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ON CERTAIN MIXED-TYPE BOUNDARY-VALUE PROBLEMS OF ELASTOSTATICS

—with a simple example of Melin's inequality for a system—

By

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Introduction.

Let Ω be a bounded domain in $\boldsymbol{R}^n$, $n\geqq 2$, with C^∞-boundary $\Gamma=\partial\Omega$. For a vector function $\boldsymbol{u}=(u_i(x))$ with values in $\boldsymbol{C}^n$, we introduce differential systems A and B by

(0.1) $$(A\boldsymbol{u})_i=-\sum_{j,k,h}\partial_j(a_{ijkh}(x)\varepsilon_{kh}(\boldsymbol{u})) \quad \text{in } \Omega,$$

(0.2) $$(B\boldsymbol{u})_i=\sum_{j,k,h}\nu_j(x)a_{ijkh}(x)\varepsilon_{kh}(\boldsymbol{u})|_\Gamma \quad \text{on } \Gamma$$

where $\partial_j=\partial_{x_j}=\partial/\partial x_j$, $\varepsilon_{ij}(\boldsymbol{u})=(\partial_j u_i+\partial_i u_j)/2$ and $\boldsymbol{\nu}=(\nu_i(x))$ denotes the unit outer normal to Γ. Here we assume that $a_{ijkh}(x)$ are real-valued C^∞-functions on $\bar{\Omega}$ with the property of *symmetry*

(0.3) $$a_{ijkh}(x)=a_{khij}(x)=a_{jikh}(x) \quad \text{on } \bar{\Omega}$$

and the property of *strong convexity*

(0.4) $$\sum_{i,j,k,h}a_{ijkh}(x)s_{kh}s_{ij}\geqq c_1\sum_{i,j}s_{ij}^2 \quad \text{on } \bar{\Omega}, \quad c_1>0: \text{const},$$

for all $n\times n$ real symmetric matrices (s_{ij}). (Throughout this note, Latin indices i, j, k, h take their values in the set $\{1, \cdots, n\}$; small letters $\boldsymbol{u}, \boldsymbol{\phi}$, etc. in boldface represent column vectors.)

Then the fundamental equations of linear elastostatics are expressed as follows:

(0.5) $$A\boldsymbol{u}=\boldsymbol{f} \quad \text{in } \Omega$$

with the mixed boundary condition

(0.6) $$B\boldsymbol{u}=\boldsymbol{\phi} \quad \text{on } \Gamma_N, \qquad \boldsymbol{u}|_\Gamma=\boldsymbol{\psi} \quad \text{on } \Gamma_D$$

where Γ_N and Γ_D are open subsets of Γ into which Γ is divided by a 1-codimensional C^1-submanifold Σ of Γ: $\Gamma=\Gamma_N\cup\Sigma\cup\Gamma_D$ (disjoint union). The

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problem of seeking a solution $\boldsymbol{u}=(u_i)$ of (0.5) with (0.6) for given data $\boldsymbol{f}=(f_i)$, $\boldsymbol{\phi}=(\phi_i)$ and $\boldsymbol{\psi}=(\psi_i)$ has been studied well (see, *e.g.*, Duvaut & Lions [2; Théorème 3.3, Chap. 3]).

We are concerned with the equation (0.5) not only with (0.6) but also with another boundary condition

$$(0.7) \qquad B_\alpha \boldsymbol{u} := \alpha(x) B\boldsymbol{u} + (1-\alpha(x))\boldsymbol{u}|_\Gamma = \boldsymbol{\phi} \qquad \text{on } \Gamma,$$

where we assume that $\alpha=\alpha(x)$ is a C^∞-function on Γ such that

$$0 \leqq \alpha(x) \leqq 1 \quad \text{and} \quad \alpha(x) \not\equiv 1 \qquad \text{on } \Gamma.$$

For the case $\alpha(x)\equiv 1$, see [2; Théorème 3.4, Chap. 3]. We are more interested in the latter boundary condition (0.7), which may possibly change its order on Γ. For the future use, we consider

$$(S_\alpha)_\lambda \qquad A_\lambda \boldsymbol{u} = \boldsymbol{f} \quad \text{in } \Omega, \qquad B_\alpha \boldsymbol{u} = \boldsymbol{\phi} \quad \text{on } \Gamma$$

where $A_\lambda = \lambda \mathrm{I} + A$, $\lambda \geqq 0$ a parameter, I the identity. In this paper, we will study the following problems:

(I) Is there a solution $\boldsymbol{u}$ of $(S_\alpha)_\lambda$ for given data $\{\boldsymbol{f}, \boldsymbol{\phi}\}$? How about the uniqueness and regularity if there exists a solution?

(II) If problem $(S_\alpha)_{\lambda=0}$ with data $\{\boldsymbol{f}, \alpha\boldsymbol{\phi}+(1-\alpha)\boldsymbol{\psi}\}$ has a unique solution $\boldsymbol{u}_\alpha$, can we construct a weak solution $\boldsymbol{u}$ of (0.5) with (0.6), namely, of the problem

$$(S) \qquad A\boldsymbol{u} = \boldsymbol{f} \quad \text{in } \Omega \quad \text{with} \quad B\boldsymbol{u} = \boldsymbol{\phi} \quad \text{on } \Gamma_N, \quad \boldsymbol{u}|_\Gamma = \boldsymbol{\psi} \quad \text{on } \Gamma_D$$

as a limit of $\boldsymbol{u}_\alpha$ when $\alpha(x)$ converges to the defining function of Γ_N in a suitable sense?

We will give affirmative answers to Problems (I) and (II); they will be stated in Theorems I (in § 1) and II (in § 3), respectively.

In connection with our problems, consider the dynamic problem corresponding to (S) when a_{ijkh} and Σ are time-independent. Theorem I enables us to construct a weak solution of this problem with $\{\boldsymbol{\phi}, \boldsymbol{\psi}\}=\{\boldsymbol{o}, \boldsymbol{o}\}$ by the method of Inoue [6]. Under slightly more general assumptions allowing the time-dependence of a_{ijkh} (but not of Σ) and non-zero $\{\boldsymbol{\phi}, \boldsymbol{\psi}\}$, Duvaut & Lions showed the existence of a unique weak solution of that problem by the Faedo-Galerkin method in [2; Théorème 4.1, Chap. 3], and proposed that "L'abandon de cette hypothèse (Σ ne dépend pas du temps) semble conduire à des problèmes ouverts et fort intéressants". Subsequently, Inoue asserted in [7] that "we may believe that the method developed in this paper will be useful to solve the problem posed by Duvaut & Lions". We may say that this paper is the first step to make sure of his words (see Ito [9]).

The plan of this paper is as follows: §§ 1, 2 are devoted to Problem (I). To examine it we reduce problem $(S_\alpha)_\lambda$ to the study of a system of pseudo-differential equations on Γ of *non-elliptic* type. And we obtain key estimates by means of *Melin's inequality* for a certain system of pseudo-differential operators. That is the same manner as Fujiwara & Uchiyama [4], Taira [13], etc., took in studying non-elliptic boundary-value problems for the Laplacian. Although the theorem of Melin [11; Theorem 3.1] is not fit for our matrix-valued operator unlike their scalar cases, we can extend it to our matrix-valued operator of a simple form (see Theorem 2.4 and the note following it). After those, we deduce Theorem I, which is a system version of Taira [13; Theorem 1], from the key estimates using the method of Agmon & Nirenberg developed in Fujiwara [3], Taira [14]. In § 3 we answer Problem (II). In § 4 we consider a slightly more general case. Finally, in Appendix, we prove Theorem 2.4.

§ 1. Reduction to the Boundary.

The purpose of this section is to reduce problem $(S_\alpha)_\lambda$ to a system of pseudo-differential equations on Γ.

Sobolev spaces and pseudo-differential operators. First, we mention the Sobolev spaces, in the framework of which we study our problems. Let M be $\boldsymbol{R}^n$, a bounded domain in $\boldsymbol{R}^n$ with C^∞-boundary, or an oriented compact C^∞-Riemannian manifold. We denote by $H^\sigma(M)$ the complex-valued Sobolev space of order $\sigma\in\boldsymbol{R}$ with norm $\|\cdot\|_{\sigma,M}$. When M is an oriented compact manifold or $\boldsymbol{R}^n$, we utilize the following particular norm on $H^\sigma(M)$:

$$\|u\|_{\sigma,M}^2=\int_M |\Lambda_M^\sigma u|^2 dv_M \quad \text{with} \quad \Lambda_M=(1-\Delta_M)^{1/2};$$

and the inner product $(\cdot,\cdot)_M$ on $L^2(M)=H^0(M)$ can be extended to a continuous sesquilinear form on $H^{-\sigma}(M)\times H^\sigma(M)$ by

$$(u, v)_M=\int_M \Lambda_M^{-\sigma}u\cdot\overline{\Lambda_M^\sigma v}\, dv_M \qquad \text{for} \quad u\in H^{-\sigma}(M),\ v\in H^\sigma(M).$$

Here, Δ_M and dv_M denote the Laplace-Beltrami operator and the volume element on M, respectively. We will express various function spaces of (n-)vector functions in boldface: $\boldsymbol{C}^\infty$, $\boldsymbol{L}^2$, $\boldsymbol{H}^\sigma$, etc. The same notation as above will be used for the norm of $\boldsymbol{H}^\sigma(M)$ and the inner product on $\boldsymbol{H}^{-\sigma}(M)\times\boldsymbol{H}^\sigma(M)$.

Secondly, we shortly refer to pseudo-diffential operators. For details, see, *e.g.*, Hörmander [5]. Let $m\in\boldsymbol{R}$ and let M be an oriented C^∞-Riemannian manifold.

A classical pseudo-differential operator $P\in\Psi_{phg}^{m}(M)$ (*regarded as acting on sections of the half density bundle on* M) has its *principal symbol* $p_m(x, \xi)$ and *subprincipal symbol* $p_{m-1}^{s}(x, \xi)$, invariantly defined on the cotangent bundle $T^*(M)\setminus 0$ on M with the zero section removed; $p_m(x, \xi)$ (resp. $p_{m-1}^{s}(x, \xi)$) is homogeneous in $\xi\neq 0$ of degree m (resp. $m-1$). For example, those symbols of $\Lambda_M^{\sigma}\in\Psi_{phg}^{\sigma}(M)$ are given by $|\xi|_M^{\sigma}$ and 0, respectively, where $|\xi|_M$ denotes the length of $\xi\in T_x^*(M)$ with respect to the metric on M.

By a matrix-valued pseudo-differential operator $P\in\boldsymbol{\Psi}_{phg}^{m}(M)$, we mean that all its elements belong to $\Psi_{phg}^{m}(M)$. The principal and subprincipal symbols of P are defined by the matrices of those symbols of its elements. Let $P\in\boldsymbol{\Psi}_{phg}^{m}(M)$ and $Q\in\boldsymbol{\Psi}_{phg}^{\mu}(M)$ be $l\times l$ matrix-valued, and let p_m and p_{m-1}^{s}, q_μ and $q_{\mu-1}^{s}$ be respectively their principal and subprincipal symbols. The adjoint and composition formulae are as follows: (i) The principal and subprincipal symbols of the formal adjoint $P^*\in\boldsymbol{\Psi}_{phg}^{m}(M)$ of P are given by $p_m(x, \xi)^*$ and $p_{m-1}^{s}(x, \xi)^*$, respectively. In particular, if $P=P^*$, then p_m and p_{m-1}^{s} are both Hermitian matrices. (ii) The principal and subprincipal symbols of $PQ\in\boldsymbol{\Psi}_{phg}^{m+\mu}(M)$ are given respectively by $p_m(x, \xi)q_\mu(x, \xi)$ and

$$p_m(x, \xi)q_{\mu-1}^{s}(x, \xi)+p_{m-1}^{s}(x, \xi)q_\mu(x, \xi)-\frac{\sqrt{-1}}{2}\{p_m(x, \xi), q_\mu(x, \xi)\}$$

where $\{\cdot, \cdot\}$ denote the Poisson brackets: $\{p_m, q_\mu\}=\sum_j\left(\frac{\partial p_m}{\partial\xi_j}\frac{\partial q_\mu}{\partial x_j}-\frac{\partial p_m}{\partial x_j}\frac{\partial q_\mu}{\partial\xi_j}\right)$.

Throughout this paper, by c, C, $C(*)$, etc., we denote positive constants independent of the various functions or variables found in given inequalities; they may change from line to line.

Uniqueness of solution. We state Korn's inequality, which is useful for the existence theorems in elasticity. For the proof, see, *e.g.*, Duvaut & Lions [2; Théorèmes 3.1 et 3.3, Chap. 3], also Ito [8]. After that, the uniqueness of solution of problem $(S_\alpha)_\lambda$ is proved.

THEOREM 1.1. *Let* Ω *be a bounded domain in* $\boldsymbol{R}^n$ *with* C^1*-boundary* Γ.

(i) *For any open subset* $\gamma(\neq\emptyset)$ *of* Γ, *there exists a constant* $c_K(\gamma)=c_K(\gamma, \Omega)>0$ *such that*

$$\sum_{i,j}\int_\Omega|\varepsilon_{ij}(\boldsymbol{u})|^2dx\geqq c_K(\gamma)\|\boldsymbol{u}\|_{1,\Omega}^2 \quad \text{for all } \boldsymbol{u}\in\boldsymbol{H}^1(\Omega) \text{ with } \boldsymbol{u}|_\gamma=\boldsymbol{o}. \tag{1.1}$$

(ii) *There exists a constant* $c_K=c_K(\Omega)>0$ *such that*

$$\sum_{i,j}\int_\Omega|\varepsilon_{ij}(\boldsymbol{u})|^2dx+\|\boldsymbol{u}\|_{0,\Omega}^2\geqq c_K\|\boldsymbol{u}\|_{1,\Omega}^2 \quad \text{for all } \boldsymbol{u}\in\boldsymbol{H}^1(\Omega). \tag{1.2}$$

PROPOSITION 1.2. *Let* $\lambda \geqq 0$. *If* $\boldsymbol{u} \in \boldsymbol{H}^2(\Omega)$ *is a solution of problem* $(S_\alpha)_\lambda$ *with* $\{\boldsymbol{f}, \boldsymbol{\phi}\}=\{\boldsymbol{o}, \boldsymbol{o}\}$, *then* $\boldsymbol{u}=\boldsymbol{o}$.

PROOF. Denoting the sesquilinear form associated with A by

$$a(\boldsymbol{u}, \boldsymbol{v})=\sum_{i,j,k,h} \int_\Omega a_{ijkh}(x) \varepsilon_{kh}(\boldsymbol{u}) \overline{\varepsilon_{ij}(\boldsymbol{v})}\, dx$$

$$=\sum_{i,j,k,h} \int_\Omega a_{ijkh}(x) \partial_h u_k \cdot \overline{\partial_j v_i}\, dx \quad \text{(by (0.3))},$$

we have Green's formula for A

(1.3) $\qquad (A\boldsymbol{u}, \boldsymbol{v})_\Omega = a(\boldsymbol{u}, \boldsymbol{v})-(B\boldsymbol{u}, \boldsymbol{v})_\Gamma \qquad$ for all $\boldsymbol{u} \in \boldsymbol{H}^2(\Omega)$, $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$.

Since $\lambda \geqq 0$ and $B_\alpha \boldsymbol{u}=\boldsymbol{o}$ on Γ, we have by (1.3)

$$(A_\lambda \boldsymbol{u}, \boldsymbol{u})_\Omega \geqq a(\boldsymbol{u}, \boldsymbol{u})+\int_{\alpha(x)\neq 0} \frac{1-\alpha(x)}{\alpha(x)} |\boldsymbol{u}|^2 dv_\Gamma \geqq a(\boldsymbol{u}, \boldsymbol{u}).$$

And since $A_\lambda \boldsymbol{u}=\boldsymbol{o}$ in Ω, we have using (0.4)

$$0=a(\boldsymbol{u}, \boldsymbol{u}) \geqq c_1 \sum_{i,j} \int_\Omega |\varepsilon_{ij}(\boldsymbol{u})|^2 dx \geqq 0.$$

Hence $(\varepsilon_{ij}(\boldsymbol{u}))=0$, so that $B_\alpha \boldsymbol{u}=(1-\alpha(x))\boldsymbol{u}=\boldsymbol{o}$ on Γ, and $\boldsymbol{u}=\boldsymbol{o}$ on $\{x \in \Gamma;\, \alpha(x)<1\} \neq \emptyset$. Thus it follows from (1.1) that $\boldsymbol{u}=\boldsymbol{o}$. □

Operator $\boldsymbol{T}(\lambda)$. When $\alpha(x) \equiv 0$ or >0 on Γ, $(S_\alpha)_\lambda$ is a boundary-value problem of *elliptic* type.

LEMMA 1.3. *Let* $\lambda \geqq 0$ *and* $\sigma \geqq 2$. *If* $\alpha(x) \equiv 0$ (*resp.* >0) *on* Γ, *then for any* $\boldsymbol{f} \in \boldsymbol{H}^{\sigma-2}(\Omega)$ *and* $\boldsymbol{\phi} \in \boldsymbol{H}^{\sigma-1/2}(\Gamma)$ (*resp.* $\boldsymbol{H}^{\sigma-2/3}(\Gamma)$) *there exists a unique solution* $\boldsymbol{u} \in \boldsymbol{H}^\sigma(\Omega)$ *of problem* $(S_\alpha)_\lambda$. *And the mapping*: $\boldsymbol{u} \to \{\boldsymbol{f}, \boldsymbol{\phi}\}$ *is an isomorphism between the corresponding Sobolev spaces.*

PROOF. We have by (0.4) and (1.2)

$$a(\boldsymbol{u}, \boldsymbol{u}) \geqq C_1 \|\boldsymbol{u}\|_{1,\Omega}^2 - C_2 \|\boldsymbol{u}\|_{0,\Omega}^2 \qquad \text{for all} \quad \boldsymbol{u} \in \boldsymbol{H}^1(\Omega).$$

This inequality implies that the differential system A is *strongly elliptic* on $\bar{\Omega}$ and the boundary-value problem $\{A, B\}$ satisfies the *strong complementing condition* on Γ (see Simpson & Spector [12]), and accordingly the boundary-value problems $\{A, \textit{Dirichlet}\}$ and $\{A, B\}$ are *elliptic* in the sense of Hörmander [5; Definition 20.1.1]. In addition, these are formally self-adjoint boundary-value problems as easily seen, so that for $\sigma \geqq 2$ the mappings

$$(1.4)\qquad \begin{cases} \boldsymbol{H}^{\sigma}(\Omega)\ni \boldsymbol{u} \longrightarrow \{A\boldsymbol{u},\ \boldsymbol{u}|_{\Gamma}\}\in \boldsymbol{H}^{\sigma-2}(\Omega)\times \boldsymbol{H}^{\sigma-1/2}(\Gamma), \\ \boldsymbol{H}^{\sigma}(\Omega)\ni \boldsymbol{u} \longrightarrow \{A\boldsymbol{u},\ B\boldsymbol{u}\}\in \boldsymbol{H}^{\sigma-2}(\Omega)\times \boldsymbol{H}^{\sigma-3/2}(\Gamma) \end{cases}$$

are Fredholm operators with index 0. Therefore, we conclude from Proposition 1.2 that the following compact perturbations of (1.4):

$$\begin{cases} \boldsymbol{H}^{\sigma}(\Omega)\ni \boldsymbol{u} \longrightarrow \{A_{\lambda}\boldsymbol{u},\ \boldsymbol{u}|_{\Gamma}\}\in \boldsymbol{H}^{\sigma-2}(\Omega)\times \boldsymbol{H}^{\sigma-1/2}(\Gamma), \\ \boldsymbol{H}^{\sigma}(\Omega)\ni \boldsymbol{u} \longrightarrow \left\{A_{\lambda}\boldsymbol{u},\ \left(B+\dfrac{1-\alpha(x)}{\alpha(x)}\mathrm{I}\right)\boldsymbol{u}\right\}\in \boldsymbol{H}^{\sigma-2}(\Omega)\times \boldsymbol{H}^{\sigma-3/2}(\Gamma), \text{ if } \alpha(x)>0, \end{cases}$$

are isomorphisms. □

Let $\lambda\geqq 0$ and $\sigma\geqq 2$. Using Lemma 1.3, the Dirichlet problem

$$A_{\lambda}\boldsymbol{u}=\boldsymbol{o} \quad \text{in } \Omega, \qquad \boldsymbol{u}|_{\Gamma}=\boldsymbol{\phi} \quad \text{on } \Gamma$$

admits a unique solution $\boldsymbol{u}\in \boldsymbol{H}^{\sigma}(\Omega)$ for any $\boldsymbol{\phi}\in \boldsymbol{H}^{\sigma-1/2}(\Gamma)$. Define a mapping $P(\lambda)$ by $\boldsymbol{u}=P(\lambda)\boldsymbol{\phi}$; $P(\lambda)$ is an isomorphism: $\boldsymbol{H}^{\sigma-1/2}(\Gamma)\to \boldsymbol{H}^{\sigma}(\Omega)$, which we call the *Poisson operator* (for A_{λ}). Then $T(\lambda):=BP(\lambda)$ defines a continuous linear operator: $\boldsymbol{H}^{\sigma-1/2}(\Gamma)\to \boldsymbol{H}^{\sigma-3/2}(\Gamma)$, which makes sense for any $\sigma\in \boldsymbol{R}$ because $T(\lambda)\in \boldsymbol{\Psi}^{1}_{phg}(\Gamma)$ as will be shown below. We now state some properties of $T(\lambda)$ as a pseudo-differential operator.

PROPOSITION 1.4. *Let $\lambda\geqq 0$. The mapping $T(\lambda)$ is an $n\times n$ matrix-valued pseudo-differential operator $\in \boldsymbol{\Psi}^{1}_{phg}(\Gamma)$ with λ-independent principal symbol $t_1(x,\xi)$ and subprincipal symbol $t_0^s(x,\xi)$ defined on $T^*(\Gamma)\setminus 0$. Moreover, $T(\lambda)$ is formally self-adjoint (which implies that $t_1(x,\xi)$ is Hermitian) and is strongly elliptic in the sense that there exists a constant $c_2>0$ such that*

$$(1.5)\qquad t_1(x,\xi)\geqq c_2|\xi|_{\Gamma}\mathrm{I} \quad \text{on } T^*(\Gamma)\setminus 0, \qquad \mathrm{I}: \text{the identity matrix.}$$

PROOF. Applying Theorem 20.1.5 in [5] to our case and using the existence of a unique solution for $(S_{\alpha\equiv 0})_{\lambda\geqq 0}$, we can show that: (i) $P(\lambda)$ admits an extension to a continuous linear operator: $\boldsymbol{H}^{\sigma-1/2}(\Gamma)\to \boldsymbol{H}^{\sigma}(\Omega)$ for any $\sigma\in \boldsymbol{R}$; (ii) $BP(\lambda)$ is a pseudo-differential operator $\in \boldsymbol{\Psi}^{1}_{phg}(\Gamma)$ with λ-independent principal and subprincipal symbols.

Putting $\boldsymbol{u}=P(\lambda)\boldsymbol{\phi}$, $\boldsymbol{v}=P(\lambda)\boldsymbol{\psi}$ in (1.3) for $\boldsymbol{\phi},\boldsymbol{\psi}\in \boldsymbol{C}^{\infty}(\Gamma)$, we obtain

$$(1.6)\qquad (T(\lambda)\boldsymbol{\phi},\boldsymbol{\psi})_{\Gamma}=a(P(\lambda)\boldsymbol{\phi},P(\lambda)\boldsymbol{\psi})+\lambda(P(\lambda)\boldsymbol{\phi},P(\lambda)\boldsymbol{\psi})_{\Omega},$$

which implies the formal self-adjointness of $T(\lambda)$. And if $\boldsymbol{\phi}=\boldsymbol{\psi}$ in (1.6) particularly, we have by (0.4) and (1.2)

$$(1.7)\qquad \begin{aligned}(T(\lambda)\boldsymbol{\phi},\boldsymbol{\phi})_{\Gamma} &\geqq c_1c_K\|P(\lambda)\boldsymbol{\phi}\|^2_{1,\Omega}+(\lambda-c_1)\|P(\lambda)\boldsymbol{\phi}\|^2_{0,\Omega} \\ &\geqq c_2\|\boldsymbol{\phi}\|^2_{1/2,\Gamma}-C\|\boldsymbol{\phi}\|^2_{-1/2,\Gamma},\end{aligned}$$

where the last inequality is due to the trace theorem and the property (i) of $P(\lambda)$. Since the principal symbol of $\Lambda_\Gamma^{1/2}$ is $|\xi|_\Gamma^{1/2}$, we conclude from (1.7) that

$$t_1(x,\xi)\eta\cdot\bar{\eta}\geqq c_2|\xi|_\Gamma|\eta|^2 \qquad \text{for all} \quad (x,\xi)\in T^*(\Gamma)\setminus 0,\ \eta\in \boldsymbol{C}^n.$$

This indicates the strong ellipticity of $T(\lambda)$. □

EXAMPLE. When an elastic body is homogeneous and isotropic, the elasticity coefficients a_{ijkh} are given by

$$a_{ijkh}=\lambda\delta_{ij}\delta_{kh}+\mu(\delta_{ik}\delta_{jh}+\delta_{ih}\delta_{jk}) \tag{1.8}$$

where λ, $\mu\in\boldsymbol{R}$ are the *Lamé moduli*, δ_{ij} the Kronecker delta. The condition $\mu>0$ and $n\lambda+2\mu>0$ is equivalent to condition (0.4):

$$\sum_{i,j,k,h} a_{ijkh}s_{kh}s_{ij}\geqq\min\{2\mu,\ n\lambda+2\mu\}\sum_{i,j}s_{ij}^2 \quad \text{for all } (s_{ij}) \text{ as in (0.4)};$$

and the associated A of (0.1) is strongly elliptic if $\mu>0$ and $\lambda+2\mu>0$; in fact, the symbol $a(\xi)=(\sum_{j,h}a_{ijkh}\xi_j\xi_h)_{i,k}$ of A satisfies

$$a(\xi)\eta\cdot\bar{\eta}\geqq\min\{\mu,\ \lambda+2\mu\}|\xi|^2|\eta|^2 \qquad \text{for all} \quad \xi\in\boldsymbol{R}^n,\ \eta\in\boldsymbol{C}^n. \tag{1.9}$$

Consider a homogeneous isotropic elastic body occupying $\overline{\boldsymbol{R}_+^n}$. Let P be the Poisson operator which assigns to $\phi\in C_0^\infty(\boldsymbol{R}^{n-1})$ the bounded solution $u\in C^\infty(\overline{\boldsymbol{R}_+^n})$ of the Dirichlet problem

$$Au=o \quad \text{in } \boldsymbol{R}_+^n, \qquad u|_{\partial\boldsymbol{R}_+^n}=\phi \quad \text{on } \partial\boldsymbol{R}_+^n\cong\boldsymbol{R}^{n-1}.$$

Then, $T:=BP$ belongs to $\Psi_{phg}^1(\boldsymbol{R}^{n-1})$ and its symbol is calculated as

$$\frac{\mu(\lambda+\mu)}{\lambda+3\mu}\begin{pmatrix}
\frac{\lambda+3\mu}{\lambda+\mu}|\xi|+\frac{\xi_1^2}{|\xi|} & \frac{\xi_1\xi_2}{|\xi|} & \cdots\cdots & \frac{\xi_1\xi_{n-1}}{|\xi|} & \frac{-2\sqrt{-1}\mu}{\lambda+\mu}\xi_1 \\
\frac{\xi_1\xi_2}{|\xi|} & \frac{\lambda+3\mu}{\lambda+\mu}|\xi|+\frac{\xi_2^2}{|\xi|} & \ddots & \vdots & \vdots \\
\vdots & \ddots & \ddots & \frac{\xi_{n-2}\xi_{n-1}}{|\xi|} & \vdots \\
\frac{\xi_1\xi_{n-1}}{|\xi|} & \cdots\cdots \ \frac{\xi_{n-2}\xi_{n-1}}{|\xi|} & & \frac{\lambda+3\mu}{\lambda+\mu}|\xi|+\frac{\xi_{n-1}^2}{|\xi|} & \frac{-2\sqrt{-1}\mu}{\lambda+\mu}\xi_{n-1} \\
\frac{2\sqrt{-1}\mu}{\lambda+\mu}\xi_1 & \cdots\cdots & & \frac{2\sqrt{-1}\mu}{\lambda+\mu}\xi_{n-1} & \frac{2(\lambda+2\mu)}{\lambda+\mu}|\xi|
\end{pmatrix}$$

where $\xi=(\xi_1,\cdots,\xi_{n-1})\neq 0$ (see Ito [8; Theorem 4.4]). Since the eigenvalues of this Hermitian matrix are given by

$$\underbrace{\mu|\xi|,\ \cdots,\ \mu|\xi|}_{n-2},\ 2\mu|\xi|,\ \frac{2\mu(\lambda+\mu)}{\lambda+3\mu}|\xi|,$$

it is positive definite if $\mu>0$ and $\lambda+\mu>0$, when the sesquilinear form $a(\cdot, \cdot)$ associated with (1.8) is coercive on $H^1(R_+^n)$ in view of (1.9); more precisely, we have

$$a(u, u)\geqq\frac{4\mu(\lambda+\mu)}{(3\lambda+5\mu+\sqrt{9\lambda^2+14\lambda\mu+9\mu^2})}\sum_{i,j}\|\partial_j u_i\|_{R_+^n}^2 \qquad \text{for all} \quad u\in H^1(R_+^n),$$

where the constant is best possible (see Ito [8; Theorem 4.6]).

Reduction to the boundary. Define a function space $H_{(\alpha)}^{\sigma}(\Gamma)$ by

$$H_{(\alpha)}^{\sigma}(\Gamma)=\{\phi=\alpha(x)\phi_1+(1-\alpha(x))\phi_0\,;\ \phi_1\in H^{\sigma}(\Gamma),\ \phi_0\in H^{\sigma+1}(\Gamma)\}.$$

The following lemma, whose proof we leave to the reader, is fundamental concerning this space.

LEMMA 1.5. *The $H_{(\alpha)}^{\sigma}(\Gamma)$ is a Banach space equipped with the norm*

$$\|\phi\|_{\alpha;\sigma,\Gamma}:=\inf\{\|\phi_1\|_{\sigma,\Gamma}+\|\phi_0\|_{\sigma+1,\Gamma}\,;\ \phi=\alpha(x)\phi_1+(1-\alpha(x))\phi_0 \ \textit{with}\ \ \phi_1\in H^{\sigma}(\Gamma),\ \phi_0\in H^{\sigma+1}(\Gamma)\}.$$

And we have the continuous inclusion relations

$$H^{\sigma+1}(\Gamma)=H_{(\alpha\equiv 0)}^{\sigma}(\Gamma)\subset H_{(\alpha)}^{\sigma}(\Gamma)\subset H_{(\alpha\equiv 1)}^{\sigma}(\Gamma)=H^{\sigma}(\Gamma)\,;$$

if $\alpha(x)>0$ on Γ, then $H_{(\alpha)}^{\sigma}(\Gamma)=H^{\sigma}(\Gamma)$ as Banach spaces.

Now we can answer Problem (I) by means of the space $H_{(\alpha)}^{\sigma}(\Gamma)$.

THEOREM I. *Let $\lambda\geqq 0$ and $\sigma\geqq 2$. For any $f\in H^{\sigma-2}(\Omega)$ and $\phi\in H_{(\alpha)}^{\sigma-3/2}(\Gamma)$, there exists a unique solution $u\in H^{\sigma}(\Omega)$ of problem $(S_\alpha)_\lambda$. Furthermore, the mapping*

$$\{A_\lambda, B_\alpha\}: H^{\sigma}(\Omega)\ni u \longrightarrow \{A_\lambda u, B_\alpha u\}\in H^{\sigma-2}(\Omega)\times H_{(\alpha)}^{\sigma-3/2}(\Gamma) \tag{1.10}$$

is an (algebraic and topological) isomorphism.

Theorem I will be proved in §2. Here we reduce $(S_\alpha)_\lambda$ to a system of pseudo-differential equations on the boundary Γ.

PROPOSITION 1.6. *Assume that, for any $\phi\in H^{\sigma-1/2}(\Gamma)$, the problem*

$$T_\alpha(\lambda)\varphi=\phi \qquad \text{on } \Gamma \tag{1.11}$$

admits a unique solution $\varphi\in H^{\sigma-1/2}(\Gamma)$ where $T_\alpha(\lambda)=\alpha(x)T(\lambda)+(1-\alpha(x))\mathrm{I}$, $\lambda\geqq 0$. Then Theorem I follows.

PROOF. By definition, $\boldsymbol{\phi}\in\boldsymbol{H}^{\sigma-3/2}_{(\alpha)}(\Gamma)$ can be written in the form $\boldsymbol{\phi}=\alpha(x)\boldsymbol{\phi}_1+(1-\alpha(x))\boldsymbol{\phi}_0$ with some $\{\boldsymbol{\phi}_1, \boldsymbol{\phi}_0\}\in\boldsymbol{H}^{\sigma-3/2}(\Gamma)\times\boldsymbol{H}^{\sigma-1/2}(\Gamma)$. By Lemma 1.3, the boundary-value problem

$$A_\lambda \boldsymbol{v}=\boldsymbol{f} \quad \text{in } \Omega, \qquad B\boldsymbol{v}+\boldsymbol{v}=\boldsymbol{\phi}_1-\boldsymbol{\phi}_0 \quad \text{on } \Gamma$$

has a unique solution $\boldsymbol{v}\in\boldsymbol{H}^\sigma(\Omega)$. Thus we see that $\boldsymbol{u}\in\boldsymbol{H}^\sigma(\Omega)$ is a unique solution of $(S_\alpha)_\lambda$ if and only if $\boldsymbol{w}:=\boldsymbol{u}-\boldsymbol{v}\in\boldsymbol{H}^\sigma(\Omega)$ is that of the boundary-value problem

$$A_\lambda \boldsymbol{w}=\boldsymbol{o} \quad \text{in } \Omega, \qquad B_\alpha \boldsymbol{w}=(2\alpha(x)-1)\boldsymbol{v}|_\Gamma+\boldsymbol{\phi}_0 \quad \text{on } \Gamma. \tag{1.12}$$

Moreover, since $\boldsymbol{w}=P(\lambda)\boldsymbol{\phi}$ with $\boldsymbol{\phi}:=\boldsymbol{w}|_\Gamma$, the solution $\boldsymbol{w}\in\boldsymbol{H}^\sigma(\Omega)$ of (1.12) corresponds one-to-one to the solution $\boldsymbol{\phi}\in\boldsymbol{H}^{\sigma-1/2}(\Gamma)$ of

$$T_\alpha(\lambda)\boldsymbol{\phi}=(2\alpha(x)-1)\boldsymbol{v}|_\Gamma+\boldsymbol{\phi}_0 \quad \text{on } \Gamma. \tag{1.13}$$

By assumption, (1.13) admits a unique solution $\boldsymbol{\phi}\in\boldsymbol{H}^{\sigma-1/2}(\Gamma)$, which indicates the unique existence of solution for $(S_\alpha)_\lambda$. That (1.10) is an isomorphism is due to the closed graph theorem. □

§2. Solvability of Problem $(S_\alpha)_\lambda$.

Operator $\tilde{T}$. To examine the solvability of (1.11), we use a method due to Agmon & Nirenberg: we introduce an auxiliary variable $y\in S:=\boldsymbol{R}^1/2\pi\boldsymbol{Z}$, the unit circle (see Fujiwara [3], Taira [14]). We consider the differential operator $\tilde{A}:=A-\partial_y^2$ in $\Omega\times S$. The boundary operator B of (0.2) is regarded as defined on $\partial(\Omega\times S)=\Gamma\times S$. The following lemma corresponds to Lemma 2.3.

LEMMA 2.1. *Let $\sigma\geqq 2$, $\tilde{\boldsymbol{f}}\in\boldsymbol{H}^{\sigma-2}(\Omega\times S)$ and $\tilde{\boldsymbol{\phi}}\in\boldsymbol{H}^{\sigma-1/2}(\Omega\times S)$. Then the Dirichlet problem*

$$\tilde{A}\boldsymbol{u}=\tilde{\boldsymbol{f}} \ \text{ in } \Omega\times S, \qquad \tilde{\boldsymbol{u}}|_{\Gamma\times S}=\tilde{\boldsymbol{\phi}} \ \text{ on } \Gamma\times S \tag{2.1}$$

admits a unique solution $\tilde{\boldsymbol{u}}\in\boldsymbol{H}^\sigma(\Omega\times S)$, and the mapping:

$$\boldsymbol{H}^\sigma(\Omega\times S)\ni\tilde{\boldsymbol{u}} \longrightarrow \{\tilde{A}\tilde{\boldsymbol{u}}, \tilde{\boldsymbol{u}}|_{\Gamma\times S}\}\in\boldsymbol{H}^{\sigma-2}(\Omega\times S)\times\boldsymbol{H}^{\sigma-1/2}(\Gamma\times S)$$

is an isomorphism.

By Lemma 2.1 we can define the Poisson operator $\tilde{P}$ which assigns to $\tilde{\boldsymbol{\phi}}\in\boldsymbol{H}^{\sigma-1/2}(\Gamma\times S)$, $\sigma\geqq 2$, the unique solution $\boldsymbol{u}\in\boldsymbol{H}^\sigma(\Omega\times S)$ of (2.1) with $\tilde{\boldsymbol{f}}=\boldsymbol{o}$; $\tilde{P}$ is an isomorphism: $\boldsymbol{H}^{\sigma-1/2}(\Gamma\times S)\to\boldsymbol{H}^\sigma(\Omega\times S)$. Then $\tilde{T}:=B\tilde{P}$ defines a continuous linear operator: $\boldsymbol{H}^{\sigma-1/2}(\Gamma\times S)\to\boldsymbol{H}^{\sigma-3/2}(\Gamma\times S)$, which makes sense for any $\sigma\in\boldsymbol{R}$. The following proposition for $\tilde{T}$ corresponds to Proposition 1.4 for $T(\lambda)$.

PROPOSITION 2.2. *The mapping* $\tilde{T}$ *is an* $n\times n$ *matrix-valued pseudo-differential operator* $\in \Psi^1_{phg}(\Gamma\times S)$, *whose principal symbol we write* $\tilde{t}_1(x, \xi; y, \eta)$, $(x, \xi; y, \eta) \in (T^*\Gamma\times T^*S)\setminus 0 \cong T^*(\Gamma\times S)\setminus 0$. *Moreover,* $\tilde{T}$ *enjoys the property of formal self-adjointness (which implies* $\tilde{t}_1(x, \xi; y, \eta)$ *is a Hermitian matrix) and the property of strong ellipticity in the sense that there exists a constant* $c_3>0$ *such that*

$$\tilde{t}_1(x, \xi; y, \eta) \geqq c_3|(\xi, \eta)|_{\Gamma\times S}\mathrm{I} \quad on \quad T^*(\Gamma\times S)\setminus 0 \tag{2.2}$$

where $|(\xi, \eta)|_{\Gamma\times S}=\sqrt{|\xi|_\Gamma^2+\eta^2}$.

A priori estimates. We set $\tilde{T}_\alpha=\alpha(x)\tilde{T}+(1-\alpha(x))\mathrm{I}\ (\in\Psi^1_{phg}(\Gamma\times S))$. The following estimates for $\tilde{T}_\alpha$ and its formal adjoint $\tilde{T}_\alpha^*$ play an important role in proving Theorem I.

PROPOSITION 2.3. *Let* $\sigma\in\boldsymbol{R}$. *There exists a constant* $C=C(\alpha, \sigma)>0$ *such that for all* $\tilde{\phi}\in C^\infty(\Gamma\times S)$

$$\|\tilde{\phi}\|_{\sigma-1/2,\Gamma\times S}\leqq C(\|\tilde{T}_\alpha\tilde{\phi}\|_{\sigma-1/2,\Gamma\times S}+\|\tilde{\phi}\|_{\sigma-1,\Gamma\times S}),$$

$$\|\tilde{\phi}\|_{-\sigma+1/2,\Gamma\times S}\leqq C(\|\tilde{T}_\alpha^*\tilde{\phi}\|_{-\sigma+1/2,\Gamma\times S}+\|\tilde{\phi}\|_{-\sigma,\Gamma\times S}).$$

To prove Proposition 2.3, we utilize Melin's inequality (see Melin [11] and Hörmander [5]) in the following form.

THEOREM 2.4. *Let* M *be an oriented compact* C^∞-*Riemannian manifold. And let* P *be an* $l\times l$ *matrix-valued pseudo-differential operator* $\in\Psi^m_{phg}(M)$, $m\in\boldsymbol{R}$. *Assume that the principal and subprincipal symbols* $p_m(x, \xi)$ *and* $p^s_{m-1}(x, \xi)$ *of* P *satisfy respectively the following conditions:*

(i) $p_m(x, \xi)$ *is expressed as* $p_m(x, \xi)=a_m(x, \xi)q_0(x, \xi)$ *with a real-valued symbol* a_m *homogeneous in* $\xi\neq 0$ *of degree* m *and an* $l\times l$ *matrix symbol* q_0 *homogeneous in* $\xi\neq 0$ *of degree* 0 *such that*

$$a_m(x, \xi)\geqq 0,\quad \mathrm{Re}\,q_0(x, \xi)>0\ (positive\ definite)\quad on\ T^*(M)\setminus 0$$

where $\mathrm{Re}\,q_0$ *denotes the Hermitian part of* q_0: $\mathrm{Re}\,q_0=(q_0+q_0^*)/2$;

(ii) *The Hermitian part* $\mathrm{Re}\,p^s_{m-1}$ *of* p^s_{m-1} *satisfies*

$$\mathrm{Re}\,p^s_{m-1}(x, \xi)+\frac{1}{2}(\mathrm{Tr}^+H_{a_m}(x, \xi))\,\mathrm{Re}\,q_0(x, \xi)\geqq c_0\mathrm{I},\quad c_0\in\boldsymbol{R},$$

on the characteristic set $\Sigma_{a_m}:=\{(x, \xi)\in T^*(M)\setminus 0;\ a_m(x, \xi)=0\}$ *of* a_m. *Here,* $H=H_{a_m}$ *is the Hessian of* a_m *invariantly defined on* Σ_{a_m}, *and* Tr^+H *denotes the sum of the positive eigenvalues, each being counted with its multiplicity, of the Hamilton map of* $H/\sqrt{-1}$ *(see [5]).*

Then, for any $\varepsilon>0$ *we have Melin's inequality for* P:

(2.3) $\mathrm{Re}(Pu, u)_M \geqq (c_0-\varepsilon)\|u\|^2_{(m-1)/2, M} - C(\varepsilon)\|u\|^2_{(m-2)/2, M}$ *for all* $u \in C^\infty(M)$.

Furthermore, if $c_0>0$, *for any* $\varepsilon \in (0, c_0)$ *and* $s \in \boldsymbol{R}$ *we have the following estimates with loss of one derivtive: for all* $u \in C^\infty(M)$,

$$(2.4) \quad \begin{cases} \|Pu\|^2_{s, M} \geqq (c_0-\varepsilon)^2\|u\|^2_{s+m-1, M} - C(\varepsilon, s)\|u\|^2_{s+m-3/2, M}, \\ \|P^*u\|^2_{s, M} \geqq (c_0-\varepsilon)^2\|u\|^2_{s+m-1, M} - C(\varepsilon, s)\|u\|^2_{s+m-3/2, M}. \end{cases}$$

This simple system version of Melin's inequality is already known (essentially). When $P \in \boldsymbol{\Psi}^m_{phg}(M)$, $m>1$, satisfies (i) with $q_0(x, \xi)=\mathrm{I}$ and (ii) with $c_0>0$, Iwasaki [10] constructed the fundamental solution $E(t)$ of $(d/dt)+P$ in a certain class of pseudo-differential operators with parameter t; inequality (2.3) follows as a corollary of that. We will, however, prove Theorem 2.4 more directly in Appendix.

PROOF OF PROPOSITION 2.3. Using the composition formula, the principal and subprincipal symbols $p_1(x, \xi; y, \eta)$ and $p_0^s(x, \xi; y, \eta)$ of $P:=\tilde{T}_\alpha$ are calculated respectively as $p_1(x, \xi; y, \eta)=\alpha(x)\tilde{t}_1(x, \xi; y, \eta)$ and

$$p_0^s(x, \xi; y, \eta)=\mathrm{I}+\alpha(x)(\tilde{t}_0^s(x, \xi; y, \eta)-\mathrm{I})-\frac{\sqrt{-1}}{2}\{\alpha(x)\mathrm{I}, \tilde{t}_1(x, \xi; y, \eta)\}$$

where $\tilde{t}_0^s(x, \xi; y, \eta)$ is the subprincipal symbol of $\tilde{T}$. Put

$$a_1(x, \xi; y, \eta)=\alpha(x)|(\xi, \eta)|_{\Gamma\times S}, \quad q_0(x, \xi; y, \eta)=\tilde{t}_1(x, \xi; y, \eta)/|(\xi, \eta)|_{\Gamma\times S},$$

then P satisfies (i) of Theorem 2.4 by Proposition 2.2. Since, at all zeros $(x, \xi; y, \eta)$ of a_1, $\mathrm{Tr}^+H_{a_1}(x, \xi; y, \eta) \geqq 0$ by definition ($=0$ truth to tell) and $p_0^s(x, \xi; y, \eta)=\mathrm{I}$, P satisfies also (ii) with $c_0=1$. Consequently we obtain the desired estimates from (2.4). □

Proof of Theorem I. Following Taira [14], we associate with equation (1.11) the closed linear operator $\mathcal{T}_\alpha(\lambda): \mathcal{D}(\mathcal{T}_\alpha(\lambda)) \subset \boldsymbol{H}^{\sigma-1/2}(\Gamma) \to \boldsymbol{H}^{\sigma-1/2}(\Gamma)$ defined by

(a) $\mathcal{D}(\mathcal{T}_\alpha(\lambda))=\{\boldsymbol{\phi} \in \boldsymbol{H}^{\sigma-1/2}(\Gamma);\ T_\alpha(\lambda)\boldsymbol{\phi} \in \boldsymbol{H}^{\sigma-1/2}(\Gamma)\}$,

(b) $\mathcal{T}_\alpha(\lambda)\boldsymbol{\phi}=T_\alpha(\lambda)\boldsymbol{\phi}$ for $\boldsymbol{\phi} \in \mathcal{D}(\mathcal{T}_\alpha(\lambda))$

where $\mathcal{D}(\mathcal{T}_\alpha(\lambda))$ denotes the domain of $\mathcal{T}_\alpha(\lambda)$. We define also a closed linear operator $\tilde{\mathcal{T}}_\alpha: \mathcal{D}(\tilde{\mathcal{T}}_\alpha) \subset \boldsymbol{H}^{\sigma-1/2}(\Gamma\times S) \to \boldsymbol{H}^{\sigma-1/2}(\Gamma\times S)$ by

($\tilde{\mathrm{a}}$) $\mathcal{D}(\tilde{\mathcal{T}}_\alpha)=\{\tilde{\boldsymbol{\phi}} \in \boldsymbol{H}^{\sigma-1/2}(\Gamma\times S);\ \tilde{T}_\alpha\tilde{\boldsymbol{\phi}} \in \boldsymbol{H}^{\sigma-1/2}(\Gamma\times S)\}$,

($\tilde{\mathrm{b}}$) $\tilde{\mathcal{T}}_\alpha\tilde{\boldsymbol{\phi}}=\tilde{T}_\alpha\tilde{\boldsymbol{\phi}}$ for $\tilde{\boldsymbol{\phi}} \in \mathcal{D}(\tilde{\mathcal{T}}_\alpha)$.

Since $\tilde{\mathcal{T}}_\alpha$ is densely defined as easily seen, $\tilde{\mathcal{T}}_\alpha$ admits its *adjoint* operator $\tilde{\mathcal{T}}_\alpha^*$: $\mathcal{D}(\tilde{\mathcal{T}}_\alpha^*) \subset \boldsymbol{H}^{-\sigma+1/2}(\Gamma\times S) \to \boldsymbol{H}^{-\sigma+1/2}(\Gamma\times S)$. Similarly, $\mathcal{T}_\alpha(\lambda)$ admits its adjoint

$\mathcal{T}_\alpha(\lambda)^*$.

LEMMA 2.5. *The closed linear operator $\tilde{\mathcal{T}}_\alpha^*$ is characterized by*

($\tilde{\mathrm{a}}^*$) $\mathcal{D}(\tilde{\mathcal{T}}_\alpha^*)=\{\tilde{\boldsymbol{\phi}}\in \boldsymbol{H}^{-\sigma+1/2}(\Gamma\times\boldsymbol{S});\ \tilde{T}_\alpha^*\tilde{\boldsymbol{\phi}}\in \boldsymbol{H}^{\sigma+1/2}(\Gamma\times\boldsymbol{S})\}$,

($\tilde{\mathrm{b}}^*$) $\tilde{\mathcal{T}}_\alpha^*\tilde{\boldsymbol{\phi}}=\tilde{T}_\alpha^*\tilde{\boldsymbol{\phi}}$ *for* $\tilde{\boldsymbol{\phi}}\in\mathcal{D}(\tilde{\mathcal{T}}_\alpha^*)$.

By the definition of $\tilde{\mathcal{T}}_\alpha$, Lemma 2.5 and Proposition 2.3, we have

$$\|\tilde{\boldsymbol{\phi}}\|_{\sigma-1/2,\Gamma\times S}\leqq C(\|\tilde{\mathcal{T}}_\alpha\tilde{\boldsymbol{\phi}}\|_{\sigma-1/2,\Gamma\times S}+\|\tilde{\boldsymbol{\phi}}\|_{\sigma-1,\Gamma\times S})\qquad \text{for all}\quad \tilde{\boldsymbol{\phi}}\in\mathcal{D}(\tilde{\mathcal{T}}_\alpha),$$

$$\|\tilde{\boldsymbol{\phi}}\|_{-\sigma+1/2,\Gamma\times S}\leqq C(\|\tilde{\mathcal{T}}_\alpha^*\tilde{\boldsymbol{\phi}}\|_{-\sigma+1/2,\Gamma\times S}+\|\tilde{\boldsymbol{\phi}}\|_{-\sigma,\Gamma\times S})\qquad \text{for all}\quad \tilde{\boldsymbol{\phi}}\in\mathcal{D}(\tilde{\mathcal{T}}_\alpha^*).$$

Furthermore, since $\boldsymbol{H}^s(\Gamma\times\boldsymbol{S})\hookrightarrow\boldsymbol{H}^{s-1/2}(\Gamma\times\boldsymbol{S})$ is compact for any $s\in\boldsymbol{R}$, $\tilde{\mathcal{T}}_\alpha$ and $\tilde{\mathcal{T}}_\alpha^*$ are, as well-known, semi-Fredholm operators (*i.e.*, operator T with finite dimensional kernel $\mathcal{N}(T)$ and closed range $\mathcal{R}(T)$).

As a result, by the same argument as in [14], we arrive at:

PROPOSITION 2.6. *Let $l\in\boldsymbol{Z}$. Then mapping $\mathcal{T}_\alpha(l^2)$: $\mathcal{D}(\mathcal{T}_\alpha(l^2))\subset\boldsymbol{H}^{s-1/2}(\Gamma)\to\boldsymbol{H}^{s-1/2}(\Gamma)$ is a Fredholm operator with the property that there exist finite subsets J and J^* of $\boldsymbol{Z}$ such that*

$$\dim\mathcal{N}(\mathcal{T}_\alpha(l^2))<\infty \ \text{if}\ l\in J,\quad =0\ \text{if}\ l\in\boldsymbol{Z}\setminus J;$$

$$\dim\mathcal{R}(\mathcal{T}_\alpha(l^2))=\dim\mathcal{N}(\mathcal{T}_\alpha^*(l^2))<\infty\ \text{if}\ l\in J^*,\quad =0\ \text{if}\ l\in\boldsymbol{Z}\setminus J^*.$$

END OF PROOF OF THEOREM I. Let $\sigma\geqq2$. Since the principal and subprincipal symbols of $T(\lambda)$ are, by Proposition 1.4, independent of $\lambda\geqq0$, so are those of $T_\alpha(\lambda)$; hence $T_\alpha(\lambda_1)-T_\alpha(\lambda_2)\in\boldsymbol{\Psi}_{phg}^{-1}(\Gamma)$ for any λ_1, $\lambda_2\geqq0$. Thus, $\mathcal{D}(\mathcal{T}_\alpha(\lambda))$ is also independent of $\lambda\geqq0$, and for any λ_1, $\lambda_2\geqq0$ the mapping $\mathcal{T}_\alpha(\lambda_1)-\mathcal{T}_\alpha(\lambda_2)$ admits an extension to a compact operator: $\boldsymbol{H}^{\sigma-1/2}(\Gamma)\to\boldsymbol{H}^{\sigma-1/2}(\Gamma)$.

On the other hand, Proposition 2.6 shows that the mapping $\mathcal{T}_\alpha(\lambda_0)$, $\lambda_0=l_0^2$ with some $l_0\in\boldsymbol{Z}\setminus(J\cup J^*)$, is a Fredholm operator with index 0. Therefore, for any $\lambda\geqq0$, $\mathcal{T}_\alpha(\lambda)=\mathcal{T}_\alpha(\lambda_0)+(\mathcal{T}_\alpha(\lambda)-\mathcal{T}_\alpha(\lambda_0))$ is a compact perturbation of a Fredholm operator with index 0, and hence is a Fredholm operator with index 0.

We finally show $\dim\mathcal{N}(\mathcal{T}_\alpha(\lambda))=0$. If $\boldsymbol{\phi}\in\mathcal{D}(\mathcal{T}_\alpha(\lambda))\subset\boldsymbol{H}^{\sigma-1/2}(\Gamma)$ satisfies $T_\alpha(\lambda)\boldsymbol{\phi}=\boldsymbol{o}$ on Γ, we have by putting $\boldsymbol{u}=P(\lambda)\boldsymbol{\phi}$

$$A_\lambda\boldsymbol{u}=\boldsymbol{o}\quad\text{in}\quad\Omega,\qquad B_\alpha\boldsymbol{u}=\boldsymbol{o}\quad\text{on}\quad\Gamma.$$

Thus Proposition 1.2. gives that $\boldsymbol{u}=\boldsymbol{o}$ and $\boldsymbol{\phi}=\boldsymbol{u}|_\Gamma=\boldsymbol{o}$. It therefore follows that

$$\operatorname{codim}\mathcal{R}(\mathcal{T}_\alpha(\lambda))=\operatorname{ind}\mathcal{T}_\alpha(\lambda)-\dim\mathcal{N}(\mathcal{T}_\alpha(\lambda))=0,$$

which completes the proof of Theorem I by Proposition 1.6. □

§ 3. Weak Solution of Problem (S).

In this section, we construct a weak solution to (S) using Theorem I (*cf.* Duvaut & Lions [2; Théorème 3.3, Chap. 3]).

DEFINITION. Suppose $\boldsymbol{f}\in \boldsymbol{L}^2(\Omega)$, $\boldsymbol{\phi}\in \boldsymbol{L}^2(\Gamma)$ and $\boldsymbol{\varphi}\in \boldsymbol{H}^{1/2}(\Gamma)$ in (S). We call $\boldsymbol{u}\in \boldsymbol{H}^1(\Omega)$ a *weak solution* of problem (S) if $\boldsymbol{u}|_\Gamma=\boldsymbol{\varphi}$ on Γ_D and

$$a(\boldsymbol{u}, \boldsymbol{\eta})=(\boldsymbol{f}, \boldsymbol{\eta})_\Omega+(\boldsymbol{\phi}, \boldsymbol{\eta})_{\Gamma_N} \quad \text{for all} \quad \boldsymbol{\eta}\in \boldsymbol{H}^1_0(\Omega\cup\Gamma_N). \tag{3.1}$$

Here $\boldsymbol{H}^1_0(\Omega\cup\Gamma_N)$ denotes the closure of $\boldsymbol{C}^\infty_0(\Omega\cup\Gamma_N):=\{\boldsymbol{u}\in \boldsymbol{C}^\infty(\bar{\Omega})\,;\ \mathrm{supp}\,\boldsymbol{u}\subset \Omega\cup\Gamma_N\}$ in $\boldsymbol{H}^1(\Omega)$. Since the interface $\Sigma=\bar{\Gamma}_N\cap\bar{\Gamma}_D$ between Γ_N and Γ_D is of class C^1, this space is characterized as

$$\boldsymbol{H}^1_0(\Omega\cup\Gamma_N)=\{\boldsymbol{u}\in \boldsymbol{H}^1(\Omega)\,;\ \boldsymbol{u}|_\Gamma=\boldsymbol{o} \quad \text{on } \Gamma_D\}\,.$$

See the Proof of Lemma 10 in Browder [1].

Let $\boldsymbol{f}\in \boldsymbol{L}^2(\Omega)$, $\boldsymbol{\phi}\in \boldsymbol{L}^2(\Gamma)$ and $\boldsymbol{\varphi}\in \boldsymbol{H}^{1/2}(\Gamma)$ be given. We begin with constructing a collection of approximate solutions of (S) by means of Theorem I. We may assume, without loss of generality, that $\mathrm{supp}\boldsymbol{\varphi}\subset\Gamma\setminus\bar{\gamma}_0$ with γ_0 an open subset of Γ such that $\bar{\gamma}_0\subset\Gamma_N$. Choose sequences $\{\boldsymbol{\phi}_m\}$ in $\boldsymbol{H}^{1/2}(\Gamma)$ and $\{\boldsymbol{\varphi}_m\}$ in $\boldsymbol{H}^{3/2}(\Gamma)$ with $\mathrm{supp}\boldsymbol{\varphi}_m\subset\Gamma\setminus\bar{\gamma}_0$ so that

$$\boldsymbol{\phi}_m \longrightarrow \boldsymbol{\phi} \quad \text{in} \quad \boldsymbol{H}^{-1/2}(\Gamma), \quad \boldsymbol{\varphi}_m \longrightarrow \boldsymbol{\varphi} \quad \text{in} \quad \boldsymbol{H}^{1/2}(\Gamma) \quad \text{as} \quad m\longrightarrow\infty\,. \tag{3.2}$$

Now, let $\{\varepsilon_m\}_{m=1}^\infty$ be an arbitrary decreasing sequence tending to 0 such that $\gamma_1\neq\emptyset$ where $\gamma_m=\{x\in\Gamma_D\,;\ \mathrm{dist}_\Gamma(x, \Gamma_N)\geqq\varepsilon_m\}$. It is easy to construct a family $\{\alpha_m(x)\}$ in $C^\infty(\Gamma)$ such that $0\leqq\alpha_m(x)\leqq 1$ on Γ and $\alpha_m(x)=1$ on Γ_N, $=0$ on γ_m. We set $B_m=B_{\alpha_m}$.

For each m, consider the approximate problem $(S)_m$ of (S) given by

$$(S)_m \qquad A\boldsymbol{u}=\boldsymbol{f} \quad \text{in } \Omega, \qquad B_m\boldsymbol{u}=\alpha_m(x)\boldsymbol{\phi}_m+(1-\alpha_m(x))\boldsymbol{\varphi}_m \quad \text{on } \Gamma.$$

By applying Theorem I, we get the unique solutien $\boldsymbol{u}_m\in \boldsymbol{H}^2(\Omega)$ of $(S)_m$.

THEOREM II. *The sequence $\{\boldsymbol{u}_m\}$ in $\boldsymbol{H}^2(\Omega)$ obtained above is $\boldsymbol{H}^1(\Omega)$-weakly convergent. The limit $\boldsymbol{u}\in \boldsymbol{H}^1(\Omega)$ gives an unique weak solution of problem (S). Moreover, it satisfies the estimate*

$$\|\boldsymbol{u}\|_{1,\Omega}\leqq C(\|\boldsymbol{f}\|_{-1,\Omega\cup\Gamma_N}+\|\boldsymbol{\phi}\|_{-1/2,\Gamma}+\|\boldsymbol{\varphi}\|_{1/2,\Gamma}) \tag{3.3}$$

where $\|\cdot\|_{-1,\Omega\cup\Gamma_N}$ denotes the norm of the dual space of $\boldsymbol{H}^1_0(\Omega\cup\Gamma_N)$.

PROOF. According to Theorem I, the boundary-value problem

$$Av=\boldsymbol{o} \quad \text{in } \Omega, \qquad v|_\Gamma=\boldsymbol{\phi}_m \quad \text{on } \Gamma$$

$$(\text{resp. } Aw=\boldsymbol{f} \quad \text{in } \Omega, \qquad B_m w=\alpha_m(x)(\boldsymbol{\phi}_m-Bv_m) \quad \text{on } \Gamma)$$

admits a unique solution v_m (resp. w_m)$\in \boldsymbol{H}^2(\Omega)$. Since v_m+w_m is a solution of $(S)_m$, it follows from the uniqueness property that $u_m=v_m+w_m$. By Green's formula (1.3), the solutions v_m and w_m satisfy

$$(3.4) \qquad \begin{cases} a(v_m, v_m)=(Bv_m, v_m)_\Gamma, \\ a(w_m, w_m)=(\boldsymbol{f}, w_m)_\Omega+\int_{\alpha_m(x)\neq 0}\left(\boldsymbol{\phi}_m-Bv_m-\dfrac{1-\alpha_m(x)}{\alpha_m(x)}w_m\right)\cdot\overline{w_m}\,dv_\Gamma. \end{cases}$$

Noting that $v_m|_{\gamma_0}=\boldsymbol{o}$ and $w_m|_{\gamma_1}=\boldsymbol{o}$, we obtain from (0.6) and (1.1) that $a(v, v)\geqq C_1\|v\|^2_{1,\Omega}$ for $v=v_m, w_m$. Using this and the fact that $BP(0)=T(0)\in \boldsymbol{\Psi}^1_{phg}(\Gamma)$, we have from (3.4)

$$(3.5) \qquad \begin{cases} C_1\|v_m\|^2_{1,\Omega}\leqq \|Bv_m\|_{-1/2,\Gamma}\|v_m\|_{1/2,\Gamma}\leqq \dfrac{C_1}{2}\|v_m\|^2_{1,\Omega}+C\|\boldsymbol{\phi}_m\|^2_{1/2,\Gamma}, \\ C_1\|w_m\|^2_{1,\Omega}\leqq (\boldsymbol{f}, w_m)_\Omega+(\|\boldsymbol{\phi}_m\|_{-1/2,\Gamma}+\|Bv_m\|_{-1/2,\Gamma})\|w_m\|_{1/2,\Gamma} \\ \qquad \leqq \dfrac{C_1}{2}\|w_m\|^2_{1,\Omega}+C(\|\boldsymbol{f}\|^2_{0,\Omega}+\|\boldsymbol{\phi}_m\|^2_{-1/2,\Gamma}+\|\boldsymbol{\phi}_m\|^2_{1/2,\Gamma}). \end{cases}$$

Thus (3.2) and (3.5) yield

$$\|u_m\|_{1,\Omega}\leqq \|v_m\|_{1,\Omega}+\|w_m\|_{1,\Omega}\leqq C(\|\boldsymbol{f}\|_{0,\Omega}+\|\boldsymbol{\phi}\|_{-1/2,\Gamma}+\|\boldsymbol{\phi}\|_{1/2,\Gamma}).$$

This shows that, for any subsequence $\{u_{m'}\}$ of $\{u_m\}$, some subsequence $\{u_{m''}\}$ of $\{u_{m'}\}$ has a weak limit u^0 in $\boldsymbol{H}^1(\Omega)$. If u^0 is a unique weak solution of (S), which will be shown below, then we see from the uniqueness that the sequence $\{u_m\}$ itself converges weakly to u^0 in $\boldsymbol{H}^1(\Omega)$.

Now, since $\alpha_m(x)=1$ on Γ_N, we have by (1.3)

$$(3.6) \qquad a(u_{m''}, \eta)=(\boldsymbol{f}, \eta)_\Omega+(\boldsymbol{\phi}_{m''}, \eta)_\Gamma \qquad \text{for all} \quad \eta\in \boldsymbol{H}^1_0(\Omega\cup\Gamma_N).$$

And, for any $\zeta\in \boldsymbol{C}^\infty(\Gamma)$ with support in Γ_D, we have

$$(u_{m''}, \zeta)_\Gamma=(\boldsymbol{\phi}_{m''}, \zeta)_\Gamma+(\alpha_{m''}(\boldsymbol{\phi}_{m''}-\boldsymbol{\phi}_{m''}-Bu_{m''}+u_{m''}), \zeta)_\Gamma;$$

hence $(u_{m''}, \zeta)_\Gamma=(\boldsymbol{\phi}_{m''}, \zeta)_\Gamma$ if m'' is so large that $\varepsilon_{m''}<\mathrm{dist}_\Gamma(\mathrm{supp}\zeta, \Sigma)$. Letting $m''\to\infty$ here and in (3.6), we see that u is a weak solution of problem (S). Furthermore, the uniqueness of weak solution is shown as follows: Let $u^1\in \boldsymbol{H}^1(\Omega)$ be a weak solution of (S) with $\{\boldsymbol{f}, \boldsymbol{\phi}, \boldsymbol{\phi}\}=\{\boldsymbol{o}, \boldsymbol{o}, \boldsymbol{o}\}$. Then, by definition, $u^1\in \boldsymbol{H}^1_0(\Omega\cup\Gamma_N)$ and $a(u^1, u^1)=0$, so that (0.6) and Korn's inequality (1.2) give us that $u^1=\boldsymbol{o}$.

Similarly, we see that the sequences $\{v_m\}$, $\{w_m\}$ are also $\boldsymbol{H}^1(\Omega)$-weakly convergent and that their limits $v^0, w^0\in \boldsymbol{H}^1(\Omega)$ satisfy $u^0=v^0+w^0$. Since $w^0\in \boldsymbol{H}^1_0(\Omega\cup\Gamma_N)$, we have, as $m\to\infty$,

$$(\boldsymbol{f}, \boldsymbol{w}_m)_\Omega \longrightarrow (\boldsymbol{f}, \boldsymbol{w}^0)_\Omega \leqq \|\boldsymbol{f}\|_{-1, \Omega\cup\Gamma_N} \|w^0\|_{1,\Omega}.$$

Therefore, the desired estimate (3.3) follows immediately by letting $m\to\infty$ in estimate (3.5). □

COROLLARY 3.1. *Let* $\boldsymbol{f}\in\boldsymbol{H}^{s-2}(\Omega)$, $\boldsymbol{\phi}\in\boldsymbol{H}^{s-3/2}(\Gamma)$ *and* $\boldsymbol{\psi}\in\boldsymbol{H}^{s-1/2}(\Gamma)$ *for* $s\geqq 2$. *And let* $\boldsymbol{u}_m\in\boldsymbol{H}^s(\Omega)$ *be the unique solution of problem* $(S)_m$ *with* $\boldsymbol{\phi}_m=\boldsymbol{\phi}$, $\boldsymbol{\psi}_m=\boldsymbol{\psi}$ *for each* m. *Then the sequence* $\{\boldsymbol{u}_m\}$ *converges to the weak solution* $\boldsymbol{u}$ *of problem* (S) *weakly in* $\boldsymbol{H}^s_{loc}(\bar{\Omega}\setminus\Sigma)$, *that is,* $\{\boldsymbol{u}_m\}$ *cenverges to* $\boldsymbol{u}$ *weakly in* $\boldsymbol{H}^s(\Omega')$ *for any subdomain* Ω' *of* Ω *(with* C^∞*-boundary) such that* $\bar{\Omega}'\subset\bar{\Omega}\setminus\Sigma$. *Furthermore, we have the estimate*

$$\|\boldsymbol{u}\|_{s,\Omega'}\leqq C(\Omega', s)(\|\boldsymbol{f}\|_{s-2,\Omega}+\|\boldsymbol{\phi}\|_{s-3/2,\Gamma}+\|\boldsymbol{\psi}\|_{s-1/2,\Gamma}).$$

PROOF. Although the claim can be shown by the general theory of elliptic systems, our proof is an application of Theorem I.

Let Ω' be any such domain in Ω as stated above. All we have to do is to show that there exists a constant $C=C(\Omega', s)>0$ such that

$$\|\boldsymbol{u}_m\|_{s,\Omega'}\leqq C(\|\boldsymbol{f}\|_{s-2,\Omega}+\|\boldsymbol{\phi}\|_{s-3/2,\Gamma}+\|\boldsymbol{\psi}\|_{s-1/2,\Gamma}) \tag{3.7}$$

for large m. Indeed, the rest of the proof is similar to the latter half of Proof of Theorem II.

Now we show estimate (3.7). For $1\leqq l\leqq[s]+1$, we choose functions $\eta_l\in C^\infty(\bar{\Omega})$ such that $0\leqq\eta_l\leqq 1$ on $\bar{\Omega}$ and $\eta_l=1$ on $\{\mathrm{dist}(x,\Sigma)\geqq l\delta\}$, $=0$ on $\{\mathrm{dist}(x,\Sigma)\leqq(l-1)\delta\}$ where $\delta=\mathrm{dist}(\bar{\Omega}',\Sigma)/([s]+1)$. Let m_0 be a number such that $\varepsilon_{m_0}<\delta$. Since $\alpha_m=\alpha_{m_0}$ on $\mathrm{supp}(\eta_l|_\Gamma)$ for all $m\geqq m_0$ and $2\leqq l\leqq[s]+1$, the equations in $(S)_m$ with $\{\boldsymbol{\phi}_m, \boldsymbol{\psi}_m\}=\{\boldsymbol{\phi}, \boldsymbol{\psi}\}$ multiplied by η_l are

$$\begin{cases} A(\eta_l\boldsymbol{u}_m)=[A, \eta_l]\boldsymbol{u}_m+\eta_l\boldsymbol{f} & \text{in } \Omega, \\ B_{m_0}(\eta_l\boldsymbol{u}_m)=\alpha_{m_0}([B, \eta_l]\boldsymbol{u}_m+\eta_l\boldsymbol{\phi})+(1-\alpha_{m_0})\eta_l\boldsymbol{\psi} & \text{on } \Gamma \end{cases}$$

where $[\cdot,\cdot]$ denotes the commutator. Thus an application of Theorem I shows that, for any $2\leqq t\leqq s$, $m\geqq m_0$ and $2\leqq l\leqq[s]+1$,

$$\begin{aligned} \|\eta_l\boldsymbol{u}_m\|_{t,\Omega}&\leqq C(\|[A, \eta_l]\boldsymbol{u}_m+\eta_l\boldsymbol{f}\|_{t-2,\Omega} \\ &\quad+\|\alpha_{m_0}([B, \eta_l]\boldsymbol{u}_m+\eta_l\boldsymbol{\phi})+(1-\alpha_{m_0})\boldsymbol{\psi}\|_{\alpha_{m_0};t-3/2,\Gamma}) \\ &\leqq C(\|\eta_{l-1}\boldsymbol{u}_m\|_{t-1,\Omega}+\|\boldsymbol{f}\|_{t-2,\Omega}+\|\boldsymbol{\phi}\|_{t-3/2,\Gamma}+\|\boldsymbol{\psi}\|_{t-1/2,\Gamma}). \end{aligned} \tag{3.8$_{l,t}$}$$

Using $(3.8)_{l,t}$ for $l=t=2,\cdots,[s]$ and (3.3), we have

$$\begin{aligned} \|\eta_{[s]}\boldsymbol{u}_m\|_{s-1,\Omega}&\leqq\|\eta_{[s]}\boldsymbol{u}_m\|_{[s],\Omega} \\ &\leqq C(\|\boldsymbol{f}\|_{[s]-2,\Omega}+\|\boldsymbol{\phi}\|_{[s]-3/2,\Gamma}+\|\boldsymbol{\psi}\|_{[s]-1/2,\Gamma}), \end{aligned}$$

which combined with $(3.8)_{[s]+1,s}$ gives (3.7). □

§4. Simple Generalization.

For a forthcoming paper (Ito [9]) dealing with a dynamic problem mentioned in Introduction, we give a simple extension of Theorem I.

When we consider $(S_\alpha)_\lambda$ only for large $\lambda>0$, it is essential for the arguments in §§2, 3 that the real-valued functions $a_{ijkh}(x)\in C^\infty(\bar{\Omega})$ possess the property of *symmetry*

$$a_{ijkh}(x)=a_{khij}(x) \qquad \text{on } \bar{\Omega} \tag{4.1}$$

and the property of *coerciveness*

$$\sum_{i,j,k,h}\int_\Omega a_{ijkh}(x)\partial_h u_k\cdot\overline{\partial_j u_i}\,dx\geqq c_4\|\boldsymbol{u}\|_{1,\Omega}^2-c_5\|\boldsymbol{u}\|_{0,\Omega}^2 \tag{4.2}$$

for all $\boldsymbol{u}\in \boldsymbol{H}_0^1(\Omega\cup\Gamma_\alpha)$ with $\Gamma_\alpha=\{x\in\Gamma;\ \alpha(x)\neq 0\}$.

Now we redefine differential systems A in Ω and B on Γ by

$$(A\boldsymbol{u})_i=-\sum_{j,k,h}\partial_j(a_{ijkh}(x)\partial_h u_k)+\sum_{j,k}b_{ijk}(x)\partial_k u_j+\sum_j c_{ij}(x)u_j, \tag{4.3}$$

$$(B\boldsymbol{u})_i=\Big(\sum_{j,k,h}\nu_i(x)a_{ijkh}(x)\partial_h u_k+\sum_j\tau_{ij}(x)u_j\Big)\Big|_\Gamma \tag{4.4}$$

where all the coefficients are real-valued C^∞-functions on $\bar{\Omega}$ or Γ and $a_{ijkh}(x)$ satisfy (4.1) and (4.2). We note that these conditions imply that A is strongly elliptic on $\bar{\Omega}$ and $\{A, B\}$ satisfies the strong complemention condition on $\bar{\Gamma}_\alpha$.

Let $\alpha(x)$ be as before but we allow the case $\alpha(x)\equiv 1$, and let $\omega_{ij}(x)$ be real-valued C^∞-functions on Γ such that $\omega(x)=(\omega_{ij}(x))$ is positive definite on Γ. Then Theorem I can be extended as follows.

THEOREM I′. *Let $\sigma\geqq 2$ and $\lambda\in\boldsymbol{R}$. The mapping*

$$\{A_\lambda,\ B_{\alpha,\omega}\}:\ \boldsymbol{H}^\sigma(\Omega)\ni\boldsymbol{u}\longrightarrow\{A_\lambda\boldsymbol{u},\ B_{\alpha,\omega}\boldsymbol{u}\}\in\boldsymbol{H}^{\sigma-2}(\Omega)\times\boldsymbol{H}_{(\alpha)}^{\sigma-3/2}(\Gamma). \tag{4.5}$$

is a Fredholm operator with index 0 *where $A_\lambda=\lambda\mathrm{I}+A$, $B_{\alpha,\omega}=\alpha(x)B+(1-\alpha(x))\omega(x)$. In particular, if λ is sufficiently large, then* (4.5) *is an (algebraic and topological) isomorphism.*

For the proof, we prepare the following two lemmas.

LEMMA 4.1. *Let $\sigma\geqq 2$. If λ is sufficiently large, the mapping* (4.5) *is an injection. If $\alpha(x)\equiv 0$ or >0 on Γ in addition, then it is then an isomorphism.*

PROOF. Let $\boldsymbol{u}\in\boldsymbol{H}^\sigma(\Omega)$, $\sigma\leqq 2$, be in the kernel of (4.5). Then, we have by integration by parts

$$\sum_{i,j,k,h}\int_{\Omega} a_{ijkh}(x)\partial_h u_k \cdot \overline{\partial_j u_i}\, dx + \lambda \|\boldsymbol{u}\|_{0,\Omega}^2$$

$$= -\sum_i \int_{\Omega}\Big(\sum_{j,k} b_{ijk}(x)\partial_k u_j + \sum_j c_{ij}(x)u_j\Big)\overline{u_i}\, dx$$

$$-\int_{\Gamma_\alpha}\Big(\frac{1-\alpha(x)}{\alpha(x)}\omega(x)\boldsymbol{u}\cdot\bar{\boldsymbol{u}} - \sum_{i,i}\tau_{ij}(x)u_j\overline{u_i}\Big)dv_\Gamma .$$

Using (4.2) and the positivity of $\omega(x)$, we obtain

$$c_4\|\boldsymbol{u}\|_{1,\Omega}^2 + (\lambda - c_5)\|\boldsymbol{u}\|_{0,\Omega}^2 \leqq C(\|\boldsymbol{u}\|_{1,\Omega}\|\boldsymbol{u}\|_{0,\Omega} + \|\boldsymbol{u}\|_{0,\Gamma}^2),$$

from which the first claim follows immediately. The second is due to the same argument as in Proof of Lemma 1.3. □

LEMMA 4.2. *Let $\sigma \geqq 2$, and A^0, B^0 be the first terms of A, B in* (4.3) *and* (4.4). *If λ is sufficiently large, then for any $\boldsymbol{f} \in \boldsymbol{H}^{\sigma-2}(\Omega)$ and $\boldsymbol{\phi} \in \boldsymbol{H}^{\sigma-3/2}(\Gamma)$ there exists a $\boldsymbol{u} \in \boldsymbol{H}^{\sigma}(\Omega)$ which satisfies $A_\lambda^0\boldsymbol{u} = \boldsymbol{f}$ in Ω, $B^0\boldsymbol{u} = \boldsymbol{\phi}$ in a neighborhood of Γ_α on Γ where $A_\lambda^0 = \lambda \mathrm{I} + A^0$, and the estimate*

$$\|\boldsymbol{u}\|_{\sigma,\Omega} \leqq C(\|\boldsymbol{f}\|_{\sigma-2,\Omega} + \|\boldsymbol{\phi}\|_{\sigma-3/2,\Gamma}).$$

PROOF. We can choose a bounded domain $\hat{\Omega}$ including Ω with C^∞-boundary $\hat{\Gamma}$ and C^∞-extensions $\hat{a}_{ijkh}(x)$ of $a_{ijkh}(x)$ to $\bar{\hat{\Omega}}$ so that (i) $\hat{\Gamma}$ includes an open neighborhood γ of $\bar{\Gamma}_\alpha$ in Γ and (ii) (4.1) and (4.2) are valid for $\hat{a}_{ijkh}(x)$ with Ω, Γ_α replaced by $\hat{\Omega}$, $\hat{\Gamma}$ (that is, $\hat{A}^0$ is strongly elliptic on $\bar{\hat{\Omega}}$ and $\{\hat{A}^0, \hat{B}^0\}$ is strongly complementing on $\hat{\Gamma}$ where $\hat{A}^0$, $\hat{B}^0$ are the associated A^0, B^0 with $\hat{a}_{ijkh}(x)$, $\hat{\Omega}$ and $\hat{\Gamma}$). Here we need to pay attention to the fact that the strong complementing condition at $x_0 \in \hat{\Gamma}$ depends (continuously) not only on $\hat{a}_{ijkh}(x_0)$ but also on the direction of the normal at x_0 to $\hat{\Gamma}$.

Take a nonnegative function $\zeta(x) \in C^\infty(\Gamma)$ with support in γ such that $\zeta(x)=1$ near $\bar{\Gamma}_\alpha$, and define $\hat{\boldsymbol{\phi}} \in \boldsymbol{H}^{\sigma-3/2}(\hat{\Gamma})$, for any given $\boldsymbol{\phi} \in \boldsymbol{H}^{\sigma-3/2}(\Gamma)$, by $\hat{\boldsymbol{\phi}} = \zeta(x)\boldsymbol{\phi}$ on γ, $= \boldsymbol{o}$ on $\hat{\Gamma} \setminus \gamma$. Then we have

$$\|\hat{\boldsymbol{\phi}}\|_{\sigma-3/2,\hat{\Gamma}} \leqq C\|\zeta(x)\boldsymbol{\phi}\|_{\sigma-3/2,\Gamma} \leqq C\|\boldsymbol{\phi}\|_{\sigma-3/2,\Gamma} \qquad \text{for all} \quad \boldsymbol{\phi} \in \boldsymbol{H}^{\sigma-3/2}(\Gamma).$$

Also, any $\boldsymbol{f} \in \boldsymbol{H}^{\sigma-2}(\Omega)$ admits an extension $\hat{\boldsymbol{f}} \in \boldsymbol{H}^{\sigma-2}(\hat{\Omega})$ such that $\|\hat{\boldsymbol{f}}\|_{\sigma-2,\hat{\Omega}} \leqq C\|\boldsymbol{f}\|_{\sigma-2,\Omega}$. Now consider the boundary-value problem

$$\hat{A}_\lambda^0\hat{\boldsymbol{u}} = \hat{\boldsymbol{f}} \quad \text{in } \hat{\Omega}, \qquad \hat{B}^0\hat{\boldsymbol{u}} = \hat{\boldsymbol{\phi}} \quad \text{on } \hat{\Gamma}. \tag{4.6}$$

By an argument similar to Proof of Lemma 1.3 (see also the preceding lemma), we have, for a sufficiently large λ, a unique solution $\hat{\boldsymbol{u}} \in \boldsymbol{H}^{\sigma}(\hat{\Omega})$ of problem (4.6), which satisfies the estimate

$$\|\hat{u}\|_{\sigma,\hat{\Omega}}\leqq C(\|\hat{f}\|_{\sigma-2,\hat{\Omega}}+\|\hat{\phi}\|_{\sigma-3/2,\hat{\Gamma}})\leqq C'(\|f\|_{\sigma-2,\Omega}+\|\phi\|_{\sigma-3/2,\Gamma}).$$

Thus $u:=\hat{u}|_{\Omega}$ is a desired one. □

PROOF OF THEOREM I′. By Lemma 4.1 and the compactness of the map:

$$H^{\sigma}(\Omega)\ni u \longrightarrow \{(\sum_{j,k} b_{ijk}\partial_k u_j+\sum_j c_{ij}u_j),\ (\alpha\sum_j \tau_{ij}u_j)\}\in H^{\sigma-2}(\Omega)\times H^{\sigma-3/2}_{(\alpha)}(\Gamma),$$

we have only to show that, for sufficiently large λ,

$$\{A_{\lambda}^0,\ B_{\alpha,\omega}^0\}:\ H^{\sigma}(\Omega)\ni u \longrightarrow \{A_{\lambda}^0 u,\ B_{\alpha,\omega}^0 u\}\in H^{\sigma-2}(\Omega)\times H^{\sigma-3/2}_{(\alpha)}(\Gamma)$$

is an isomorphism where $B_{\alpha,\omega}^0=\alpha(x)B^0+(1-\alpha(x))\omega(x)$.

Applying Lemma 4.1 in the case $\{A_{\lambda},\ B_{\alpha,\omega}\}=\{A_{\lambda}^0,\ Dirichlet\}$, we can define the Poisson operator $P^0(\lambda)$ for A_{λ}^0 if $\lambda\geqq\lambda_1$ with λ_1 large enough. Then Proposition 1.4 is valid for $T^0(\lambda):=B^0P^0(\lambda)$, $\lambda\geqq\lambda_1$, except that the principal symbol $t_1^0(x,\xi)$ of $T^0(\lambda)$ is strongly elliptic on $\bar{\Gamma}_{\alpha}$ in the sense that for some $c_2^0>0$

$$t_1^0(x,\xi)\geqq c_2^0|\xi|_{\Gamma}\mathrm{I} \qquad \text{for all}\quad (x,\xi)\in\bigcup_{x\in\bar{\Gamma}_{\alpha}} T_x^*(\Gamma)\setminus\{0\}\subset T^*(\Gamma)\setminus 0$$

And, by virtue of Lemma 4.2, Proposition 1.6 is also valid if we replace $T_{\alpha}(\lambda)$, $\lambda\geqq 0$, with $T_{\alpha,\omega}^0(\lambda)=\alpha(x)T^0(\lambda)+(1-\alpha(x))\omega(x)$. Moreover, the argument in §2 will be justified in this case if we replace A, $P(\lambda)$ and $T_{\alpha}(\lambda)$ with $A_{\lambda_1}^0$, $P^0(\lambda)$ and $T_{\alpha,\omega}^0(\lambda)$, respectively; we have only to remark that, in Proposition 2.2, the corresponding principal symbol $\tilde{t}_1^0(x,\xi;y,\eta)$ satisfies only the following condition:

$$\tilde{t}_1^0(x,\xi;y,\eta)\geqq c_3^0|(\xi,\eta)|_{\Gamma\times S}\mathrm{I},\qquad c_3^0>0:\ \text{const},$$

$$\text{for all}\quad (x,\xi;y,\eta)\in\bigcup_{(x,y)\in\bar{\Gamma}_{\alpha}\times S} T_{(x,y)}^*(\Gamma\times S)\setminus\{0\}\subset T^*(\Gamma\times S)\setminus 0,$$

which is weaker than (2.2) but sufficient for our argument. □

Appendix. Proof of Theorem 2.4.

For simplicity, we abbreviate $(\cdot,\ \cdot)_M$ and $\|\cdot\|_{s,M}$ as $(\cdot,\ \cdot)$ and $\|\cdot\|_s$, respectively.

PROOF OF INEQUALITY (2.3).

First step (Reduction to the case $P=P^*$, $c_0=0$ and $q_0(x,\xi)=\mathrm{I}$). It suffices to consider the case $P=P^*$ and $c_0=0$. In fact,

$$\mathrm{Re}(Pu,\ u)=((\mathrm{Re}\,P-c_0\Lambda_M^{m-1}\mathrm{I})u,\ u)+c_0\|u\|_{(m-1)/2}^2$$

where $\mathrm{Re}\,P=(P+P^*)/2$, and the principal and subprincipal symbols of $\mathrm{Re}\,P-c_0\Lambda_M^{m-1}\mathrm{I}$ are given respectively by

$$\mathrm{Re}\,p_m(x,\xi)=a_m(x,\xi)\mathrm{Re}\,q_0(x,\xi),\qquad \mathrm{Re}\,p_{m-1}^s(x,\xi)-c_0|\xi|_M^{m-1}\mathrm{I}.$$

Assume that $P=P^*$ and $c_0=0$, so $q_0=q_0^*$. Let Q_1 (iesp. $Q_2)\in\Psi^0_{phg}(M)$ be a formally self-adjoit pseudo-differential operator with principal symbol $q_0^{-1/2}$ (resp. $q_0^{1/2}$) and subprincipal symbol 0 (resp. $(\sqrt{-1}/2)q_0^{1/2}\{q_0^{-/2}, q_0^{1/2}\}$). Since $Q_1Q_2\equiv Q_2Q_1\equiv \mathrm{I} \bmod \Psi^{-2}_{phg}(M)$, we have

(A.1) $\quad (Pu, u)\geqq(Q_1PQ_1Q_2u, Q_2u)-C\|u\|^2_{(m-2)/2} \quad$ for all $u\in C^\infty(M)$.

On the other hand, the principal symbol $\tilde{p}_m(x, \xi)$ and subprincipal symbol $\tilde{p}^s_{m-1}(x, \xi)$ of $\tilde{P}:=Q_1PQ_1$ are given respectively by $\tilde{p}_m=a_m\mathrm{I}$ and

$$\tilde{p}^s_{m-1}=q_0^{-1/2}p^s_{m-1}q_0^{-1/2}-\frac{\sqrt{-1}}{2}(\{q_0^{-1/2}, a_mq_0\}q_0^{-1/2}+\{a_mq_0^{1/2}, q_0^{-1/2}\}).$$

Since a_m vanishes to the second order on $\Sigma:=\Sigma_{a_m}$, condition (ii) of Theorem 2.4 is equivalent to

$$\tilde{p}^s_{m-1}(x, \xi)+\frac{1}{2}(\mathrm{Tr}^+H(x, \xi))\mathrm{I}\geqq 0 \qquad \text{on } \Sigma$$

where $H=H_{a_m}$. Now, suppose that Theorem 2.4 is valid for the case $q_0=\mathrm{I}$. Then we have for any $\varepsilon>0$

(A.2) $\quad (\tilde{P}Q_2u, Q_2u)\geqq-\varepsilon\|Q_2u\|^2_{(m-1)/2}-C(\varepsilon)\|Q_2u\|^2_{(m-2)/2}$

$$\geqq-\varepsilon C_1\|u\|^2_{(m-1)/2}-C(\varepsilon)\|u\|^2_{(m-2)/2} \qquad \text{for all } u\in C^\infty(M)$$

where the constant $C_1>0$ depends only on Q_2. The desired inequality (2.3) follows immediately from inequalities (A.1) and (A.2).

Second step (Proof of the case $P=P^*$, $c_0=0$ and $q_0(x, \xi)=\mathrm{I}$). Fix an $\varepsilon>0$ arbitrarily. We first show that, for any $(x_0, \xi_0)\in T^*(M)\setminus 0$, there exists a conic neighborhood $\Gamma_0\subset T^*(M)\setminus 0$ of (x_0, ξ_0) with the following property: Let $\phi_0(x, \xi)$ be any real-valued symbol homogeneous in $\xi\neq 0$ of degree 0 with support in Γ_0. Then we have

(A.3) $\quad (P\Phi u, \Phi u)\geqq-\varepsilon\|\Phi u\|^2_{(m-1)/2}-C(\varepsilon, \Phi)\|u\|^2_{(m-2)/2} \quad$ for all $u\in C^\infty(M)$

where $\Phi\in\Psi^0_{phg}(M)$ is any formally self-adjoint pseudo-differential operator with principal symbol ϕ_0 and subprincipal symbol 0.

When $(x_0, \xi_0)\notin\Sigma$, there is a conic neighborhood Γ_0 of (x_0, ξ_0) such that $a_m(x, \xi)\geqq 2\delta|\xi|^m_M$ on Γ_0 for some $\delta>0$, so by the Gårding inequality

$$(P\Phi u, \Phi u)\geqq\delta\|\Phi u\|^2_{m/2}-C(\Phi)\|u\|^2_{(m-2)/2} \qquad \text{for all } u\in C^\infty(M)$$

with any $\Phi\in\Psi^0_{phg}(M)$ as above.

When $(x_0, \xi_0)\in\Sigma$, we define a symbol $a^s_{m-1}(x, \xi)$ by

$$a^s_{m-1}(x, \xi)=\Big(\frac{\varepsilon}{4}-\mathrm{Tr}^+H(x_0, \xi_0/|\xi_0|_M)\Big)|\xi|^{m-1}_M.$$

Then, by the continuity of $\mathrm{Tr}^+H(x, \xi)$ on Σ, there exists a conic neighborhood Γ_0 of (x_0, ξ_0) such that

$$\text{(A.4)} \qquad \frac{\varepsilon}{8}|\xi|_M^{m-1} \leqq a_{m-1}^s(x, \xi)+\mathrm{Tr}^+H(x, \xi) \leqq \frac{\varepsilon}{2}|\xi|_M^{m-1} \quad \text{on} \quad \Gamma_0 \cap \Sigma.$$

If $A \in \Psi_{phg}^m(M)$ is a formally self-adjoint pseudo-differential operator with principal symbol a_m and subprincipal symbol a_{m-1}^s, the usual Melin's inequality (see Hörmander [5; Theorem 22.3.3]) gives

$$\text{(A.5)} \quad (A\Phi u, \Phi u) \geqq -\varepsilon\|\Phi u\|_{(m-1)/2}^2 - C(\varepsilon, \Phi)\|u\|_{(m-2)/2}^2 \quad \text{for all} \quad u \in C^\infty(M).$$

On the other hand, $R := P - A\mathrm{I} \in \Psi_{phg}^{m-1}(M)$ is a formally self-adjoint pseudo-differential operator with principal symbol $r_{m-1} := p_{m-1}^s - a_{m-1}^s \mathrm{I}$, which satisfy by virtue of (A.4) and condition (ii)

$$r_{m-1} = (p_{m-1}^s - (\mathrm{Tr}^+H)\mathrm{I}) - (a_{m-1}^s - \mathrm{Tr}^+H)\mathrm{I} \geqq -\frac{\varepsilon}{2}|\xi|_M^{m-1}\mathrm{I} \quad \text{on} \quad \Gamma_0 \cap \Sigma.$$

Thus, by shrinking Γ_0 if necessary, we have $r_{m-1} \geqq -\varepsilon|\xi|_M^{m-1}\mathrm{I}$ on Γ_0, so that by the sharp Gårding inequality

$$(R\Phi u, \Phi u) \geqq -\varepsilon\|\Phi u\|_{(m-1)/2}^2 - C(\varepsilon, \Phi)\|u\|_{(m-2)/2}^2 \quad \text{for all} \quad u \in C^\infty(M)$$

with any Φ as above. This and (A.5) show (A.3) in this case.

To complete the proof, we choose finite number of real-valued symbols $\phi_{0j}(x, \xi) \geqq 0$ homogeneous in $\xi \neq 0$ of degree 0 with so small support that (A.3) is valid for each Φ_j and $\sum_j \phi_{0j}^2 = 1$ in $T^*(M) \setminus 0$ where $\Phi_j \in \Psi_{phg}^0(M)$ is a formally self-adjoint pseudo-differential operator with principal symbol ϕ_{0j} and subprincipal symbol 0. Since $\sum_j \Phi_j^2 - \mathrm{I} \in \Psi_{phg}^{-2}(M)$ and $[[P, \Phi_j], \Phi_j] \in \Psi_{phg}^{m-2}(M)$, we therefore obtain that

$$\begin{aligned}(Pu, u) &= \sum_j (P\Phi_j u, \Phi_j u) + \mathrm{Re}((\mathrm{I} - \sum_j \Phi_j^2)Pu, u) + \frac{1}{2}\sum_j ([[P, \Phi_j], \Phi_j]u, u) \\ &\geqq -\varepsilon \sum_j \|\Phi_j u\|_{(m-1)/2}^2 - C(\varepsilon)\|u\|_{(m-1)/2}^2 \geqq -\varepsilon\|u\|_{(m-1)/2}^2 - C(\varepsilon)\|u\|_{(m-2)/2}^2\end{aligned}$$

for all $u \in C^\infty(M)$. □

PROOF OF INEQUALITIES (2.4). Let $\sigma \in \boldsymbol{R}$. The principal and subprincipal symbols of $\Lambda_M^{-\sigma} P \Lambda_M^\sigma$ are given respectively by p_m and $p_{m-1}^s + \sigma\sqrt{-1}|\xi|_M\{|\xi|_M, p_m\}$. Since $\{|\xi|_M, p_m\} = 0$ on Σ, it follows from (2.3) that for any $\varepsilon \in (0, c_0)$

$$\mathrm{Re}(\Lambda_M^{-\sigma} P \Lambda_M^\sigma v, v) \geqq (c_0 - \varepsilon)\|v\|_{(m-1)/2}^2 - C(\varepsilon, \sigma)\|v\|_{(m-2)/2}^2.$$

By putting $v = \Lambda_M^{(m-1)/2} u$ and $\sigma = s - (m-1)/2$ in the above, we have

$$(c_0-\varepsilon)\|u\|^2_{s+m-1}-C(\varepsilon, s)\|u\|^2_{s+m-3/2}\leqq \mathrm{Re}(\Lambda^s_M Pu, \Lambda^{s+m-1}_M u)$$
$$\leqq \frac{1}{2\delta}\|Pu\|^2_s+\frac{\delta}{2}\|u\|^2_{s+m-1} \qquad \text{for all} \quad u\in C^\infty(M).$$

Putting $\delta=c_0-\varepsilon$, we obtain the former of (2.4). As for the latter, we have only to note that, if P satisfies (i) and (ii), so does P^*. □

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