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REALIZATIONS OF INVOLUTIVE AUTOMORPHISMS σ AND G^σ OF EXCEPTIONAL LINEAR LIE GROUPS G, PART III, $G=E_8$

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M. Berger [1] classified involutive automorphisms σ of simple Lie algebras $\mathfrak{g}$ and determined the type of the subalgebras $\mathfrak{g}^\sigma$ of fixed points. In the preceding papers [Y1], [Y2], we found involutive automorphisms σ and realized the subgroups G^σ of fixed points explicitly for the connected exceptional universal linear Lie groups G of type G_2, F_4, E_6 and E_7. In this paper we consider the case of type E_8. Our results are as follows.

G	G^σ	σ			
$E_8{}^C$	$(SL(2, C)\times E_7{}^C)/\boldsymbol{Z}_2$	υ			
	$Ss(16, C)$	$\tilde{\lambda}\gamma$			
$E_8{}^C$	E_8	$\tau\tilde{\lambda}$			
E_8	$(SU(2)\times E_7)/\boldsymbol{Z}_2$	υ			
	$Ss(16)$	$\tilde{\lambda}\gamma$			
$E_8{}^C$	$E_{8(8)}$	$\tau\gamma$	$\tau\tilde{\lambda}\upsilon\gamma$	$\tau\sigma\gamma$	$\tau\upsilon\gamma$
$E_{8(8)}$	$(SL(2, \boldsymbol{R})\times E_{7(7)})/\boldsymbol{Z}_2\times 2$	υ			
	$(SU(2)\times E_{7(-5)})/\boldsymbol{Z}_2$		υ		
	$Ss(16)$	$\tilde{\lambda}\gamma$			
	$Sso(8, 8)\times 2$			$\tilde{\lambda}\gamma$	
	$Sso^*(16)\times 2$				$\tilde{\lambda}\gamma$
$E_8{}^C$	$E_{8(-24)}$	τ	$\tau\tilde{\lambda}\upsilon$	$\tau\tilde{\lambda}\gamma$	$\tau\tilde{\lambda}\upsilon\sigma$
$E_{8(-24)}$	$(SL(2, \boldsymbol{R})\times E_{7(-25)})/\boldsymbol{Z}_2\times 2$	υ			
	$(SU(2)\times E_7)/\boldsymbol{Z}_2$		υ		
	$(SU(2)\times E_{7(-5)})/\boldsymbol{Z}_2$			υ	
	$So(4, 12)$				σ
	$Sso^*(16)\times 2$		$\tilde{\lambda}\gamma$		

This paper is a continuation of [Y1], [Y2] and we use the same notations as them. So the numbering of sections and theorems starts from 5.1 and 5.1.1,

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respectively.

Groups E_8

5.1. The complex Lie algebra $\mathfrak{e}_8{}^C$

In the 248-dimensional C-vector space

$$\mathfrak{e}_8{}^C=\mathfrak{e}_7{}^C\oplus\mathfrak{P}^C\oplus\mathfrak{P}^C\oplus C\oplus C\oplus C,$$

we define the Lie bracket $[R_1, R_2]$ by

$$[(\Phi_1, P_1, Q_1, r_1, s_1, t_1), (\Phi_2, P_2, Q_2, r_2, s_2, t_2)]=(\Phi, P, Q, r, s, t)$$

where

$$\begin{cases} \Phi=[\Phi_1, \Phi_2]+P_1\times Q_2-P_2\times Q_1, \\ P=\Phi_1P_2-\Phi_2P_1+r_1P_2-r_2P_1+s_1Q_2-s_2Q_1, \\ Q=\Phi_1Q_2-\Phi_2Q_1-r_1Q_2+r_2Q_1+t_1P_2-t_2P_1, \\ r=-\dfrac{1}{8}\{P_1, Q_2\}+\dfrac{1}{8}\{P_2, Q_1\}+s_1t_2-s_2t_1, \\ s=\dfrac{1}{4}\{P_1, P_2\}+2r_1s_2-2r_2s_1, \\ t=-\dfrac{1}{4}\{Q_1, Q_2\}-2r_1t_2+2r_2t_1, \end{cases}$$

then $\mathfrak{e}_8{}^C$ becomes a simple C-Lie algebra of type E_8 [16], [17]. For $R\in\mathfrak{e}_8{}^C$, $\mathrm{ad}R$ ($(\mathrm{ad}R)R_1=[R, R_1]$, $R_1\in\mathfrak{e}_8{}^C$) is denoted by $\Theta(R)$. Hereafter we often identify R and $\Theta(R)$, $\mathfrak{e}_8{}^C$ and $\mathrm{Der}_C(\mathfrak{e}_8{}^C)=\{\Theta(R)\,|\,R\in\mathfrak{e}_8{}^C\}$, respectively. The group $E_8{}^C$ is defined to be the automorphism group of the Lie algebra $\mathfrak{e}_8{}^C$:

$$E_8{}^C=\{\alpha\in\mathrm{Iso}_C(\mathfrak{e}_8{}^C)\,|\,\alpha[R_1, R_2]=[\alpha R_1, \alpha R_2]\}.$$

5.2. Involutions of the Lie group $E_8{}^C$

We arrange here main involutions used in this chapter E_8. We define C-linear transformations γ, σ, λ, υ, ω of $\mathfrak{e}_8{}^C$ by

$$\gamma(\Phi, P, Q, r, s, t)=(\gamma\Phi\gamma, \gamma P, \gamma Q, r, s, t),$$

$$\sigma(\Phi, P, Q, r, s, t)=(\sigma\Phi\sigma, \sigma P, \sigma Q, r, s, t),$$

$$\lambda(\Phi, P, Q, r, s, t)=(\lambda\Phi\lambda^{-1}, \lambda P, \lambda Q, r, s, t)$$

where γ, σ, λ of right sides are the same ones as $\gamma\in G_2{}^C\subset F_4{}^C\subset E_6{}^C\subset E_7{}^C$, $\sigma\in F_4{}^C\subset E_6{}^C\subset E_7{}^C$, $\lambda\in E_7{}^C$,

$$\upsilon(\Phi, P, Q, r, s, t)=(\Phi, -P, -Q, r, s, t),$$

$$\omega(\Phi, P, Q, r, s, t)=(\Phi, Q, -P, -r, -t, -s).$$

Then $\gamma, \sigma, \lambda, \upsilon, \omega \in E_8{}^C$ and $\gamma^2=\sigma^2=\upsilon^2=1$, $\lambda^2=\omega^2=\upsilon$. ($\upsilon$ is nothing but the element -1 of the center of $E_7{}^C$ (see Proposition 5.4.1), however we use υ in $E_8{}^C$ to avoid confusions because -1 has many differnt meanings. Note that $\upsilon=\phi(-E)$, $\omega=\phi(J)$ using ϕ of Proposition 5.4.2). We put

$$\tilde{\lambda}=\lambda\omega=\omega\lambda.$$

Then $\tilde{\lambda}\in E_8{}^C$ and $\tilde{\lambda}^2=1$. The complex conjugation in $\mathfrak{e}_8{}^C$ is denoted by τ:

$$\tau(\Phi, P, Q, r, s, t)=(\tau\Phi\tau, \tau P, \tau Q, \tau r, \tau s, \tau t).$$

These transformations $\gamma, \sigma, \upsilon, \tilde{\lambda}, \tau$ of $\mathfrak{e}_8{}^C$ induce involutive automorphisms $\tilde{\gamma}, \tilde{\sigma}, \tilde{\upsilon}, \tilde{\tilde{\lambda}}, \tilde{\tau}$ of $E_8{}^C$:

$$\tilde{\gamma}(\alpha)=\gamma\alpha\gamma, \quad \tilde{\sigma}(\alpha)=\sigma\alpha\sigma, \quad \tilde{\upsilon}(\alpha)=\upsilon\alpha\upsilon, \quad \tilde{\tilde{\lambda}}(\alpha)=\tilde{\lambda}\alpha\tilde{\lambda}, \quad \tilde{\tau}(\alpha)=\tau\alpha\tau, \qquad \alpha\in E_8{}^C.$$

5.3. Lie groups of type E_8

We define $\boldsymbol{R}$-Lie algebras $\mathfrak{e}_{8(8)}$, $\mathfrak{e}_{8(-24)}$ by

$$\mathfrak{e}_{8(8)}=\mathfrak{e}_{7(7)}\oplus\mathfrak{P}'\oplus\mathfrak{P}'\oplus\boldsymbol{R}\oplus\boldsymbol{R}\oplus\boldsymbol{R} \qquad (\mathfrak{e}_{7(7)}=(\mathfrak{e}_7{}^C)^{\tau\gamma}),$$

$$\mathfrak{e}_{8(-24)}=\mathfrak{e}_{7(-25)}\oplus\mathfrak{P}\oplus\mathfrak{P}\oplus\boldsymbol{R}\oplus\boldsymbol{R}\oplus\boldsymbol{R} \qquad (\mathfrak{e}_{7(-25)}=(\mathfrak{e}_7{}^C)^{\tau})$$

(Theorem 4.5.2) with the Lie brackets as $\mathfrak{e}_8{}^C$, respectively. The connected linear Lie groups of type E_8 are obtained as

$$E_8{}^C=\{\alpha\in \mathrm{Iso}_C(\mathfrak{e}_8{}^C)\,|\,\alpha[R_1, R_2]=[\alpha R_1, \alpha R_2]\},$$

$$E_8=\{\alpha\in \mathrm{Iso}_C(\mathfrak{e}_8{}^C)\,|\,\alpha[R_1, R_2]=[\alpha R_1, \alpha R_2],\ \langle\alpha R_1, \alpha R_2\rangle=\langle R_1, R_2\rangle\},$$

$$E_{8(8)}=\{\alpha\in \mathrm{Iso}_{\boldsymbol{R}}(\mathfrak{e}_{8(8)})\,|\,\alpha[R_1, R_2]=[\alpha R_1, \alpha R_2]\},$$

$$E_{8(-24)}=\{\alpha\in \mathrm{Iso}_{\boldsymbol{R}}(\mathfrak{e}_{8(-24)})\,|\,\alpha[R_1, R_2]=[\alpha R_1, \alpha R_2]\}$$

here $\langle R_1, R_2\rangle=-(1/15)B_8(\tau\tilde{\lambda}R_1, R_2)$ where B_8 is the Killing form of $\mathfrak{e}_8{}^C$ [18], [21]. $E_8{}^C$, E_8 are simply connected (see Appendix).

LEMMA 5.3.1. $(\mathfrak{e}_8{}^C)_{\tau\gamma}\cong\mathfrak{e}_{8(8)}$, $(\mathfrak{e}_8{}^C)_{\tau}=\mathfrak{e}_{8(-24)}$.

THEOREM 5.3.2. $(E_8{}^C)^{\tau\tilde{\lambda}}=E_8$, $(E_8{}^C)^{\tau\gamma}\cong E_{8(8)}$, $(E_8{}^C)^{\tau}=E_{8(-24)}$.

PROOF. As for $E_{8(8)}$, $E_{8(-24)}$, these are direct results of Lemma 5.3.1. E_8 is nothing but its definition.

Note that $\gamma\in G_2\subset F_4\subset E_6\subset E_7\subset E_8$ ($E_7=(E_7{}^C)^{\tau\lambda}\subset(E_8{}^C)^{\tau\tilde{\lambda}}$ (see Proposition 5.4.1) $=E_8$), $\sigma\in F_4\subset E_6\subset E_7\subset E_8$, $\upsilon\in E_7\subset E_8$, $\tilde{\lambda}\in E_8$. We know that the simply connected compact Lie group E_8 has two classes of involutive elements up to conjugation.

LEMMA 5.3.3. *For $\sigma_1, \sigma_2\in E_8$ such that $\sigma_1{}^2=\sigma_2{}^2=1$, we have*

$$\sigma_1\sim\sigma_2 \Longleftrightarrow (E_8)^{\sigma_1}\cong(E_8)^{\sigma_2} \Longleftrightarrow (\mathfrak{e}_8)^{\sigma_1}\cong(\mathfrak{e}_8)^{\sigma_2},$$

moreover if and only if $\dim(\mathfrak{e}_8)^{\sigma_1}=\dim(\mathfrak{e}_8)^{\sigma_2}$.

PROOF is due to [19].

PROPOSITION 5.3.4. (1) *$\tilde{\lambda}$, υ, γ, $\upsilon\sigma$ are conjugate in E_8 with one another.*
(2) *$\tilde{\lambda}\gamma$ and $\upsilon\gamma$ are conjugate in E_8.*

PROOF. We can easily calculate $\dim(\mathfrak{e}_8)^{\tilde{\lambda}}=\dim(\mathfrak{e}_8)^{\upsilon}=\dim(\mathfrak{e}_8)^{\gamma}=\dim(\mathfrak{e}_8)^{\upsilon\sigma}=136$ and $\dim(\mathfrak{e}_8)^{\tilde{\lambda}\gamma}=\dim(\mathfrak{e}_8)^{\upsilon\gamma}=120$, hence Proposition 5.3.4 follows from Lemma 5.3.3.

REMARK. The author can not find any element $\delta\in E_8$ which gives the conjugation: $\delta\tilde{\lambda}=\upsilon\delta$ etc..

5.4. Subgroups of type $A_1\oplus E_7$ of Lie groups of type E_8

We consider a subgroup $(E_8{}^C)_{1,1^-,1_-}$ of $E_8{}^C$:

$$(E_8{}^C)_{1,1^-,1_-}=\{\alpha\in E_8{}^C \mid \alpha 1=1,\ \alpha 1^-=1^-,\ \alpha 1_-=1_-\}$$

where $1=(0, 0, 0, 1, 0, 0)$, $1^-=(0, 0, 0, 0, 1, 0)$, $1_-=(0, 0, 0, 0, 0, 1)\in\mathfrak{e}_8{}^C$.

PROPOSITION 5.4.1. $(E_8{}^C)_{1,1^-,1_-}\cong E_7{}^C$.

PROOF ([17]). For $\beta\in E_7{}^C$ we correspond $\alpha\in E_8{}^C$,

$$\alpha(\Phi, P, Q, r, s, t)=(\beta\Phi\beta^{-1}, \beta P, \beta Q, r, s, t)$$

for $(\Phi, P, Q, r, s, t)\in\mathfrak{e}_8{}^C$. It is easy to verify $\alpha\in(E_8{}^C)_{1,1^-,1_-}$. Conversely let $\alpha\in(E_8{}^C)_{1,1^-,1_-}$. From the conditions $\alpha 1=1$, $\alpha 1^-=1^-$, $\alpha 1_-=1_-$, α has the form

$$\alpha=\begin{pmatrix} \beta_1 & \beta_{12} & \beta_{13} & 0 & 0 & 0 \\ \beta_{21} & \beta_2 & \beta_{23} & 0 & 0 & 0 \\ \beta_{31} & \beta_{32} & \beta_3 & 0 & 0 & 0 \\ a_1 & b_1 & c_1 & 1 & 0 & 0 \\ a_2 & b_2 & c_2 & 0 & 1 & 0 \\ a_3 & b_3 & c_3 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{array}{l} \beta_1\in \mathrm{Hom}_C(\mathfrak{e}_7{}^C, \mathfrak{e}_7{}^C), \\ \beta_2, \beta_3, \beta_{23}, \beta_{32}\in \mathrm{Hom}_C(\mathfrak{P}^C, \mathfrak{P}^C), \\ \beta_{21}, \beta_{31}\in \mathrm{Hom}_C(\mathfrak{e}_7{}^C, \mathfrak{P}^C), \\ \beta_{12}, \beta_{13}\in \mathrm{Hom}_C(\mathfrak{P}^C, \mathfrak{e}_7{}^C), \\ a_i\in \mathrm{Hom}_C(\mathfrak{e}_7{}^C, C), \\ b_i, c_i\in \mathrm{Hom}_C(\mathfrak{P}^C, C). \end{array}$$

From $[\alpha\Phi, 1]=\alpha[\Phi, 1]=0$ we have $\beta_{21}=\beta_{31}=0$, $a_2=a_3=0$, and from $[\alpha\Phi, 1^-]=\alpha[\Phi, 1^-]=0$ we have $a_1=0$. Put $P^-=(0, P, 0, 0, 0, 0)$, $Q_-=(0, 0, Q, 0, 0, 0)$. From $[\alpha P^-, 1]=-\alpha P^-$ we have $\beta_{12}=\beta_{32}=0$, $b_1=b_2=b_3=0$. Similarly from $[\alpha Q_-, 1]=\alpha Q_-$ we have $\beta_{13}=\beta_{23}=0$, $c_1=c_2=c_3=0$. Operate α on $[P^-, Q_-]=(P\times Q, 0, 0, -(1/8)\{P, Q\}, 0, 0)$, then

$$\beta_1(P\times Q)=\beta_2P\times\beta_3Q, \qquad \{\beta_2P, \beta_3Q\}=\{P, Q\}. \tag{i}$$

Again operate α on $[P^-, Q^-]=(1/4)\{P, Q\}1$, then

$$\{\beta_2P, \beta_2Q\}=\{P, Q\}. \tag{ii}$$

Operate α on $[\Phi, P^-]=(\Phi P)^-$, then

$$(\beta_1\Phi)(\beta_2P)=\beta_2(\Phi P). \tag{iii}$$

From (i), (ii) we have $\{\beta_2P, \beta_3Q\}=\{\beta_2P, \beta_2Q\}$ for all $P, Q\in\mathfrak{P}^C$, hence $\beta_2=\beta_3$ (put$=\beta$). In (iii) put $\beta^{-1}P$ instead of P, then $\beta_1\Phi=\beta\Phi\beta^{-1}$. Therefore, from (i) we have $\beta(P\times Q)\beta^{-1}=\beta_1(P\times Q)=\beta P\times\beta Q$. Thus $\beta\in E_7{}^C$.

PROPOSITION 5.4.2. *$(E_8{}^C)^\nu$ has a subgroup $\phi(SL(2, C))$ which is isomorphic to the group $SL(2, C)$. Where $\phi(A)$, $A=\begin{pmatrix} a & c \\ b & d\end{pmatrix}\in SL(2, C)$, is the C-linear transformation of $\mathfrak{e}_8{}^C$ defined by*

$$\phi\left(\begin{pmatrix} a & c \\ b & d\end{pmatrix}\right)=\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & a1 & c1 & 0 & 0 & 0 \\ 0 & b1 & d1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1+2bc & -ab & cd \\ 0 & 0 & 0 & -2ac & a^2 & -c^2 \\ 0 & 0 & 0 & 2bd & -b^2 & d^2 \end{pmatrix}.$$

PROOF. For $A=\begin{pmatrix} a & c \\ b & d\end{pmatrix}=\exp\begin{pmatrix} r & s \\ t & -r\end{pmatrix}$, we have

$$\phi(A)=\exp\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & r1 & s1 & 0 & 0 & 0 \\ 0 & t1 & -r1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -t & s \\ 0 & 0 & 0 & -2s & 2r & 0 \\ 0 & 0 & 0 & 2t & 0 & -2r \end{pmatrix}$$

$$=\exp(\Theta(0, 0, 0, r, s, t)).$$

LEMMA 5.4.3. *$\phi: SL(2, C)\to(E_8{}^C)^\nu$ of Proposition 5.4.2 satisfies*

$$\tau\phi(A)\tau=\phi(\tau A), \qquad \omega\phi(A)\omega^{-1}=\phi({}^tA^{-1}),$$

$$\lambda\phi(A)\lambda^{-1}=\sigma\phi(A)\sigma=\gamma\phi(A)\gamma=\phi(A).$$

THEOREM 5.4.4. $(E_8{}^C)^\upsilon\cong(SL(2, C)\times E_7{}^C)/\boldsymbol{Z}_2$, $\boldsymbol{Z}_2=\{(E, 1), (-E, -1)\}$.

PROOF. We define a mapping $\psi: SL(2, C)\times E_7{}^C\to(E_8{}^C)^\upsilon$ by

$$\psi(A, \beta)=\phi(A)\beta.$$

Obviously $\psi(A, \beta)\in(E_8{}^C)^\upsilon$. Since $\phi(A)$, $A\in SL(2, C)$ and $\beta\in E_7{}^C$ are commutative, ψ is a homomorphism. $\mathrm{Ker}\psi=\{(E, 1), (-E, -1)\}=\boldsymbol{Z}_2$. Since $(E_8{}^C)^\upsilon$ is connected (Lemma 0.7) and $\dim_C(\mathfrak{sl}(2, C)\oplus\mathfrak{e}_7{}^C)=3+133=136=\dim_C(\mathfrak{e}_8{}^C)^\upsilon$ (note that $(\mathfrak{e}_8{}^C)^\upsilon=\{(\Phi, 0, 0, r, s, t)\in\mathfrak{e}_8{}^C\,|\,\Phi\in\mathfrak{e}_7{}^C, r, s, t\in C\}$), ψ is onto. Thus we have the required isomorphism.

THEOREM 5.4.5. (1) $(E_8)^\upsilon\cong(SU(2)\times E_7)/\boldsymbol{Z}_2\cong(\tau\tilde{\lambda}\upsilon)^\upsilon\sim(E_{8(-24)})^\upsilon$.
(2) $(E_{8(-24)})^\upsilon\sim(\tau\tilde{\lambda}\gamma)^\upsilon\cong(SU(2)\times E_{7(-5)})/\boldsymbol{Z}_2\cong(\tau\tilde{\lambda}\upsilon\gamma)^\upsilon\sim(E_{8(8)})^\upsilon$.

PROOF. (1) Let $\alpha\in(E_8)^\upsilon=((E_8{}^C)^{\tau\tilde{\lambda}})^\upsilon=(\tau\tilde{\lambda})^\upsilon$. By Theorem 5.4.4, there exist $A\in SL(2, C)$, $\beta\in E_7{}^C$ such that $\alpha=\phi(A)\beta$. From the condition $\tau\tilde{\lambda}\alpha=\alpha\tau\tilde{\lambda}$, we have $\phi(\tau{}^tA^{-1})\tau\tilde{\lambda}\beta\tilde{\lambda}\tau=\phi(A)\beta$ (Lemma 5.4.3). Hence

$$\begin{cases}\tau{}^tA^{-1}=A\\ \tau\tilde{\lambda}\beta\tilde{\lambda}\tau=\beta\end{cases}\quad\text{or}\quad\begin{cases}\tau{}^tA^{-1}=-A\\ \tau\tilde{\lambda}\beta\tilde{\lambda}\tau=\upsilon\beta.\end{cases}$$

The latter case is impossible because $(\tau{}^tA)A=-E$ is false. In the first case, $(\tau{}^tA)A=E$, that is, $A\in SU(2)$. For $\beta\in E_7{}^C$, $\tau\tilde{\lambda}\beta\tilde{\lambda}\tau=\beta$ is $\tau\lambda\beta\lambda^{-1}\tau=\beta$, hence $\beta\in(E_7{}^C)^{\tau\lambda}=E_7$ (Theorem 4.3.2). Thus $(E_8)^\upsilon=\psi(SU(2)\times E_7)\cong(SU(2)\times E_7)/\boldsymbol{Z}_2$.

$$E_{8(-24)}=(E_8{}^C)^\tau\cong(E_8{}^C)^{\tau\tilde{\lambda}\upsilon}.$$

In fact, since $\tilde{\lambda}\sim\upsilon$ under some $\delta\in E_8$: $\delta\tilde{\lambda}=\upsilon\delta$, $\delta\tau\tilde{\lambda}=\tau\tilde{\lambda}\delta$ (Proposition 5.3.4), $(E_8{}^C)^\tau\ni\alpha\to\delta\alpha\delta^{-1}\in(E_8{}^C)^{\tau\tilde{\lambda}\upsilon}$ gives an isomorphism. Now $(E_{8(-24)})^\upsilon\sim(\tau\tilde{\lambda}\upsilon)^\upsilon=(\tau\tilde{\lambda})^\upsilon$.

(2) $$E_{8(-24)}=(E_8{}^C)^\tau\cong(E_8{}^C)^{\tau\tilde{\lambda}\gamma}.$$

In fact, since $\tilde{\lambda}\sim\gamma$ under some $\delta\in E_8$: $\delta\tilde{\lambda}=\gamma\delta$, $\delta\tau\tilde{\lambda}=\tau\tilde{\lambda}\delta$ (Proposition 5.3.4), $(E_8{}^C)^\tau\ni\alpha\to\delta\alpha\delta^{-1}\in(E_8{}^C)^{\tau\tilde{\lambda}\gamma}$ gives an isomorphism. Let $\alpha\in((E_8{}^C)^{\tau\tilde{\lambda}\gamma})^\upsilon=(\tau\tilde{\lambda}\gamma)^\upsilon$, $\alpha=\phi(A)\beta$, $A\in SL(2, C)$, $\beta\in E_7{}^C$. From $\tau\tilde{\lambda}\gamma\alpha=\alpha\tau\tilde{\lambda}\gamma$, we have $\phi(\tau{}^tA^{-1})\tau\tilde{\lambda}\gamma\beta\gamma\tilde{\lambda}\tau=\phi(A)\beta$ (Lemma 5.4.3). Hence

$$\begin{cases}\tau{}^tA^{-1}=A\\ \tau\tilde{\lambda}\gamma\beta\gamma\tilde{\lambda}\tau=\beta\end{cases}\quad\text{or}\quad\begin{cases}\tau{}^tA^{-1}=-A\\ \tau\tilde{\lambda}\gamma\beta\gamma\tilde{\lambda}\tau=\upsilon\beta.\end{cases}$$

As similar to (1), the latter case is impossible. In the first case, $A\in SU(2)$ and $\beta\in(E_7{}^C)^{\tau\tilde{\lambda}\gamma}=(E_7{}^C)^{\tau\lambda\gamma}=E_{7(-5)}$ (Theorem 4.3.2). Thus $(E_{8(-24)})^\upsilon\sim(\tau\tilde{\lambda}\gamma)^\upsilon\cong(SU(2)\times$

$E_{7(-5)})/\boldsymbol{Z}_2$.

$$E_{8(8)}=(E_8{}^C)^{\tau\gamma}\cong(E_8{}^C)^{\tau\tilde{\lambda}\upsilon\gamma}.$$

In fact, since $\tilde{\lambda}\gamma\sim\upsilon\gamma$ under some $\delta\in E_8$: $\delta\tilde{\lambda}\gamma=\upsilon\gamma\delta$, $\delta\tau\tilde{\lambda}=\tau\tilde{\lambda}\delta$ (Proposition 5.3.4), $(E_8{}^C)^{\tau\gamma}\ni\alpha\to\delta\alpha\delta^{-1}\in(E_8{}^C)^{\tau\tilde{\lambda}\upsilon\gamma}$ gives an isomorphism. Now $(E_{8(8)})^{\upsilon}\sim(\tau\tilde{\lambda}\upsilon\gamma)^{\upsilon}=(\tau\tilde{\lambda}\gamma)^{\upsilon}$.

THEOREM 5.4.6. (1) $(E_{8(8)})^{\upsilon}\cong(SL(2,\boldsymbol{R})\times E_{7(7)})/\boldsymbol{Z}_2\times 2$.
(2) $(E_{8(-24)})^{\upsilon}\cong(SL(2,\boldsymbol{R})\times E_{7(-25)})/\boldsymbol{Z}_2\times 2$.

RROOF. (1) Let $\alpha\in(E_{8(8)})^{\upsilon}=((E_8{}^C)^{\tau\gamma})^{\upsilon}=(\tau\gamma)^{\upsilon}$, $\alpha=\phi(A)\beta$, $A\in SL(2,C)$, $\beta\in E_7{}^C$. From $\tau\gamma\alpha=\alpha\tau\gamma$, we have $\phi(\tau A)\tau\gamma\beta\gamma\tau=\phi(A)\beta$ (Lemma 5.4.3). Hence

$$\begin{cases}\tau A=A\\ \tau\gamma\beta\gamma\tau=\beta\end{cases}\quad\text{or}\quad\begin{cases}\tau A=-A\\ \tau\gamma\beta\gamma\tau=\upsilon\beta.\end{cases}$$

In the first case, $A\in SL(2,\boldsymbol{R})$, $\beta\in(E_7{}^C)^{\tau\gamma}=E_{7(7)}$ (Theorem 4.3.2). In the latter case, $A=(iI)B$, $B\in SL(2,\boldsymbol{R})$, $\beta=\iota\beta'$, $\beta'\in E_{7(7)}$ where $\iota\in E_7$ is one defined in 4.2. Thus $(E_{8(8)})^{\upsilon}\cong(SL(2,\boldsymbol{R})\times E_{7(7)}\cup(iI)SL(2,\boldsymbol{R})\times\iota E_{7(7)})/\boldsymbol{Z}_2=(SL(2,\boldsymbol{R})\times E_{7(7)})/\boldsymbol{Z}_2\times 2$. (The explicit from of $\phi(iI,\iota)$ is

$$\phi(iI,\iota)(\Phi,P,Q,r,s,t)=(\iota\Phi\iota^{-1},i\iota P,-i\iota Q,r,-s,-t)).$$

(2) Let $\alpha\in(E_{8(-24)})^{\upsilon}=((E_8{}^C)^{\tau})^{\upsilon}=(\tau)^{\upsilon}$, $\alpha=\phi(A)\beta$, $A\in SL(2,C)$, $\beta\in E_7{}^C$. From $\tau\alpha=\alpha\tau$, we have $\phi(\tau A)\tau\beta\tau=\phi(A)\beta$ (Lemma 5.4.3). As similar to (1), $(E_{8(-24)})^{\upsilon}\cong(SL(2,\boldsymbol{R})\times E_{7(-25)})\cup(iI)SL(2,\boldsymbol{R})\times\iota E_{7(-25)})/\boldsymbol{Z}_2=(SL(2,\boldsymbol{R})\times E_{7(-25)})/\boldsymbol{Z}_2\times 2$.

5.5. Subgroups of type D_8 of Lie groups of type E_8

We define an $\boldsymbol{R}$-algebraic homomorphism l: $M(8,\boldsymbol{C})\to M(16,\boldsymbol{R})$ by

$$l\Big((x+y\boldsymbol{i})\Big)=\left(\begin{pmatrix} x & y\\ -y & x\end{pmatrix}\right),\qquad x,y\in\boldsymbol{R}$$

and l: $M(8,\boldsymbol{C}^C)\to M(16,C)$ is its complexification. These l satisfy

$$Il(X)=l(\bar{X})I,\quad Jl(X)=l(X)J,\quad {}^t(l(X))=l(X^*),\quad X\in M(8,K),\quad K=\boldsymbol{C},\boldsymbol{C}^C.$$

PROPOSITION 5.5.1. (1) $l(\mathfrak{u}(8,\boldsymbol{C}^C))=\{B\in\mathfrak{so}(16,C)\mid JB=BJ\}$,

$l(\mathfrak{S}(8,\boldsymbol{C}^C))I=\{B\in\mathfrak{so}(16,C)\mid JB=-BJ\}$.

(2) *Any element $B\in\mathfrak{so}(16,C)$ is uniquely expressed by*

$$\begin{aligned}B&=l(D')+l(S)I && D'\in\mathfrak{u}(8,\boldsymbol{C}^C),\ S\in\mathfrak{S}(8,\boldsymbol{C}^C)\\ &=l(D)+l(S)I+l(\boldsymbol{i}cE) && D\in\mathfrak{su}(8,\boldsymbol{C}^C),\ S\in\mathfrak{S}(8,\boldsymbol{C}^C),\ c\in C.\end{aligned}$$

PROOF. (1) If $D\in\mathfrak{u}(8, \boldsymbol{C}^C)$, then $Jl(D)=l(D)J$, ${}^tl(D)=l(D^*)=-l(D)$. Conversely, for $B\in\mathfrak{so}(16, C)$ such that $JB=BJ$, put $B=l(D)$, $D\in M(8, \boldsymbol{C}^C)$. Then $l(-D)=-B={}^tB={}^tl(D)=l(D^*)$, hence $D^*=-D$. Next, for $S\in\mathfrak{S}(8, \boldsymbol{C}^C)$, $Jl(S)I=l(S)JI=-l(S)IJ$, ${}^t(l(S)I)={}^tI{}^tl(S)=Il(S^*)=-Il(\bar{S})=-l(S)I$. Conversely, for $B\in\mathfrak{so}(16, C)$ such that $JB=-BJ$, consider BI. Then $JBI=BIJ$, hence we can put $BI=l(S)$, $S\in M(8, \boldsymbol{C}^C)$. Then $B=l(S)I$, ${}^tB={}^tI{}^tl(S)=Il(S^*)=l({}^tS)$, hence $-S={}^tS$, that is, $S\in\mathfrak{S}(8, \boldsymbol{C}^C)$.

(2) $B=(B-JBJ)/2+(B+JBJ)/2$ and use the above (1).

We consider Lie subalgebras $\mathfrak{so}(4, 12)$, $\mathfrak{so}(8, 8)$, $\mathfrak{so}^*(16)$ (which are isomorphic to the ordinary ones) of $\mathfrak{so}(16, C)$ as

$$\mathfrak{so}(4, 12)=\{B\in\mathfrak{so}(16, C)\,|\,l(I_2)(\tau B)l(I_2)=B\},$$

$$\mathfrak{so}(8, 8)=\{B\in\mathfrak{so}(16, C)\,|\,l(I_4')(\tau B)l(I_4')=B\},$$

$$\mathfrak{so}^*(16)=\{B\in\mathfrak{so}(16, C)\,|\,JB=(\tau B)J\}$$

where $I_4'=\mathrm{diag}(1, 1, 1, 1, -1, -1, -1, -1)$, and corresponding the above we use the following notations.

$$\mathfrak{su}(2, 6)=\{D\in\mathfrak{su}(8, \boldsymbol{C}^C)\,|\,I_2(\tau D)I_2=D\},$$

$$\mathfrak{S}(2, 6, C)=\{S\in\mathfrak{S}(8, \boldsymbol{C}^C)\,|\,I_2(\tau S)I_2=S\},$$

$$\mathfrak{su}(4, 4)=\{D\in\mathfrak{su}(8, \boldsymbol{C}^C)\,|\,I_4'(\tau D)I_4'=D\},$$

$$\mathfrak{S}(4, 4, C)=\{S\in\mathfrak{S}(8, \boldsymbol{C}^C)\,|\,I_4'(\tau S)I_4'=S\}.$$

From Proposition 5.5.1, we have easily the following

PROPOSITION 5.5.2. *Any elements* $B\in\mathfrak{so}(16)$, $\mathfrak{so}(4, 12)$, $\mathfrak{so}(8, 8)$, $\mathfrak{so}^*(16)$ *are, respectively, expressed by* $B=l(D)+l(S)I+l(\boldsymbol{i}cE)$,

(1) case $\mathfrak{so}(16)$, $D\in\mathfrak{su}(8)$, $S\in\mathfrak{S}(8, \boldsymbol{C})$, $c\in\boldsymbol{R}$,
(2) case $\mathfrak{so}(4, 12)$, $D\in\mathfrak{su}(2, 6)$, $S\in\mathfrak{S}(2, 6, \boldsymbol{C})$, $c\in\boldsymbol{R}$,
(3) case $\mathfrak{so}(8, 8)$, $D\in\mathfrak{su}(4, 4)$, $S\in\mathfrak{S}(4, 4, \boldsymbol{C})$, $c\in\boldsymbol{R}$,
(4) case $\mathfrak{so}^*(16)$, $D\in\mathfrak{su}(8)$, $S\in i\mathfrak{S}(8, \boldsymbol{C})$, $c\in\boldsymbol{R}$.

Recall the C-linear isomorphism χ: $\mathfrak{P}^C=\mathfrak{J}^C\oplus\mathfrak{J}^C\oplus C\oplus C\to\mathfrak{S}(8, \boldsymbol{C}^C)$,

$$\chi(X, Y, \xi, \eta)=\Big(k\Big(gX-\frac{\xi}{2}E\Big)\Big)J+\boldsymbol{i}\Big(k\Big(g(\gamma Y)-\frac{\eta}{2}E\Big)\Big)J$$

which is used to define the homomorphism ϕ: $SU(8, \boldsymbol{C}^C)\to(E_7{}^C)^{\lambda\gamma}$, $\phi(A)P=\chi^{-1}(A(\chi P){}^tA)$, $P\in\mathfrak{P}^C$ (Theorem 4.5.3).

LEMMA 5.5.3. *For* $S\in\mathfrak{S}(8, \boldsymbol{C}^C)$, *we have*

$$\gamma\chi^{-1}S=\chi^{-1}(I_2SI_2),\quad \tau\chi^{-1}S=\chi^{-1}(I_2(\tau S)I_2),\quad \sigma\chi^{-1}S=\chi^{-1}(I_4'SI_4').$$

PROOF. $\chi\gamma\chi^{-1}S=\chi\gamma(X, Y, \xi, \eta)=\chi(\gamma X, \gamma Y, \xi, \eta)$

$$=k\Big(g(\gamma X)-\frac{\xi}{2}E\Big)J+ik\Big(gY-\frac{\eta}{2}E\Big)J$$

$$=k\Big(I_1(gX)I_1-\frac{\xi}{2}E\Big)J+ik\Big(I_1(g(\gamma Y))I_1-\frac{\eta}{2}E\Big)J$$

$$=I_2\Big(k\Big(gX-\frac{\xi}{2}E\Big)J+ik\Big(g(\gamma Y)-\frac{\eta}{2}E\Big)J\Big)I_2$$

$$=I_2\chi(X, Y, \xi, \eta)I_2=I_2SI_2.$$

Using $g(\tau X)=I_1(\tau(gX))I_1$, $g(\sigma X)=I_2'(gX)I_2'$, $I_2'=\mathrm{diag}(1, 1, -1, -1)$, other formulae are similarly obtained.

LEMMA 5.5.4. *For* $S_1, S_2\in\mathfrak{S}(8, \boldsymbol{C}^C)$, *we have*

(1) $\mathrm{tr}(S_1\bar{S}_2-S_2\bar{S}_1)=4i\{\chi^{-1}S_1, \chi^{-1}S_2\}$.

(2) $\phi_*\Big(S_1\bar{S}_2-S_2\bar{S}_1-\dfrac{1}{8}\mathrm{tr}(S_1\bar{S}_2-S_2\bar{S}_1)E\Big)=4(\lambda\gamma\chi^{-1}S_1\times\chi^{-1}S_2-\lambda\gamma\chi^{-1}S_2\times\chi^{-1}S_1)$

where $\phi_*: \mathfrak{sp}(4, \boldsymbol{H}^C)\to(\mathfrak{e}_6{}^C)^{\lambda\gamma}$ *is one defined in* 3.5.

PROOF is in [21, Proposition 6].

THEOREM 5.5.5. $(\mathfrak{e}_8{}^C)^{\tilde{\lambda}\gamma}\cong\mathfrak{so}(16, C)$.

PROOF. $(\mathfrak{e}_8{}^C)^{\tilde{\lambda}\gamma}=\{\Theta\in\mathrm{Der}_C(\mathfrak{e}_8{}^C)\,|\,\tilde{\lambda}\gamma\Theta=\Theta\tilde{\lambda}\gamma\}$

$$=\{\Theta(\Phi, \lambda\gamma Q, Q, 0, s, -s)\,|\,\Phi\in(\mathfrak{e}_7{}^C)^{\lambda\gamma}, Q\in\mathfrak{P}^C, s\in C\}.$$

We define a mapping $\zeta: \mathfrak{so}(16, C)\to(\mathfrak{e}_8{}^C)^{\tilde{\lambda}\gamma}$ by

$$\zeta(l(D)+l(S)I+l(icE))=\Theta(\phi_*(D), 2\lambda\gamma\chi^{-1}S, 2\chi^{-1}S, 0, 2c, -2c)$$

where $D\in\mathfrak{su}(8, \boldsymbol{C}^C)$, $S\in\mathfrak{S}(8, \boldsymbol{C}^C)$, $c\in C$ (Proposition 5.5.1) and $\phi_*: \mathfrak{su}(8, \boldsymbol{C}^C)\to(\mathfrak{e}_7{}^C)^{\lambda\gamma}$, $\chi: \mathfrak{P}^C\to\mathfrak{S}(8, \boldsymbol{C}^C)$ are ones defined in the beginning. Clearly ζ is bijective. We have to prove that ζ is a homomorphism, that is,

$$\zeta[X, Y]=[\zeta X, \zeta Y],\qquad X, Y=l(D), l(S)I, l(icE).$$

For example, to prove $\zeta[l(S_1)I, l(S_2)I]=[\zeta(l(S_1)I), \zeta(l(S_2)I)]$, we use Lemma 5.5.4. The details of calculations are in [21].

THEOREM 5.5.6. (1) $(\mathfrak{e}_{8(8)})^{\tilde{\lambda}\gamma}\cong\mathfrak{so}(16)\cong(\mathfrak{e}_8)^{\tilde{\lambda}\gamma}$.

(2) $(\mathfrak{e}_{8(-24)})^{\tilde{\lambda}\gamma}\cong\mathfrak{so}(4, 12)$.

(3) $(\mathfrak{e}_{8(8)})^{\lambda\gamma} \sim ((\mathfrak{e}_8{}^C)^{\tau\sigma\gamma})^{\lambda\gamma} \cong \mathfrak{so}(8, 8)$.

(4) $(\mathfrak{e}_{8(8)})^{\lambda\gamma} \sim ((\mathfrak{e}_8{}^C)^{\tau\upsilon\gamma})^{\lambda\gamma} \cong \mathfrak{so}^*(16) \cong ((\mathfrak{e}_8{}^C)^{\tau\lambda\upsilon})^{\lambda\gamma} \sim (\mathfrak{e}_{8(-24)})^{\lambda\gamma}$.

PROOF. (1) Let $\Theta \in (\mathfrak{e}_{8(8)})^{\lambda\gamma} = ((\mathfrak{e}_8{}^C)^{\tau\gamma})^{\lambda\gamma}$. By Theorem 5.5.5, there exist $D \in \mathfrak{su}(8, \boldsymbol{C}^C)$, $S \in \mathfrak{S}(8, \boldsymbol{C}^C)$, $c \in C$ such that

$$\Theta = \Theta(\phi_*(D), 2\lambda\gamma\chi^{-1}S, 2\chi^{-1}S, 0, 2c, -2c).$$

From the condition $\tau\gamma\Theta(R)\gamma\tau = \Theta(R)$, $R \in \mathfrak{e}_8{}^C$, that is, $\Theta(\tau\gamma R) = \Theta(R)$, we have

$$\tau\gamma\phi_*(D)\gamma\tau = \phi_*(D), \qquad \tau\gamma\chi^{-1}S = \chi^{-1}S, \qquad \tau c = c.$$

Hence $D \in \mathfrak{su}(8)$ (Theorem 4.5.5), $\chi^{-1}(\tau S) = \chi^{-1}S$ (Lemma 5.5.3), $\tau S = S$, that is, $S \in \mathfrak{S}(8, \boldsymbol{C})$ and $c \in \boldsymbol{R}$. Therefore, $\Theta \in \zeta(l(\mathfrak{su}(8)) + l(\mathfrak{S}(8, \boldsymbol{C}))I + l(i\boldsymbol{R}E)) = \zeta(\mathfrak{so}(16))$ (Proposition 5.5.2. (1)). Thus $(\mathfrak{e}_{8(8)})^{\lambda\gamma} \cong \mathfrak{so}(16)$. $(\mathfrak{e}_8)^{\lambda\gamma} = ((\mathfrak{e}_8{}^C)^{\tau\lambda})^{\lambda\gamma} = ((\mathfrak{e}_8{}^C)^{\tau\gamma})^{\lambda\gamma}$.

(2) Let $\Theta \in (\mathfrak{e}_{8(-24)})^{\lambda\gamma} = ((\mathfrak{e}_8{}^C)^{\tau})^{\lambda\gamma}$, $\Theta = \Theta(\phi_*(D), 2\lambda\gamma\chi^{-1}S, 2\chi^{-1}S, 0, 2c, -2c)$ as in (1). From $\tau\Theta = \Theta\tau$, we have

$$\tau\phi_*(D)\tau = \phi_*(D), \qquad \tau\chi^{-1}S = \chi^{-1}S, \qquad \tau c = c.$$

Hence $D \in \mathfrak{su}(2, 6)$ (Theorem 4.5.5), $\chi^{-1}(I_2(\tau S)I_2) = \chi^{-1}S$ (Lemma 5.5.3), $I_2(\tau S)I_2 = S$, that is, $S \in \mathfrak{S}(2, 6, \boldsymbol{C})$ and $c \in \boldsymbol{R}$. Therefore, from Proposition 5.5.2. (2), we have $(\mathfrak{e}_{8(-24)})^{\lambda\gamma} \cong \mathfrak{so}(4, 12)$.

(3) $$E_{8(8)} = (E_8{}^C)^{\tau\gamma} \cong (E_8{}^C)^{\tau\sigma\gamma}$$

because $\gamma \sim \sigma\gamma$ under $\delta \in F_4$ (Proposition 2.2.3) $\subset E_6 \subset E_7 \subset E_8$: $\delta\gamma = \sigma\gamma\delta$, $\delta\tau = \tau\delta$. Now let $\Theta \in ((\mathfrak{e}_8{}^C)^{\tau\sigma\gamma})^{\lambda\gamma}$, $\Theta = \Theta(\phi_*(D), 2\lambda\gamma\chi^{-1}S, 2\chi^{-1}S, 0, 2c, -2c)$ as in (1). From $\tau\sigma\gamma\Theta = \Theta\tau\sigma\gamma$, we have

$$\tau\sigma\gamma\phi_*(D)\gamma\sigma\tau = \phi_*(D), \qquad \tau\sigma\gamma\chi^{-1}S = \chi^{-1}S, \qquad \tau c = c.$$

Hence $D \in \mathfrak{su}(4, 4)$ (Theorem 4.5.7), $\chi^{-1}(I_4'(\tau S)I_4') = \chi^{-1}S$ (Lemma 5.5.3), $I_4'(\tau S)I_4' = S$, that is, $S \in \mathfrak{S}(4, 4, \boldsymbol{C})$ and $c \in \boldsymbol{R}$. Therefore, from Proposition 5.5.2. (3), we have $(\mathfrak{e}_{8(8)})^{\lambda\gamma} \sim ((\mathfrak{e}_8{}^C)^{\tau\sigma\gamma})^{\lambda\gamma} \cong \mathfrak{so}(8, 8)$.

(4) $$E_{8(8)} = (E_8{}^C)^{\tau\gamma} \cong (E_8{}^C)^{\tau\upsilon\gamma}.$$

In fact, consider $\delta: \mathfrak{e}_8{}^C \to \mathfrak{e}_8{}^C$,

$$\delta(\Phi, P, Q, r, s, t) = (\Phi, iP, -iQ, r, -s, -t)$$

(δ is $\phi(iI)$ of Proposition 5.4.2), then $\delta \in E_8$, $\delta\gamma = \gamma\delta$, $\delta\tau = \tau\upsilon\delta$, and $(E_8{}^C)^{\tau\gamma} \ni \alpha \to \delta\alpha\delta^{-1} \in (E_8{}^C)^{\tau\upsilon\gamma}$ gives an isomorphism. Now let $\Theta \in ((\mathfrak{e}_8{}^C)^{\tau\upsilon\gamma})^{\lambda\gamma}$, $\Theta = \Theta(\phi_*(D), 2\lambda\gamma\chi^{-1}S, 2\chi^{-1}S, 0, 2c, -2c)$ as in (1). From $\tau\upsilon\gamma\Theta = \Theta\tau\upsilon\gamma$, we have

$$\tau\gamma\phi_*(D)\gamma\tau = \phi_*(D), \qquad -\tau\gamma\chi^{-1}S = \chi^{-1}S, \qquad \tau c = c.$$

Hence $D\in\mathfrak{su}(8)$ (Theorem 4.5.5), $-\chi^{-1}(\tau S)=\chi^{-1}S$ (Lemma 5.5.3), $-\tau S=S$, that is, $S\in i\mathfrak{S}(8, \boldsymbol{C})$ and $c\in\boldsymbol{R}$. Therefore, from Proposition 5.5.2. (4), we have $(\mathfrak{e}_{8(8)})^{\tilde{\lambda}\gamma}\sim((\mathfrak{e}_8{}^C)^{\tau\upsilon\gamma})^{\tilde{\lambda}\gamma}\cong\mathfrak{so}^*(16)$. $(\mathfrak{e}_{8(-24)})^{\tilde{\lambda}\gamma}\sim((\mathfrak{e}_8{}^C)^{\tau\tilde{\lambda}\upsilon})^{\tilde{\lambda}\gamma}$ (Theorem 5.4.5. (1))$=((\mathfrak{e}_8{}^C)^{\tau\upsilon\gamma})^{\tilde{\lambda}\gamma}$.

THEOREM 5.5.7. (1) *$(E_8{}^C)^{\tilde{\lambda}\gamma}\cong Ss(16, C)$ $(=Spin(16, C)/\boldsymbol{Z}_2$ and not $SO(16, C))$.*
(2) $(E_8)^{\tilde{\lambda}\gamma}\cong Ss(16)$ $(=Spin(16)/\boldsymbol{Z}_2$ and not $SO(16))$.

PROOF is in [21]. The outline of the proof is as follows. The group $(E_8{}^C)^{\tilde{\lambda}\gamma}$ is connected (Lemma 0.7) and the center $z((E_8{}^C)^{\tilde{\lambda}\gamma})$ is $\boldsymbol{Z}_2=\{1, \tilde{\lambda}\gamma\}$, hence $(E_8{}^C)^{\tilde{\lambda}\gamma}$ is isomorphic to one of $SO(16, C)$ or $Ss(16, C)$. $(E_8{}^C)^{\tilde{\lambda}\gamma}$ has the 128-dimensional irreducible C-representation $(\mathfrak{e}_8{}^C)_{-\tilde{\lambda}\gamma}$, however $SO(16, C)$ has no 128-dimensional irreducible C-representation. Therefore $(E_8{}^C)^{\tilde{\lambda}\gamma}$ must be $Ss(16, C)$. As for $(E_8)^{\tilde{\lambda}\gamma}$, the argument is similar to $(E_8{}^C)^{\tilde{\lambda}\gamma}$.

According to Theorem 5.5.6, we use the following notations.

$$So^*(16)=((E_8{}^C)^{\tau\tilde{\lambda}\upsilon})^{\tilde{\lambda}\gamma}\sim(E_{8(-24)})^{\tilde{\lambda}\gamma}, \qquad Sso^*(16)=(So^*(16))_0,$$
$$So(4, 12)=((E_8{}^C)^{\tau})^{\tilde{\lambda}\gamma}=(E_{8(-24)})^{\tilde{\lambda}\gamma},$$
$$So(8, 8)=((E_8{}^C)^{\tau\sigma\gamma})^{\tilde{\lambda}\gamma}\sim(E_{8(8)})^{\tilde{\lambda}\gamma}, \qquad Sso(8, 8)=(So(8, 8))_0.$$

THEOREM 5.5.8. *$(E_{8(-24)})^{\tilde{\lambda}\gamma}\sim(\tau\tilde{\lambda}\upsilon)^{\tilde{\lambda}\gamma}=So^*(16)=Sso^*(16)\times 2\cong(\tau\upsilon\gamma)^{\tilde{\lambda}\gamma}\sim(E_{8(8)})^{\tilde{\lambda}\gamma}$. The Cartan decomposition of $So^*(16)$ is*

$$So^*(16)\simeq((SO(2)\times SU(8))/\boldsymbol{Z}_4\times 2)\times\boldsymbol{R}^{56}$$

where $\boldsymbol{Z}_4=\{(E, E), (E, -E), (-E, iE), (-E, -iE)\}$.

PROOF. The maximal compact subgroup $So^*(16)_K$ of $So^*(16)$ is

$$So^*(16)_K=So^*(16)\cap E_8=((E_8{}^C)^{\tau\tilde{\lambda}\upsilon})^{\tilde{\lambda}\gamma})^{\tau\tilde{\lambda}}=((\tau\tilde{\lambda}\upsilon)^{\tilde{\lambda}\gamma})^{\tau\tilde{\lambda}}=((\upsilon)^{\tilde{\lambda}\gamma})^{\tau\tilde{\lambda}}$$
$$=((\tau\tilde{\lambda})^{\upsilon})^{\tilde{\lambda}\gamma}=((E_8)^{\upsilon})^{\tilde{\lambda}\gamma}\cong((SU(2)\times E_7)/\boldsymbol{Z}_2)^{\tilde{\lambda}\gamma}$$
$$(=((\tau\tilde{\lambda})^{\tilde{\lambda}\gamma})^{\upsilon}=((E_8)^{\tilde{\lambda}\gamma})^{\upsilon}=(Ss(16))^{\upsilon}).$$

Let $\alpha\in So^*(16)_K=((E_8)^{\upsilon})^{\tilde{\lambda}\gamma}$. By Theorem 5.4.5, there exist $A\in SU(2)$, $\beta\in E_7$ such that $\alpha=\phi(A)\beta$. From the condition $\tilde{\lambda}\gamma\alpha=\alpha\tilde{\lambda}\gamma$, we have $\phi({}^tA^{-1})\tilde{\lambda}\gamma\beta\gamma\tilde{\lambda}=\phi(A)\beta$ (Lemma 5.4.3). Hence

$$\begin{cases} {}^tA^{-1}=A \\ \tilde{\lambda}\gamma\beta\gamma\tilde{\lambda}=\beta \end{cases} \quad\text{or}\quad \begin{cases} {}^tA^{-1}=-A \\ \tilde{\lambda}\gamma\beta\gamma\tilde{\lambda}=\upsilon\beta . \end{cases}$$

In the first case, $A\in SO(2)$, $\beta\in(E_7)^{\tilde{\lambda}\gamma}=(E_7)^{\lambda\gamma}\cong SU(8)/Z_2$ (where $Z_2=\{E, -E\}$) (Theorem 4.5.5). In the latter case, $A=(iI)B$, $B\in SO(2)$, $\beta=\iota\beta'$, $\beta'\in(E_7)^{\lambda\gamma}$, Thus $SO^*(16)_K\cong(SO(2)\times(E_7)^{\lambda\gamma}\cup(iI)SO(2)\times\iota(E_7)^{\lambda\gamma})/\boldsymbol{Z}_2$ (where $\boldsymbol{Z}_2=\{(E, 1), (-E,$

$-1)\}) = (SO(2) \times (E_7)^{\lambda\gamma})/\boldsymbol{Z}_2 \times 2 \cong (SO(2) \times SU(8)/Z_2)/\boldsymbol{Z}_2 \times 2$ (where $\boldsymbol{Z}_2 = \{(E, E), (-E, iE)Z_2\}) \cong (SO(2) \times SU(8))/\boldsymbol{Z}_4 \times 2$. (As for the explicit form of $\phi(iI, \iota)$, see Theorem 5.4.6). $(E_{8(8)})^{\tilde{\lambda}\gamma} \sim (\tau\upsilon\gamma)^{\tilde{\lambda}\gamma}$ (Theorem 5.5.6. (4))$=(\tau\tilde{\lambda}\upsilon)^{\tilde{\lambda}\gamma}$.

THEOREM 5.5.9. *$(E_{8(-24)})^{\tilde{\lambda}\gamma} \sim (\tau\tilde{\lambda}\upsilon\sigma)^{\sigma} = So(4, 12)$ is connected. The Cartan decomposition of $So(4, 12)$ is*

$$So(4, 12) \simeq (SU(2) \times SU(2) \times Spin(12))/(Z_2 \times \boldsymbol{Z}_2) \times \boldsymbol{R}^{48}$$

where $Z_2 \times \boldsymbol{Z}_2 = \{(E, E, 1), (E, -E, -\sigma)\} \times \{(E, E, 1), (-E, -E, \sigma)\} = \{(E, E, 1), (E, -E, -\sigma), (-E, -E, \sigma), (-E, E, -1)\}$.

PROOF.
$$E_{8(-24)} = (E_8{}^C)^{\tau} \cong (E_8{}^C)^{\tau\tilde{\lambda}\upsilon\sigma}.$$

In fact, since $\tilde{\lambda} \sim \upsilon\sigma$ under some $\delta \in E_8$: $\delta\tilde{\lambda} = \upsilon\sigma\delta$, $\delta\tau\tilde{\lambda} = \tau\tilde{\lambda}\delta$ (Proposition 5.3.4), $(E_8{}^C)^{\tau} \ni \alpha \to \delta\alpha\delta^{-1} \in (E_8{}^C)^{\tau\tilde{\lambda}\upsilon\sigma}$ gives an isomorphism. Consider the subgroup $((E_8{}^C)^{\tau\tilde{\lambda}\upsilon\sigma})^{\sigma} = (\tau\tilde{\lambda}\upsilon\sigma)^{\sigma} = (\tau\tilde{\lambda}\upsilon)^{\sigma}$ of $(\tau\tilde{\lambda}\upsilon\sigma)$. Since $\dim((\mathfrak{e}_8{}^C)^{\tau\tilde{\lambda}\upsilon})^{\sigma}) = 120$, the type of $(\tau\tilde{\lambda}\upsilon)^{\sigma}$ must be D_8. Moreover, as is shown in the following, the type of the maximal compact subgroup of $(\tau\tilde{\lambda}\upsilon)^{\sigma}$ is $D_2 \oplus D_6$, hence the type of the Lie algebra of $(\tau\tilde{\lambda}\upsilon)^{\sigma}$ is $\mathfrak{so}(4, 12)$. Hence we put here $So(4, 12) = (\tau\tilde{\lambda}\upsilon)^{\sigma}$. Now the maximal compact subgroup $So(4, 12)_K$ of $So(4, 12)$ is

$$So(4, 12)_K = So(4, 12) \cap E_8 = ((E_8{}^C)^{\tau\tilde{\lambda}\upsilon})^{\sigma})^{\tau\tilde{\lambda}} = ((\tau\tilde{\lambda}\upsilon)^{\sigma})^{\tau\tilde{\lambda}} = ((\tau\tilde{\lambda})^{\upsilon})^{\sigma}$$
$$= ((E_8)^{\upsilon})^{\sigma} \cong ((SU(2) \times E_7)/\boldsymbol{Z}_2)^{\sigma}.$$

Let $\alpha \in So(4, 12)_K = ((E_8)^{\upsilon})^{\sigma}$. By Theorem 5.4.5, there exist $A \in SU(2)$, $\beta \in E_7$ such that $\alpha = \phi(A)\beta$. From the condition $\sigma\alpha = \alpha\sigma$, we have $\phi(A)\sigma\beta\sigma = \phi(A)\beta$ (Lemma 5.4.3). Hence $\sigma\beta\sigma = \beta$, that is, $\beta \in (E_7)^{\sigma} \cong (SU(2) \times Spin(12))/Z_2$ (where $Z_2 = \{(E, 1), (-E, -\sigma)\}$) (Theorem 4.6.14). Thus $((\tau\tilde{\lambda})^{\upsilon})^{\sigma} \cong (SU(2) \times (E_7)^{\sigma})/\boldsymbol{Z}_2$ (where $\boldsymbol{Z}_2 = \{(E, 1), (-E, -1)\}) \cong (SU(2) \times (SU(2) \times Spin(12))/Z_2)/\boldsymbol{Z}_2$ (where $\boldsymbol{Z}_2 = \{(E, (E, 1)), (-E, (-E, \sigma)Z_2)\}) \cong (SU(2) \times SU(2) \times Spin(12))/(Z_2 \times \boldsymbol{Z}_2)$.

REMARK. The maximal compact subgroup $So(4, 12)_K$ of $So(4, 12) = (\tau)^{\tilde{\lambda}\gamma}$ is

$$So(4, 12)_K = ((\tau)^{\tilde{\lambda}\gamma})^{\tau\tilde{\lambda}} = ((\tau)^{\tilde{\lambda}})^{\gamma} = ((\tau\tilde{\lambda})^{\tilde{\lambda}})^{\gamma} = ((\tau\tilde{\lambda})^{\tilde{\lambda}\gamma})^{\gamma} = (Ss(16))^{\gamma}.$$

The author can not give any isomorphism between $((\tau\tilde{\lambda})^{\tilde{\lambda}})^{\gamma}$ and $((\tau\tilde{\lambda})^{\upsilon})^{\sigma}$ directly.

Before determine the maximal compact subgroup $So(8, 8)_K$ of $So(8, 8)$, recall the construction of the spinor group $Spin(n)$ using the Clifford algebra $C(\boldsymbol{R}^n{}_-)$ [15]. Let $C(\boldsymbol{R}^n{}_-)$ be the Clifford algebra generated by $e_1, \cdots, e_n$ with relations $e_i{}^2 = -1$. Let $\boldsymbol{R}^n$ be the $\boldsymbol{R}$-vector space spanned by $e_1, \cdots, e_n$ and put $S^{n-1} = \{a \in \boldsymbol{R}^n \mid a^2 = -1\}$. Now the spinor group $Spin(n)$ is defined by

$$Spin(n)=\{a_1 \cdots a_{2m} \mid a_i \in S^{n-1},\ m=1, 2, \cdots\}.$$

We use here the notation $Spin(n)=Spin(e_1, \cdots, e_n)$. In the case of $n \equiv 0 \pmod 4$, the center $z(Spin(n))$ of $Spin(n)$ is $\boldsymbol{Z}_2 \times \boldsymbol{Z}_2=\{1, -1\} \times \{1, e_1 \cdots e_n\}=\{1, -1, e_1 \cdots e_n, -e_1 \cdots e_n\}$ and we know

$$Spin(n)/\{1, -1\}=SO(n), \qquad Spin(n)/\{1, e_1 \cdots e_n\}=Ss(n).$$

THEOREM 5.5.10. $(E_{8(8)})^{\tilde{\lambda}\gamma} \sim (\tau\sigma\gamma)^{\tilde{\lambda}\gamma}=So(8, 8)=Sso(8, 8)\times 2$. *The Cartan decomposition of* $So(8, 8)$ *is*

$$So(8, 8) \simeq ((Spin(8)\times Spin(8))/(\boldsymbol{Z}_2\times \boldsymbol{Z}_2)\times 2)\times \boldsymbol{R}^{64}$$

where $\boldsymbol{Z}_2 \times \boldsymbol{Z}_2=\{(1, 1), (-1, -1)\} \times \{(1, 1), (e_1 \cdots e_8, e_9 \cdots e_{16})\}=\{(1, 1), (-1, -1), (e_1 \cdots e_8, e_9 \cdots e_{16}), (-e_1 \cdots e_8, -e_9 \cdots e_{16})\}$.

PROOF. Consider the group $(E_8{}^C)^{\tau\sigma\gamma}=(\tau\sigma\gamma) \cong E_{8(8)}$ (Theorem 5.5.6. (2)). J. Sekiguchi [20] shows that $(\tau\sigma\gamma)/((\tau\sigma\gamma)^{\tilde{\lambda}\gamma})_0$ is simply connected and the fundamental group of $(\tau\sigma\gamma)/(\tau\sigma\gamma)^{\tilde{\lambda}\gamma}$ is $\boldsymbol{Z}_2$, hence $Sso(8, 8)=(\tau\sigma\gamma)^{\lambda\gamma}$ must have two connected components. The maximal compact subgroup $So(8, 8)_K$ of $So(8, 8)$ is

$$So(8, 8)_K=So(8, 8)\cap E_8=(((E_8{}^C)^{\tau\sigma\gamma})^{\tilde{\lambda}\gamma})^{\tau\tilde{\lambda}}=((\tau\sigma\gamma)^{\tilde{\lambda}\gamma})^{\tau\tilde{\lambda}}=((\tau\sigma\gamma)^{\tau\gamma})^{\tau\tilde{\lambda}}$$
$$=((\sigma)^{\tau\gamma})^{\tau\tilde{\lambda}}=((\sigma)^{\tilde{\lambda}\gamma})^{\tau\tilde{\lambda}}=((\tau\tilde{\lambda})^{\tilde{\lambda}\gamma})^{\sigma}=(Ss(16))^{\sigma}.$$

We shall find a subgroup of type $D_4 \oplus D_4$ in $Ss(16)$. In the spinor group $Spin(16)=Spin(e_1, \cdots, e_{16})$, consider two subgroups $Spin(8)=Spin(e_1, \cdots, e_8)$, $Spin(8)=Spin(e_9, \cdots, e_{16})$. We define a mapping

$$Spin(8)\times Spin(8) \xrightarrow{\ \psi\ } Spin(16) \xrightarrow{\ \pi\ } Ss(16)$$

by $\psi(\alpha, \beta)=\alpha\beta$ and let π be the natural projection. Then the image of $\pi\psi$ is isomorphic to the group $(Spin(8)\times Spin(8))/(\boldsymbol{Z}_2\times \boldsymbol{Z}_2)$.

REMARK. The author can not find any element of $So(8, 8)$ which does not be contained in $Sso(8, 8)$, and can not realize the subgroup $(Spin(8)\times Spin(8))/(\boldsymbol{Z}_2\times \boldsymbol{Z}_2)$ in $So(8, 8)$ concretely.

Appendix

The Cartan decompositions of the exceptional linear Lie groups of type E_8 are given as follows.

E_8: simply connected compact Lie group of type E_8,

$E_8{}^C \simeq E_8 \times \boldsymbol{R}^{248}$,

$$E_{8(8)} \simeq Ss(16) \times \boldsymbol{R}^{128},$$

$$E_{8(-24)} \simeq (SU(2) \times E_7)/\boldsymbol{Z}_2 \times \boldsymbol{R}^{112}.$$

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