rangle = \langle R_{WX}Y, Z\rangle - \langle W, Z\rangle\langle X, Y\rangle + \langle W, Y\rangle\langle X, Z\rangle, \tag{2.9}$$

$$\langle (\tilde{\nabla}_X B)(Y, Z), \xi\rangle = -\langle B(Y, Z), \phi X\rangle. \tag{2.10}$$

For such a C-totally real submanifold M, we state the definitions as follows:

DEFINITION [11]. We say that the mean curvatare vector field of M is *η-parallel* if

$$\textstyle\sum_{\alpha=1}^n \langle (\tilde{\nabla}_W B)(e_\alpha, e_\alpha), \phi X\rangle = 0. \tag{2.11}$$

We say that the second fundamental form of M is *η-parallel* if

$$\langle \tilde{\nabla}_W B\rangle (Y, Z), \phi X\rangle = 0. \tag{2.12}$$

If M has η-parallel mean curvature vector field, then the equations (2.8) and (2.10) yield

$$\begin{aligned} &\textstyle\sum_{\alpha=1}^n \langle (\tilde{\nabla}^2_{WX} B)(e_\alpha, e_\alpha), \phi Y\rangle \\ &= -\textstyle\sum_{\alpha=1}^n [\langle (\tilde{\nabla}_X B)(e_\alpha, e_\alpha), D_W \phi Y\rangle + 2\langle (\tilde{\nabla}_X B)(\nabla_W e_\alpha, e_\alpha), \phi Y\rangle] \\ &= -\textstyle\sum_{\alpha=1}^n [-\langle W, Y\rangle\langle B(e_\alpha, e_\alpha), \phi X\rangle + 2\langle \tilde{\nabla}_X B)(\nabla_W e_\alpha, e_\alpha), \phi Y\rangle]. \end{aligned}$$

Taking the normal coordinate system, we can state the following.

LEMMA 2.1. *If M has η-parallel mean curvature vector field, then we have*

$$\textstyle\sum_{\alpha=1}^n \langle (\tilde{\nabla}^2_{WX} B)(e_\alpha, e_\alpha), \phi Y\rangle = -\sum_{\alpha=1}^n \langle W, Y\rangle\langle B(e_\alpha, e_\alpha), \phi X\rangle. \tag{2.13}$$

§ 3. C-totally real submanifolds.

Throughout this section let M be an n-dimensional curvature-invariant C-totally real submanifold in a $(2n+1)$-dimensional Sasakian manifold. We denote the components of the second fundamental form B by

$$h_{\alpha\beta\gamma}:=\langle B(e_\alpha, e_\beta), \phi e_\gamma\rangle \quad \text{for } 1\leqq\alpha, \beta, \gamma\leqq n. \tag{3.1}$$

As M is C-totally real, we find that h is symmetric, i.e.,

$$h_{\alpha\beta\gamma}=h_{\alpha\gamma\beta}=h_{\beta\alpha\gamma} \quad \text{for } 1\leqq\alpha, \beta, \gamma\leqq n. \tag{3.2}$$

The components of first and second covariant derivatives of B with respect to $\phi\Gamma(M)$ are respectively expressed as

$$(\nabla_\alpha h)_{\beta\gamma\delta}:=\langle(\tilde{\nabla}_\alpha B)(e_\beta, e_\gamma), \phi e_\delta\rangle \quad \text{for } 1\leqq\alpha, \beta, \gamma, \delta\leqq n, \tag{3.3}$$

$$(\nabla^2_{\alpha\beta}h)_{\gamma\delta\varepsilon}:=\langle(\tilde{\nabla}^2_{\alpha\beta}B)(e_\gamma, e_\delta), \phi e_\varepsilon\rangle \quad \text{for } 1\leqq\alpha, \beta, \gamma, \delta, \varepsilon\leqq n. \tag{3.4}$$

Since M is curvature-invariant, then, from (2.5) and (3.3), we find that ∇h is symmetric with respect to $\phi\Gamma(M)$, i.e.,

$$(\nabla_\alpha h)_{\beta\gamma\delta}=(\nabla_\beta h)_{\alpha\gamma\delta} \quad \text{for } 1\leqq\alpha, \beta, \gamma, \delta\leqq n. \tag{3.5}$$

We consider a function f on $S(M)$ defined by $f(m, x)=h_{xxx}$ for any point $(m, x)\in S(M)$ and then prove the following Lemma to use later.

LEMMA 3.1. *Let M be an n-dimensional curvature-invariant C-totally real submanifold in a $(2n+1)$-dimensional Sasakian manifold N. If M has η-parallel mean curvature vector field, then we have $L(1/3, -1/3)(f)=0$.*

PROOF. We take any point (m, x) of $S(M)$. For each α, $1\leqq\alpha\leqq n$, let $\gamma_\alpha(s)$ be a geodesic in M such that $\gamma_\alpha(0)=m$ and $\gamma'_\alpha(0)=e_\alpha$. Then we denote a vector field by parallel translating of x along γ_α as the same letter x. By virtue of (2.7)—(2.10), we obtain

$$\begin{aligned}\left(\frac{\partial^2 f}{\partial x^2_\alpha}\right)(m, x)&=\langle\phi x, D_\alpha(\tilde{\nabla}_\alpha B)(x, x)\rangle+\langle D_\alpha\phi x, (\tilde{\nabla}_\alpha B)(x, x)\rangle \quad \text{at } m\\ &=\langle\phi x, (\tilde{\nabla}^2_{\alpha\alpha}B)(x, x)\rangle+x_\alpha\langle\phi e_\alpha, B(x, x)\rangle \quad \text{at } m\\ &=(\nabla^2_{\alpha\alpha}h)_{xxx}+x_\alpha h_{\alpha xx},\end{aligned}$$

where we have put $x_\alpha:=\langle e_\alpha, x\rangle$, which implies

$$\textstyle\sum_{\alpha=1}^n\left(\frac{\partial^2 f}{\partial x^2_\alpha}\right)(m, x)=\sum_{\alpha=1}^n(\nabla^2_{\alpha\alpha}h)_{xxx}+h_{xxx}. \tag{3.6}$$

From (2.6), (2.9), (3.2) and (3.5), we can verify

$$(\nabla^2_{\alpha\alpha}h)_{xxx}=(\nabla^2_{\alpha x}h)_{\alpha xx}$$

$$=\langle\phi x, (\tilde{\nabla}^2_{x\alpha}B)(x, e_\alpha)\rangle+\langle\phi x, R^D_{\alpha x}B(x, e_\alpha)\rangle$$

$$-\langle\phi x, B(R_{\alpha x}x, e_\alpha)\rangle-\langle\phi x, B(x, R_{\alpha x}e_\alpha)\rangle \quad \text{at } m$$

$$=\langle\phi x, (\tilde{\nabla}^2_{xx}B)(e_\alpha, e_\alpha)\rangle-\langle B(x, e_\alpha), R^D_{\alpha x}\phi x\rangle$$

$$-\langle B(x, e_\alpha), \phi R_{\alpha x}x\rangle-\langle B(x, x), \phi R_{\alpha x}e_\alpha\rangle \quad \text{at } m$$

$$=(\nabla^2_{xx}h)_{\alpha\alpha x}+\sum^n_{\beta=1}[-2h_{\beta\alpha x}R_{\alpha xx\beta}-h_{\beta xx}R_{\alpha x\alpha\beta}$$

$$+\delta_{\alpha\beta}h_{\beta\alpha x}-h_{\beta\alpha x}x_\alpha x_\beta],$$

from which follows that

(3.7) $$\sum^n_{\alpha=1}(\nabla^2_{\alpha\alpha}h)_{xxx}=\sum^n_{\alpha=1}[(\nabla^2_{xx}h)_{\alpha\alpha x}-2\sum^n_{\beta=1}h_{\beta\alpha x}R_{\alpha xx\beta}+h_{\alpha xx}\rho_{\alpha x}+h_{\alpha\alpha x}]-h_{xxx}.$$

Thus it is shown from (3.6) and (3.7) that

(3.8) $$\sum^n_{\alpha=1}\left(\frac{\partial^2 f}{\partial x^2_\alpha}\right)(m, x)=\sum^n_{\alpha=1}[(\nabla^2_{xx}h)_{\alpha\alpha x}-2\sum^n_{\beta=1}R_{\alpha xx\beta}h_{\alpha\beta x}+\rho_{x\alpha}h_{\alpha xx}+h_{\alpha\alpha x}].$$

From the definition of f, we have

(3.9) $$f((\cos r)x+\left(\frac{\sin r}{r}\right)\sum_{\gamma>1}u_\gamma e_\gamma)$$

$$=(\cos r)^3h_{xxx}+3(\cos r)^2\left(\frac{\sin r}{r}\right)\sum_{\gamma>1}u_\gamma h_{\gamma xx}$$

$$+3(\cos r)\left(\frac{\sin r}{r}\right)^2\sum_{\gamma,\delta>1}u_\gamma u_\delta h_{\delta\gamma x}+\left(\frac{\sin r}{r}\right)^3\sum_{\gamma,\delta,\varepsilon>1}u_\gamma u_\delta u_\varepsilon h_{\varepsilon\delta\gamma}$$

$$=(\cos r)^3h_{xxx}+3(\cos r)^2\left(\frac{\sin r}{r}\right)\sum_{\gamma>1}u_\gamma h_{\gamma xx}$$

$$+(\cos r)\left(\frac{\sin r}{r}\right)^2\sum_{\gamma>1}(3h_{\gamma\gamma x}-h_{xxx})u_\gamma{}^2$$

$$+6(\cos r)\left(\frac{\sin r}{r}\right)^2\sum_{\gamma>\delta>1}u_\gamma u_\delta h_{\gamma\delta x}+\left(\frac{\sin r}{r}\right)^3\sum_{\gamma,\delta,\varepsilon>1}u_\gamma u_\delta u_\varepsilon h_{\varepsilon\delta\gamma}$$

because of $r^2=\sum^n_{\gamma=2}u^2_\gamma$. Applying Lemma A to (3.9), we find

(3.10) $$\frac{\partial f}{\partial y_\alpha}(m, x)=3h_{\alpha xx} \quad \text{for } 2\leqq\alpha\leqq n,$$

(3.11) $$\frac{\partial^2 f}{\partial y_\alpha\partial y_\beta}(m, x)=-3h_{xxx}\delta_{\alpha\beta}+6h_{\alpha\beta x} \quad \text{for } 2\leqq\alpha, \beta\leqq n.$$

We see from (3.8), (3.10) and (3.11) that

$$L(1/3,\ -1/3)(f)(m,\ x)=\sum_{\alpha=1}^{n}[(\nabla_{xx}^{2}h)_{\alpha\alpha x}+h_{\alpha\alpha x}]. \tag{3.12}$$

On the other hand, the equation (2.13) is rewritten as

$$\sum_{\alpha=1}^{n}(\nabla_{\beta\delta}^{2}h)_{\alpha\alpha\gamma}=-\sum_{\alpha=1}^{n}\delta_{\beta\gamma}h_{\delta\alpha\alpha} \quad \text{for } 1\leqq\beta,\ \gamma,\ \delta\leqq n. \tag{3.13}$$

Combining (3.12) with (3.13), we have

$$L(1/3,\ -1/3)(f)(m,\ x)=0.$$

THEOREM 3.1. *Let M be an n-dimensional compact curvature-invariant C-totally real submanifold in a $(2n+1)$-dimensional Sasakian manifold with η-parallel mean curvature vector field. If the sectional curvature of M is positive, then M is totally geodesic.*

PROOF. As M has positive sectional curvature, $L(1/3,\ -1/3)$ is elliptic. From the above hypothesis we have $L(1/3,\ -1/3)(f)=0$. By maximum principle [10], f is constant on $S(M)$. Since f is an odd function, it must be zero. Thus M is totally geodesic.

COROLLARY 3.2. *Let M be an n-dimensional compact C-totally real submanifold in a $(2n+1)$-dimensional Sasakian space form with η-parallel mean curvature vector field. If the sectional curvature of M is positive, then M is totally geodesic.*

PROOF. If the ϕ-sectional curvature of Sasakian space form N is denoted by c, then the Riemannian curvature tensor $\bar{R}$ of N restricted to M is given by

$$\bar{R}_{WX}Y=\frac{c+3}{3}[\langle Y,\ X\rangle W-\langle Y,\ W\rangle X],$$

which means clearly that M is curvature-invariant. By Theorem 3.1, M is totally geodesic.

REMARK 1. If the normal connection of M is flat, then, from (2.9), M is of constant curvature 1, so that we have the same result as those in Theorem 3.1 or Corollary 3.2.

REMARK 2. As a Corollary of Theorem 3.1, we can state the Blair-Ogiue's Theorem in the introduction of this paper.

THEOREM 3.3. *Let M be an n-dimensional compact curvature-invariant C-totally real submanifold in a $(2n+1)$-dimensional Sasakian manifold with η-parallel mean curvature vector field. If the sectional curvature of M is non-negative, then M has η-parallel second fundamental form.*

PROOF. By use of Lemma 3.1, we have $L(1/3, -1/3)(f)=0$. Applying Lemma B, we find that $\mathrm{grad}^h f$ must be identically zero. From (3.2) and (3.5), the fact that $\mathrm{grad}^h f=0$ is equivalent to saying that the second fundamental form is η-parallel.

COROLLARY 3.4. *Let M be an n-dimensional compact C-totally real submanifold in a $(2n+1)$-dimensional Sasakian space form with η-parallel mean curvature vector field. If the sectional curvature of M is non-negative, then M has η-parallel second fundamental form.*

References

[1] Blair, D. E., Contact manifolds in Riemannian geometry, Springer, 509, 1976.
[2] Blair, D. E. and Ogiue, K., Positively curved integral submanifolds of a contact distribution, Illinois J. Math., **19** (1975), 628-631.
[3] Gray, A., Einstein-like manifolds which are not Einstein, Geom. Dedicata, **7** (1978), 259-280.
[4] Ohnita, Y., Totally real submanifolds with nonnegative sectional curvature, Proc. Amer. Math. Soc., **97** (1986), 474-478.
[5] Ros, A., Positively curved Kaehler submanifolds, Proc. Amer. Math. Soc., **93** (1985), 329-331.
[6] Urbano, F., Totally real minimal submanifolds of a complex projective space, Proc. Amer. Math. Soc., **93** (1985), 332-334.
[7] Yamaguchi, S. and Ikawa, T., On compact minimal C-totally real submanifolds, Tensor N. S., **29** (1975), 30-34.
[8] Yamaguchi, S. Kon, M. and Ikawa, T., On C-totally real submanifolds, J. Diff. Geom., **11** (1976), 59-64.
[9] Yamaguchi, S. Kon, M. and Miyahara, Y., A theorem on C-totally real minimal surface, Proc. Amer. Math. Soc., **54** (1976), 276-280.
[10] Yano, K. and Bochner, S., Curvature and Betti Numbers, Ann. Math. Studies **32**, Princeton University Press, Princeton, New Jersey, 1953.
[11] Yano, K. and Kon, M., Anti-invariant submanifolds, Marcel Dekker, 1976.

Author's address:
Department of Mathematics
Faculty of Science
Science University of Tokyo
Tokyo 162
JAPAN.
Correct address:
Mathematical Laboratory
Science University of Tokyo,
Yamaguchi College
Onoda, Yamaguchi 756 JAPAN.