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SPACES WITH A PROPERTY RELATED TO UNIFORMLY LOCAL FINITENESS

By

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Throughout this paper a space always means a topological space.

A collection $\mathcal{A}$ of subsets of a space X is said to be *uniformly locally finite* if there is a normal open cover $\mathcal{U}$ of X such that each member of $\mathcal{U}$ intersects only finitely many members of $\mathcal{A}$. If every locally finite collection of subsets of X is uniformly locally finite, then X is said to have *property* (U). These notions are defined in K. Morita [10], and it is pointed out there that every M-space or every strongly normal (=collectionwise normal and countably paracompact) space is a space with property (U), and such a space is expandable in the sense of L. L. Krajewski [7]. Hence for normal spaces property (U), expandability and strong normality all coincide with each other by a well-known theorem of M. Katětov [6], and so a question was posed by Morita [10] to find a condition which, together with expandability, is equivalent to property (U).

The purpose of this paper is to investigate spaces with property (U), mainly by defining a new notion of U-embedding which is a generalization of P-embedding; a subspace A of a space X is said to be *U-embedded* in X if every uniformly locally finite collection of subsets of A is uniformly locally finite also in X. In §1 we treat spaces having a property that every discrete collection of subsets is uniformly locally finite, which we call spaces with property $(U)^*$. By C. H. Dowker [1], collectionwise normal spaces are precisely those spaces any of whose closed set is P-embedded. Being motivated with this result we shall establish a theorem that a space X has property $(U)^*$ iff any closed set of X is U-embedded in X, and then it will be shown that a space has property (U) iff it has property $(U)^*$ and is a cb-space in the sense of J. Mack [9]; the latter is a quite analogue to a theorem of Krajewski [7] that a space is expandable iff it is discretely expandable and countably paracompact. In §2 we shall give another description of spaces with property (U), which is an answer to the question of Morita above, by defining spaces with weak property (U) that include all M'-spaces [5] and all extremally disconnected spaces.

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§ 1. $U^{\mathfrak{m}}$-embedding

Let $\mathcal{A}$ be a collection of subsets and $\mathcal{U}$ an open cover of a space X. Then we say, for convenience, that $\mathcal{A}$ is locally finite with respect to $\mathcal{U}$ in case every member of $\mathcal{U}$ intersects only a finite number of members of $\mathcal{A}$. The following lemma, which is proved in [10], [12] will be useful.

LEMMA 1.1. *For a collection $\mathcal{A}=\{A_\alpha|\alpha\in\Omega\}$ of subsets of a space X the following are equivalent.*

(a) *$\mathcal{A}$ is uniformly locally finite.*

(b) *There is a normal open cover $\mathcal{V}$ of X with* Card $\mathcal{V}$ = Card Ω *such that $\mathcal{A}$ is locally finite with respect to $\mathcal{V}$.*

(c) *There are cozero-sets G_α and zero-sets F_α, $\alpha\in\Omega$ of X such that $A_\alpha\subset F_\alpha\subset G_\alpha$ for each $\alpha\in\Omega$ and $\{G_\alpha|\alpha\in\Omega\}$ is locally finite.*

The union of a locally finite collection $\mathcal{A}$ of zero-sets is not always a zero-set. If $\mathcal{A}$ is uniformly locally finite, then the union is a zero-set. This fact was proved earlier in [11] and will be frequently used in the present paper.

Let $\mathfrak{m}$ denote an infinite cardinal number. A subspace A of a space X is said to be $P^{\mathfrak{m}}$-embedded in X if for any normal open cover $\mathcal{U}$ of A with cardinality $\leqq\mathfrak{m}$ there is a normal open cover $\mathcal{V}$ of X such that $\mathcal{V}\cap A=\{V\cap A|V\in\mathcal{V}\}$ refines $\mathcal{U}$. If A is $P^{\mathfrak{m}}$-embedded in X for every $\mathfrak{m}$, A is said to be P-embedded in X [15]. It is known that $P^{\aleph_0}$-embedding coinsides with C-embedding [2].

DEFINITION 1.2. A subspace A of a space X is said to be *$U^{\mathfrak{m}}$-embedded* in X if every uniformly locally finite collection of subsets of A with cardinality $\leqq\mathfrak{m}$ is uniformly locally finite in X. If A is $U^{\mathfrak{m}}$-embedded in X for every $\mathfrak{m}$, A is said to be *U-embedded* in X.

In view of Lemma 1.1 $P^{\mathfrak{m}}$- (rest. P-) embedding implies $U^{\mathfrak{m}}$- (resp. U-) embedding; in particular C-embedding implies $U^{\aleph_0}$-embedding. Clearly, in a space with property (U) any closed set is U-embedded. Hence any non-normal M-space contains a closed U-embedded, subspace which is not C-embedded, and so the converse of the implications above does not hold in general.

The following theorem may be of interest in itself when compared with the notion of $P^{\mathfrak{m}}$-embedding.

THEOREM 1.3. *A subspace A of a space X is $U^{\mathfrak{m}}$-embedded in X iff for every normal open cover $\mathcal{U}$ of A with cardinality $\leqq\mathfrak{m}$ there exists a normal open cover $\mathcal{V}$ of X such that for each $V\in\mathcal{V}$ $V\cap A$ is contained in a union of finitely many members of $\mathcal{U}$.*

PROOF. To prove the "if" part, assume that the condition is satisfied. Let $\mathcal{A}$ be a uniformly locally finite collection of subsets of A with Card $\mathcal{A} \leqq \mathfrak{m}$. Then by Lemma 1.1 $\mathcal{A}$ is locally finite with respect to a normal open cover $\mathcal{U}$ of A with Card $\mathcal{U} \leqq \mathfrak{m}$. By assumption take a normal open cover $\mathcal{V}$ of X satisfying the condition above. Then it is easy to see that $\mathcal{A}$ is locally finite with respect to $\mathcal{V}$. Hence A is $U^{\mathfrak{m}}$-embedded in X. Coversely suppose that A is $U^{\mathfrak{m}}$-embedded in X, and let $\mathcal{U}=\{U_\lambda|\lambda\in\Lambda\}$ be a normal open cover of A with Card $\Lambda \leqq \mathfrak{m}$. Since $\mathcal{U}$ is normal, there are a locally finite cozero-set cover $\{G_\lambda|\lambda\in\Lambda\}$ and a zero-set cover $\mathcal{F}=\{F_\lambda|\lambda\in\Lambda\}$ of A such that

$$F_\lambda \subset G_\lambda \subset U_\lambda,\ \lambda\in\Lambda.$$

Then by Lemma 1.1 $\mathcal{F}$ is uniformly locally finite in A, and by the assumption, so is also in X. Let $\mathcal{V}$ be a normal open cover of X so that $\mathcal{F}$ is locally finite with respect to $\mathcal{V}$. Then for each $V\in\mathcal{V}$ there are $\lambda_1, \cdots, \lambda_n \in \Lambda$ such that $V\cap A\subset F_{\lambda_1}\cup\cdots\cup F_{\lambda_n}$ since $\mathcal{F}$ covers A. Hence we have $V\cap A\subset U_{\lambda_1}\cup\cdots\cup U_{\lambda_n}$. Thus, the "only if" part is proved. This completes the proof.

In case $\mathfrak{m}=\aleph_0$, we have

COROLLARY 1.4. *A subspace A of a space X is $U^{\aleph_0}$-embedded in X iff for any countable increasing normal open cover $\mathcal{U}$ of A there exists a normal open cover $\mathcal{V}$ of X such that $\mathcal{V}\cap A$ refines $\mathcal{U}$.*

In [4] T. Ishii and H. Ohta defined the notion of C_1-embedding; a subspace A of a space X is said to be *C_1-embedded* in X if any zero-set Z_1 of X and any zero-set Z_2 of A disjoint from Z_1 are completely separated in X. It is proved there that C-embedding implies C_1-embedding. The following lemma contains this result.

LEMMA 1.5. *$U^{\aleph_0}$-embedding implies C_1-embedding.*

PROOF. Assume that a subspace A of a space X is $U^{\aleph_0}$-embedded in X. Let Z_1 be a zero-set of X and Z_2 a zero-set of A disjoint from Z_1. Let $f: X\to I$ be a continuous map such that $Z_1=\{x|f(x)=0\}$, where $I=[0,1]$. Let us put for $n\in N$ (=the set of natural numbers)

$$G_n=\{x\in X|f(x)>1/n\},\ E_n=\{x\in X|f(x)\geqq 1/n\}$$

and

$$U_n=(A\cap G_n)\cup(A-Z_2).$$

Then U_n is a cozero-set of A and we have

$$U_n\subset U_{n+1},\ n\in N;\ A=\cup\{U_n|n\in N\}.$$

Since A is $U^{\aleph_0}$-embedded, by Corollary 1.4 there are a locally finite cozero-set cover $\{V_n|n\in N\}$ and a zero-set cover $\{F_n|n\in N\}$ of X such that $V_n\cap A\subset U_n$ and $F_n\subset V_n$ for each $n\in N$. Let us set

$$Z_3=\cup\{E_n\cap F_n|Z_2\cap F_n\neq\emptyset,\ n\in N\}.$$

Note that $\{E_n\cap F_n|n\in N\}$ is a uniformly locally finite collection of zero-sets of X. Hence Z_3 is a zero-set of X, and we easily have

$$Z_3\supset Z_2,\ Z_3\cap Z_1=\emptyset.$$

Therefore Z_1 and Z_2 are completely separated in X. This completes the proof.

REMARK. Let $X=Y\cup\{p\}$ be the one-point Lindelöfication of an uncountable set Y. Then Y is C_1-embedded in X ([4], Example 6.3), but not $U^{\aleph_0}$-embedded in X. Hence the converse of Lemma 1.5 does not hold in general.

It should be noted that the condition which describes $U^{\mathfrak{m}}$-embedding in Theorem 1.3 has already appeared in [11, Theorem 2.2], where it is shown that this condition together with C^*-embedding is equivalent to $P^{\mathfrak{m}}$-embedding. Hence by Theorem 1.3, Lemma 1.5 and a result of [4] that C_1- and z-embedding is equal to C-embedding, we have the following proposition.

Recall that a subspace A of a space X is z-embedded in X if for any zero-set Z of A there is a zero-set Z' of X such that $Z'\cap A=Z$.

PROPOSITION 1.6. *$P^{\mathfrak{m}}$-embedding $\Longleftrightarrow$ $U^{\mathfrak{m}}$- and z-embedding.*

As another characterization of $U^{\mathfrak{m}}$-embedding, we prove the following theorem.

THEOREM 1.7. *A subspace A of a space X is $U^{\mathfrak{m}}$-embedded in X iff A is $U^{\aleph_0}$-embedded in X and if $\mathcal{F}$ is a discrete and uniformly locally finite collection of subsets in A with cardinality $\leqq\mathfrak{m}$ then $\mathcal{F}$ is uniformly locally finite in X.*

PROOF. We shall only prove the "if" part. Assume that the condition is satisfied. Let $\mathcal{U}$ be a normal open cover of A with Card $\mathcal{U}\leqq\mathfrak{m}$. Then there exist a cozero-set cover $\mathcal{H}=\cup\mathcal{H}_n$, a zero-set cover $\mathcal{F}=\cup\mathcal{F}_n$, a cozero-set cover $\mathcal{G}=\cup\mathcal{G}_n$ of A, where $\mathcal{H}_n=\{H_{n\alpha}|\alpha\in\Omega_n\}$, $\mathcal{F}_n=\{F_{n\alpha}|\alpha\in\Omega_n\}$, $\mathcal{G}_n=\{G_{n\alpha}|\alpha\in\Omega_n\}$ with Card $\Omega_n\leqq\mathfrak{m}$ for $n\in N$ such that

(1) $\mathcal{G}$ refines $\mathcal{U}$,

(2) $\mathcal{G}_n$ is discrete for $n=1, 2, \cdots$,

(3) $H_{n\alpha}\subset F_{n\alpha}\subset G_{n\alpha}$ for $\alpha\in\Omega_n$, $n=1, 2, \cdots$.

Let $H_n=\cup\{H_{k\alpha}|\alpha\in\Omega_k,\ k\leqq n\}$. Then $\{H_n|n\in N\}$ is an increasing cozero-set cover of A. Hence, by assumption, there is a countable cozero-set cover $\{L_n|n\in N\}$ of X

such that $L_n \cap A \subset H_n$ for $n \in N$. On the other hand, since $\mathcal{F}_k$ is discrete and uniformly locally finite in A by (2), by assumption for $\bigcup_{k \leq n} \mathcal{F}_k$ we can choose a locally finite cozero-set cover $\mathcal{V}_n = \{V_{n\lambda} | \lambda \in \Lambda_n\}$ of X so that $\bigcup_{k \leq n} \mathcal{F}_k$ is locally finite with respect to $\mathcal{V}_n$. Let

$$\mathcal{V}' = \bigcup_{n \in N} \mathcal{V}_n', \mathcal{V}_n' = \{V_{n\lambda} \cap L_n | \lambda \in \Lambda_n\}.$$

Then $\mathcal{V}'$ is a σ-locally finite cozero-set cover of X, and hence a normal open cover of X, and it is easy to see that for each $V_{n\lambda} \cap L_n \in \mathcal{V}_n'$, $V_{n\lambda} \cap L_n \cap A$ is contained in a finite union of members of $\bigcup_{k \leq n} \mathcal{F}_k$, and so, by (1), (3) it is also in a union of finitely many members of $\mathcal{U}$. Thus, by Theorem 1.3 A is $\mathcal{U}^{\mathfrak{m}}$-embedded in X. This completes the proof.

Let us now proceed to prove our results mentioned in the introduction.

DEFINITION 1.8. A space X is said to have *property* $(U^{\mathfrak{m}})^*$ (resp. $(U^{\mathfrak{m}})$) if every discrete (resp. locally finite) collection of subsets of X with cardinality $\leq \mathfrak{m}$ is uniformly locally finite in X. If X has property $(U^{\mathfrak{m}})^*$ for any $\mathfrak{m}$, X is said to have *property* $(U)^*$.

Obviously, a space has property (U) iff it has property $(U^{\mathfrak{m}})$ for any $\mathfrak{m}$.

THEOREM 1.9. *For a space X the following are equivalent.*

(a) *X has property $(U)^*$.*

(b) *Every locally finite collection of closed sets of X of finite order is uniformly locally finite.*

(c) *Every closed set of X is U-embedded in X.*

Theorem 1.9 directly follows from the following.

THEOREM 1.10. *For a space X the following are equivalent.*

(a) *X has property $(U^{\mathfrak{m}})^*$.*

(b) *Every locally finite collection of closed sets of X of finite order with cardinality $\leq \mathfrak{m}$ is uniformly locally finite.*

(c) *Every closed set of X is $U^{\mathfrak{m}}$-embedded in X.*

PROOF. (a)→(b). The method is similar to that of Katětov [6]. Suppose (a), and let $\mathcal{F} = \{F_\alpha | \alpha \in \Omega\}$ be a locally finite collection of closed sets of X of finite order and with Card $\Omega \leq \mathfrak{m}$. We shall prove (b) by induction on the order n of $\mathcal{F}$. (b) is evidently valid in case $n=1$. Assume that (b) is true for $k \leq n$, and that $\mathcal{F}$ has order $n+1$. Let Γ be the set of all finite subsets of Ω, and let us put

$$\Gamma^* = \{\gamma | \text{Card}\, \gamma = n+1,\ \gamma \in \Gamma\},$$

$$\mathcal{F}^* = \{\bigcap_{\alpha \in \gamma} F_\alpha | \gamma \in \Gamma^*\}.$$

Then $\mathcal{F}^*$ is a discrete collection of closed sets with cardinality $\leqq \mathfrak{m}$. Hence by (a) and Lemma 1.1 there are a cozero-set L_γ, a zero-set K_γ and cozero-set M_γ of X such that for $\gamma \in \Gamma^*$

$$\bigcap_{\alpha\in\gamma} F_\alpha \subset L_\gamma \subset K_\gamma \subset M_\gamma$$

and $\{M_\gamma | \gamma \in \Gamma^*\}$ is locally finite. Let

$$L_\gamma' = L_\gamma - \cup\{F_\alpha | \alpha \notin \gamma\},$$

for $\gamma \in \Gamma^*$, and

$$L = \cup\{L_\gamma' | \gamma \in \Gamma^*\}.$$

Then $\{F_\alpha - L | \alpha \in \Omega\}$ is a locally finite closed collection of order $\leqq n$ with cardinality $\leqq \mathfrak{m}$. Therefore by induction hypothesis and Lemma 1.1 there are a cozero-set H_α and zero-set D_α for $\alpha \in \Omega$ such that

$$F_\alpha - L \subset D_\alpha \subset H_\alpha,$$

and $\{H_\alpha | \alpha \in \Omega\}$ is locally finite. Let us set for $\alpha \in \Omega$

$$E_\alpha = D_\alpha \cup \cup\{K_\gamma | \gamma \in \Gamma^*, \ \alpha \in \gamma\}$$

$$G_\alpha = H_\alpha \cup \cup\{M_\gamma | \gamma \in \Gamma^*, \ \alpha \in \gamma\}.$$

Then E_α is a zero-set and G_α a cozero-set since $\{K_\gamma\}$ is uniformly locally finite and $\{M_\gamma\}$ locally finite. We shall prove that $F_\alpha \subset E_\alpha$ and $\{G_\alpha | \alpha \in \Omega\}$ is locally finite. Let $x \in F_\alpha$. If $x \notin L$, then $x \in F_\alpha - L \subset D_\alpha$. Hence $x \in E_\alpha$. If $x \in L$, then $x \in L_\gamma'$ for some $\gamma \in \Gamma^*$. Hence $F_\alpha \cap L_\gamma' \neq \emptyset$, and so $\alpha \in \gamma$. Since $x \in L_\gamma' \cap K_\gamma$ and $\alpha \in \gamma$, we have $x \in E_\alpha$. Thus $F_\alpha \subset E_\alpha$. To show that $\{G_\alpha\}$ is locally finite, let $x \in X$ and U be a neighborhood of x such that for some $\gamma_0 \in \Gamma$ and a finite subset $\{\gamma_1, \cdots, \gamma_s\}$ of Γ^* we have

$$U \cap H_\alpha = \emptyset \quad \text{if} \quad \alpha \notin \gamma_0, \text{ and}$$

$$U \cap M_\gamma = \emptyset \quad \text{if} \quad \gamma \notin \{\gamma_1, \cdots, \gamma_s\}.$$

Then we see that if $\alpha \notin \gamma_0$ and $U \cap G_\alpha \neq \emptyset$, then

$$\alpha \in \gamma_1 \cup \cdots \cup \gamma_s.$$

Hence $\{G_\alpha | \alpha \in \Omega\}$ is locally finite. Thus, by Lemma 1.1 $\mathcal{F}$ is uniformly locally finite, and (b) holds.

(b)→(c). Suppose (b), and let A be a closed set of X. To apply Theorem 1.7, first we shall prove that A is $U^{\aleph_0}$-embedded in X. Let $\mathcal{U} = \{U_n | n \in N\}$ be an increasing cozero-set cover of A. Then there are a cozero-set V_n and a zero-set F_n of A such that

$$F_n \subset U_n, \ V_n \subset F_n \subset V_{n+1}, \ n \in N,$$

$$A = \cup\{V_n | n \in N\}.$$

Let us set

$$\mathcal{F}=\{F_n-V_{n-1}|n\in N\},$$

where $V_0=0$. Then $\mathcal{F}$ is a locally finite closed collection in X since A is closed, and we have the order of $\mathcal{F}\leqq 2$. Hence by (b) there are a cozero-set G_n and a zero-set E_n of X such that

$$F_n-V_{n-1}\subset E_n\subset G_n,\ n\in N, \text{ and}$$
$$\{G_n|n\in N\} \text{ is locally finite.}$$

Let

$$H_0=X-\bigcup_n E_n\ ;\ H_n=G_1\cup\cdots\cup G_n-\bigcup_{i>n} E_i\ (n\geqq 1)$$

Then $\mathcal{H}=\{H_0, H_n|n\in N\}$ is a countable cozero-set cover of X. Hence it is normal and we have

$$A\cap H_0=\emptyset\ ;\ A\cap H_n\subset F_n\subset U_n,\ n\in N.$$

Hence A is $U^{\aleph_0}$-embedded in X by Corollary 1.4. Now again by (b) we see that A satisfies the condition in Theorem 1.7. Thus, A is $U^{\mathfrak{m}}$-embedded in X.

(c)→(a). Suppose (c), and let $\{A_\alpha\}$ be a discrete collection of subsets in X with cardinality $\leqq\mathfrak{m}$. Then the collection $\{\mathrm{Cl}\,A_\alpha\}$ is discrete and uniformly locally finite in the closed set $\cup\mathrm{Cl}\,A_\alpha$. Hence by (c), it is uniformly locally finite in X, which shows (a). This completes the proof of the theorem.

A space X is called a (weak) *cb*-space if for any decreasing sequence $\{F_n\}$ of (regular) closed sets of X with $\bigcap_n F_n=\emptyset$, there is a sequence $\{Z_n\}$ of zero-sets of X with $\bigcap_n Z_n=\emptyset$ such that $Z_n\supset F_n$ for each $n\in N$ ([8], [9]). It is known that every normal and countably paracompact space is *cb*, and a space is *cb* iff it is weak *cb* and countably paracompact. Weak *cb*-spaces are known to include all Tychonoff pseudocompact spaces, more generally all Tychonoff M'-spaces [5], and all extremally disconnected spaces. Recall that a space is extremally disconnected if the closure of every open set is open.

LEMMA 1.11. *A space satisfies property* $(U^{\aleph_0})$ *iff it is cb.*

PROOF. We shall only prove the "if" part. Suppose that a space X is *cb* and $\mathcal{F}=\{F_n|n\in N\}$ a locally finite collection of closed sets in X. Let $E_n=\bigcup_{k\geqq n} F_k$. Then $\{E_n\}$ is a decreasing sequence of closed sets with $\bigcap_n E_n=\emptyset$. Hence there is a sequence $\{Z_n\}$ of zero-sets of X with $\bigcap_n Z_n=\emptyset$ such that $Z_n\supset E_n$ for each $n\in N$. Then $\mathcal{U}=\{X-Z_n|n\in N\}$ is a countable cozero-set cover of X. Hence $\mathcal{U}$ is normal, and $\mathcal{F}$ is locally finite with respect to $\mathcal{U}$ as is easily seen. This completes the proof.

THEOREM 1.12. *A space has property* (U) *iff it satisfies property* $(U)^*$ *and is a cb-space.*

Theorem 1.12 follows directly from the following.

THEOREM 1.13. *A space has property* $(U^{\mathfrak{m}})$ *iff it satisfies property* $(U^{\mathfrak{m}})^*$ *and is a cb-space.*

PROOF. By Lemma 1.11 the "only if" part is obvious. Let a space X be a *cb*-space with property $(U^{\mathfrak{m}})^*$. Let $\mathcal{F}=\{F_\alpha|\alpha\in\Omega\}$ be a locally finite collection of subsets of X with $\mathrm{Card}\,\Omega\leqq\mathfrak{m}$. We may assume that each F_α is closed. Let us put

$$U_n=\{x\in X|\mathrm{ord}_x\mathcal{F}<n\},$$

where $\mathrm{ord}_x\mathcal{F}$ denotes the order of $\mathcal{F}$ at x. Then U_n is an open set and we have

$$U_n\subset U_{n+1},\ n\in N;\ X=\bigcup_n U_n.$$

Since X is *cb*, one can readily choose a cozero-set V_n of X so that $V_n\subset U_n$ for $n\in N$ and $X=\bigcup_n V_n$. Then $\{V_n\}$ is normal, and hence, we may assume that $\{V_n\}$ is locally finite and admits a zero-set cover $\{Z_n\}$ with $Z_n\subset V_n$ for each $n\in N$. Let $\mathcal{E}_n=\{Z_n\cap F_\alpha|\alpha\in\Omega\}$. Then we have order of $\mathcal{E}_n<n$. Hence, by assumption and Theorem 1.10, there are a cozero-set $G_{n\alpha}$ and a zero-set $Z_{n\alpha}$ such that

$$Z_n\cap F_\alpha\subset Z_{n\alpha}\subset G_{n\alpha},\ \alpha\in\Omega.$$

Let us put for $\alpha\in\Omega$

$$E_\alpha=\bigcup\{Z_{n\alpha}\cap Z_n|n\in N\}.$$

$$H_\alpha=\bigcup\{G_{n\alpha}\cap V_n|n\in N\}.$$

Then, since $\{V_n\}$ is locally finite, E_α is a zero-set and H_α a cozero-set of X, and $F_\alpha\subset E_\alpha\subset H_\alpha$. Moreover, it is easy to see that $\{H_\alpha|\alpha\in\Omega\}$ is locally finite. Thus, by Lemma 1.1 $\mathcal{F}$ is uniformly locally finite, and the proof is completed.

A space X is called $\mathfrak{m}$-expandable (resp. discretely $\mathfrak{m}$-expandable) if for any locally finite (resp. discrete) collection $\{F_\alpha|\alpha\in\Omega\}$ of closed sets in X with $\mathrm{Card}\,\Omega\leqq\mathfrak{m}$ there is a locally finite collection of $\{G_\alpha|\alpha\in\Omega\}$ of open sets of X such that $F_\alpha\subset G_\alpha$ for each $\alpha\in\Omega$. Expandable (resp. discretely expandable) spaces are defined to be an $\mathfrak{m}$-expandable (resp. a discretely $\mathfrak{m}$-expandable) space for any $\mathfrak{m}$. It is known that a space is ($\mathfrak{m}$-) expandable iff it is discretely ($\mathfrak{m}$-) expandable and countably paracompact, and $\aleph_0$-expandability coincides with countably paracompactness. These notions and facts are obtained by Krajewski [7] and Smith and Krajewski [16]. Our Theorem 1.12 as well as Lemma 1.11 may be compared with these results.

In Theorem 1.12 it is unknown whether *cb*-property can be replaced by countably paracompactness, or equivalently, any countably paracompact space with property $(U^{\aleph_0})^*$ is *cb*.

In [3] Hardy and Juhasz defined the notion of *nd*-spaces; a space X is called *nd* if for any decreasing sequence $\{F_n\}$ of nowhere dense closed sets with $\cap F_n=\emptyset$ there is a sequence $\{Z_n\}$ of zero-sets with $\cap Z_n=\emptyset$ such that $Z_n\supset F_n$ for each $n\in N$. The following lemma can be proved similarly as Lemma 1.11. Note that the union of a locally finite collection of nowhere dense closed sets is nowhere dense and closed.

LEMMA 1.14. *A space X is nd iff every countable locally finite collection of nowhere dense closed sets is uniformly locally finite.*

In [3] it is obtained that *cb*-property implies *nd*-property, and *nd*-property implies countably paracompactness (its proof seems to contain a gap, but the fact still remains true), and an example of countably paracompact but not an *nd*-space. It is also conjectured that an *nd*-space need not be *cb*.

Relating to our question above or this conjecture, we shall prove the following proposition.

PROPOSITION 1.15. *A space is cb iff it has property $(U^{\aleph_0})^*$ and is nd.*

PROOF. We shall only prove the "if" part. Let X be an *nd*-space with property $(U^{\aleph_0})^*$, and let $\{F_n\}$ be a decreasing sequence of closed sets of X with $\cap F_n=\emptyset$. Then $\{\mathrm{Bd}\,F_n|n\in N\}$ is a locally finite collection of nowhere dense closed sets, where $\mathrm{Bd}\,F_n=$the boundary of F_n. Hence by Lemma 1.14 there are cozero-sets C_n, A_n and a zero-set B_n such that $\{A_n|n\in N\}$ is locally finite and

$$\mathrm{Bd}\,F_n\subset C_n\subset B_n\subset A_n,\ n\in N.$$

Let $E_n=F_n-C_{n+1}\cup F_{n+1}$. Then $\{E_n\}$ is a discrete closed collection. Since X has $(U^{\aleph_0})^*$, there is a locally finite cozero-set collection $\{G_n\}$ and a zero-set collection $\{D_n\}$ such that $E_n\subset D_n\subset G_n$ for each $n\in N$. Let

$$Z_n=\cup\{B_k|k>n,\ k\in N\}\cup\cup\{D_k|k\geqq n,\ k\in N\}.$$

Then it is easily checked that $\{Z_n\}$ is a decreasing sequence of zero-sets of X with $\cap Z_n=\emptyset$ such that $Z_n\supset F_n$ for each $n\in N$. Thus, X is *cb* and that completes the proof.

COROLLARY 1.16. *A space has property (U) iff it has property $(U)^*$ and is nd.*

REMARK. By the result above of [8] or Proposition 1.15 we see that for *nd*-

spaces property $(U^{\aleph_0})^*$ or weak cb-property is equivalent to cb-property. However, in general, property $(U^{\aleph_0})^*$ or weak cb-property does not imply the other. Indeed, there exists a normal space which is not weak cb ([3], [13]), and every normal space has $(U^{\aleph_0})^*$. On the other hand, the Tychonoff plank is weak cb but not has property $(U^{\aleph_0})^*$.

§ 2. Property $(U)'$

DEFINITION 2.1. A space X is said to have *property* $(U)'$ if every locally finite collection $\mathfrak{F}$ of regular closed sets in X is uniformly locally finite.

Obviously by definition, every extremally disconnected space has property $(U)'$. By a modified proof of a theorem of Isiwata [5] that every Tychonoff M'-space is weak cb, we see also that every Tychonoff M'-space has property $(U)'$.

If $\mathfrak{F}$ is further assumed to be countable, similarly as Lemma 1.11 the above definition is shown to be equivalent to weak cb-property. Thus analogous to the result of [8] mentioned in § 1, we have the following theorem, which is an answer to the question of Morita in the introduction.

THEOREM 2.2. *A space has property* (U) *iff it has property* $(U)'$ *and is expandable.*

PROOF. Let $\mathfrak{F}=\{F_\alpha\}$ be a locally finite collection of subsets of an expandable space X with property (U). Then there is a locally finite collection $\{G_\alpha\}$ of open sets such that $F_\alpha\subset G_\alpha$ for each α. By property $(U)'$ $\{\mathrm{Cl}\, G_\alpha\}$ is uniformly locally finite, and so is also $\mathfrak{F}$. This completes the proof.

QUESTION. In Theorem 2.2 can property $(U)'$ be weakened to weak cb-property?

§ 3. Products and $U^{\aleph_0}$-embedding

Finally we shall give an application of Corollary 1.4.

THEOREM 3.1. *For a subset A of a space X the following are equivalent.*

(a) *A is $U^{\aleph_0}$-embedded in X.*

(b) *$A\times Y$ is $U^{\aleph_0}$-embedded in $X\times Y$ for any compact space Y.*

(c) *$A\times I$ is C_1-embedded in $X\times I$.*

PROOF. (a)→(b). Assume that A is $U^{\aleph_0}$-embedded in X and let $\mathcal{U}=\{U_n|n\in N\}$ be a countable increasing cozero-set cover of $A\times Y$. Let $p: X\times Y\to X$ be the projection. Then each $V_n=A-p(A\times Y-U_n)$ is a cozero-set of A, and

$$V_n\subset V_{n+1},\ n\in N;\ \bigcup_n V_n=A.$$

Hence by Corollary 1.4 there exists a normal open cover $\mathcal{W}$ of X such that $\mathcal{W}\cap A$ refines $\{V_n\}$. Then $\mathcal{W}'=\{W\times Y\mid W\in\mathcal{W}\}$ is a normal open cover of $X\times Y$, and $\mathcal{W}'\cap(A\times Y)$ refines $\mathcal{U}$. Thus, by Corollary 1.4 $A\times Y$ is $U^{\aleph_0}$-embedded in $X\times Y$, which shows (b).

(b)→(c). This follows from Lemma 1.5.

(c)→(a). Assume (c). Let $\mathcal{U}=\{U_n|n\in N\}$ be a countable increasing cozero-set cover of A. Then there exist a zero-set E_n and a cozero-set V_n such that

$$V_n\subset E_n\subset U_n,\ V_n\subset V_{n+1},\ E_n\subset E_{n+1},\ n\in N;\ \bigcup_n V_n=A.$$

Then $Z=\cup\{(A-U_n)\times\{1/n\}|n\in N\}$ is a zero-set of $A\times I$ and we have $Z\cap(X\times\{0\})=\emptyset$. By (c) there exist a zero-set Z' of $X\times I$ such that

$$Z\subset Z',\ Z'\cap(X\times\{0\})=\emptyset.$$

Let us put for each $n\in N$

$$G_n=\{x\in X|(x,1/n)\notin Z'\}.$$

Then G_n is a cozero-set of X and we have

$$\bigcup_n G_n=X,\ G_n\cap A\subset U_n,\ n\in N.$$

Thus, by Corollary 1.4 A is $U^{\aleph_0}$-embedded in X. This proves the theorem.

In [11], [14] it is proved that for a subset A of a space X A is $P^{\mathfrak{m}}$-embedded in X iff $A\times Y$ is C^*-embedded in $X\times Y$ for a compact Hausdorff space Y with weight $\mathfrak{m}$. If we replace $P^{\mathfrak{m}}$ and C^* by $U^{\mathfrak{m}}$ and C_1 respectively, Theorem 3.1 shows that the analogue is valid or not valid according as $\mathfrak{m}=\aleph_0$ or $\mathfrak{m}>\aleph_0$.

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