

WEAKLY NORMAL FILTERS AND LARGE CARDINALS

By

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0. Introduction.

In this paper, κ denotes an uncountable regular cardinal and λ a cardinal $\geq \kappa$. For any such pair, $P_\kappa\lambda$ is the set $\{x \subset \lambda : |x| < \kappa\}$.

An “ideal on $P_\kappa\lambda$ ” is always a “proper, nonprincipal, κ -complete, fine ideal on $P_\kappa\lambda$ ” unless specified. (An ideal I is fine if for all $\alpha < \lambda$, $\{x \in P_\kappa\lambda : \alpha \notin x\} \in I$.) For any ideal I , $I^+ = P(P_\kappa\lambda) - I$ and I^* is the filter dual to I .

DEFINITION. An ideal I as well as I^* are said to be *weakly normal* iff for every regressive function $f : P_\kappa\lambda \rightarrow \lambda$,

$$(\exists \gamma < \lambda)(\{x \in P_\kappa\lambda : f(x) < \gamma\} \in I^*).$$

The above definition is a translation of Kanamori’s “weak normality” for filters on κ in [5]. There is another weak normality presented by Mignone [10]. It is known that our notion is Mignone’s weak normality plus some saturation property and every $cf\lambda$ -saturated normal ideal on $P_\kappa\lambda$ has our weak normality.

κ is said to be λ -compact if there is a fine ultrafilter on $P_\kappa\lambda$. If κ is λ -compact, $P_\kappa\lambda$ carries many fine ultrafilters. Moreover every fine ultrafilter has a weakly normal fine ultrafilter which is Rudin-Keisler ordering below it. So, it may be a natural question whether κ is large if a weakly normal filter on $P_\kappa\lambda$ exists.

In §1, we consider a case where one can say κ is large.

Kunen-Paris [7] and Kunen [6] consider the possibility of $S(\kappa, \eta)$ holding for various κ, η where

$S(\kappa, \eta) \equiv$ There is a κ -complete η -saturated ideal on κ . §2 is devoted to an application of their methods to weakly normal ideals on $P_\kappa\lambda$.

Much of our notation is standard, and Jech [4] or [7] should be consulted.

§1. κ may be strong compact.

For a reader’s convenience, we give a proof of a lemma which appears

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in [2].

LEMMA 1.1. *If I is weakly normal, then there is no disjoint family of $cf\lambda$ -many sets in I^+ .*

PROOF. Suppose that $\{A_\alpha : \alpha < cf\lambda\} \subset I^+$ is a pairwise disjoint family. Let $\{\lambda_\alpha : \alpha < cf\lambda\}$ be a cofinal increasing sequence in λ . We can assume that $A_\alpha \subset \widehat{\{\lambda_\alpha\}}$ for all $\alpha < cf\lambda$. Define a regressive function $f : P_\kappa\lambda \rightarrow \lambda$ so that $f''A_\alpha = \{\lambda_\alpha\}$. Then $B = \{x \in P_\kappa\lambda : f(x) \leq \gamma\} \in I^*$ for some $\gamma < \lambda$. Now pick a $\lambda_\alpha > \gamma$. By the definition of f , $A_\alpha \subset f^{-1}(\{\lambda_\alpha\})$ and $f^{-1}(\{\lambda_\alpha\}) \cap B = \emptyset$. This contradicts to $A_\alpha \in I^+$. \square

If $P_\kappa\lambda$ carries a weakly normal ideal I and $cf\lambda \leq \kappa$, I is κ -saturated. So, κ is measurable in some inner model. In general, we can say at least κ is not small.

LEMMA 1.2. (See Matsubara [9]) *Let $\kappa = \delta^+$ and I an ideal on $P_\kappa\lambda$. If $\gamma \leq \lambda$, $cf\gamma > \delta$ and $X \in I^*$, then X can be decomposed into γ -many disjoint I -positive subsets.*

THEOREM 1.3. *If $P_\kappa\lambda$ carries a weakly normal ideal, then κ is weakly inaccessible.*

PROOF. There is no $cf\lambda$ -many disjoint sets in I^+ . Hence κ is weakly inaccessible by 1.2 if $cf\lambda \geq \kappa$. When $cf\lambda < \kappa$, I is $cf\lambda$ -saturated, and there is no λ -saturated ideal on $P_\kappa\lambda$ if κ is a successor cardinal. \square

If $cf\lambda$ is small, κ becomes very large. In fact we have a direct analogue of Proposition 3.8 in DiPrisco and Marek [3].

THEOREM 1.4. *Let $2^{<cf\lambda} < \kappa$. If there is a weakly normal ideal on $P_\kappa\lambda$, then κ is λ -compact.*

SKETCH OF PROOF. Assuming that every set $X \in I^+$ can be partitioned into two disjoint sets in I^+ , we construct a tree $T \subset I^+$ with fewer than κ -many branches such that $P_\kappa\lambda$ is the union of the intersections of each branches which are all sets in I .

So, there is a set $X \in I^*$ which is not a union of two disjoint sets in I^+ . Then $I|X = \{Y \subset P_\kappa\lambda : Y \cap X \in I\}$ is a prime ideal and $(I|X)^*$ is a κ -complete fine ultrafilter on $P_\kappa\lambda$. \square

COROLLARY 1.5. *Suppose that $cf\lambda = \omega$ and $P_\kappa\lambda$ carries a weakly normal ideal.*

Then κ is λ -compact.

Combining weakly compactness, the method of 1.4 yields following.

THEOREM 1.6. *If κ is weakly compact, $P_\kappa\lambda$ carries a κ -saturated ideal and $cf\lambda=\kappa$, then κ is λ -compact.*

PROOF. Assume that every $X \in I^+$ is a disjoint union of two sets in I^+ . A tree T with α th level T_α is defined as follows.

$$T_0 = \{P_\kappa\lambda\}.$$

For $X \in T_\alpha$, the immediate successors of X are two sets in I^+ such that X is their disjoint union.

For α limit, $T_\alpha = \{\bigcap S : S \text{ is an } \alpha\text{-branch}\} \cap I^+$.

Since I is κ -saturated, $|T_\alpha| < \kappa$ for each $\alpha < \kappa$. Also every $T_\alpha \neq \emptyset$ since κ is inaccessible and I is κ -complete.

By the tree property, there is a branch B through T . Let $B = \{A_\alpha : \alpha < \kappa\}$ and $A_\alpha = A_{\alpha+1} \cup A'_{\alpha+1}$. Then $\{A'_{\alpha+1} : \alpha < \kappa\}$ is an almost disjoint family which contradicts the κ -saturation of I . \square

We have shown that κ is λ -compact if there is a weakly normal ideal on $P_\kappa\lambda$ and one of (a) and (b) is satisfied.

(a) $2^{<cf\lambda} < \kappa$

(b) κ is weakly compact and $cf\lambda = \kappa$.

§2. κ may not be very large.

Next we examine the other situation and present some consistency results such that κ is not λ -compact although $P_\kappa\lambda$ bears weakly normal ideals. Theorem 1.4 and 1.6 impose limitations on κ and λ .

We need several lemmas.

LEMMA 2.1. *Let I be weakly normal. If $f : X \rightarrow \lambda$ is a regressive function and $X \in I^+$, then $\{x \in X : f(x) \leq \gamma\} \in I^+$ for some $\gamma < \lambda$.*

PROOF. We extend f to $g : P_\kappa\lambda \rightarrow \lambda$ which is also regressive and $g|X = f$. Then $Y = \{x \in P_\kappa\lambda : g(x) \leq \gamma\} \in I^*$ for some $\gamma < \lambda$ and $Z = X \cap Y \in I^+$. $g|Z = f|Z$. \square

This lemma says that our weak normality is stronger than Mignone's version of weak normality.

LEMMA 2.2. $\{x : cf(\sup x) < \kappa\} \in I^+$ for every weakly normal ideal.

PROOF. We only have to show that $\{x : \sup x \in x\} \in I$. Then we have $\{x : x \text{ is cofinal in } \sup x\} \in I^*$ and the lemma is proved.

Suppose that $\{x : \sup x \in x\} \in I^+$. By the previous lemma, there is a $\gamma < \lambda$ such that $\{x : \sup x \leq \gamma\} \in I^+$. Now $\{x : x \subset \gamma + 1\} \in I^+$ contradicting that I is fine. \square

LEMMA 2.3. Suppose that I is weakly normal and $\alpha < \min(cf\lambda, \kappa)$. Then $\{x : cf(\sup x) > \alpha\} \in I^*$.

PROOF. Assume that $\{x : cf(\sup x) \leq \alpha\} \in I^+$. Since $\alpha < \kappa$ and I is κ -complete, there is an $\alpha_0 \leq \alpha$ such that $A = \{x : cf(\sup x) = \alpha_0\} \in I^+$. $I \upharpoonright A$ is also weakly normal and $A \in (I \upharpoonright A)^*$.

Let $\{\beta_\xi^x \mid \xi < \alpha_0\}$ be a cofinal increasing sequence in $\sup x$ for each $x \in A$. $f_\xi : A \rightarrow \lambda$ can be defined so that $\beta_\xi^x < f_\xi(x) \in x$. Now, for every $\xi < \alpha_0$ we have $\gamma_\xi < \lambda$ such that $\{x \in A : f_\xi(x) \leq \gamma_\xi\} \in (I \upharpoonright A)^*$. Since $\alpha_0 \leq \alpha < cf\lambda$, $\gamma = \sup\{\gamma_\xi : \xi < \alpha_0\} < \lambda$. Then $A_\xi = \{x \in A : f_\xi(x) \leq \gamma\} \in (I \upharpoonright A)^*$ for each $\xi < \alpha_0$ and $B = \bigcap \{A_\xi : \xi < \alpha_0\} \in (I \upharpoonright A)^*$. But $\sup x \leq \gamma$ for every $x \in B$, which is a contradiction. \square

LEMMA 2.4. Let $\delta > \omega$. Suppose that I is a weakly normal ideal such that $\{x \in P_\kappa\lambda : cf(\sup x) \geq \delta\} \in I^*$ and P is a δ -c.c. forcing notion in V . Then I generates a weakly normal ideal J on $P_\kappa\lambda$ in the generic extension $V[G]$.

PROOF. In $V[G]$, J is defined by $X \in J^*$ iff $Y \subset X$ for some $Y \in I^*$. By our assumption and 2.2, $\delta < \kappa$. Hence J is an ideal on $P_\kappa\lambda$ extending I .

Let f be a regressive function on $P_\kappa\lambda$ in $V[G]$ and \underline{f} its name always denoting a regressive function on $P_\kappa\lambda$. In V , for each $x \in P_\kappa\lambda$, let $A_x = \{\alpha \in x : p \vdash \underline{f}(x) = \alpha \text{ for some } p \in P\}$.

Since P satisfies the δ chain condition, $|A_x| < \delta$. Hence $\{x : \sup A_x < \sup x\} \in I^*$ because $\{x : cf(\sup x) \geq \delta\} \in I^*$. Using the weak normality, we have a $\gamma < \lambda$ such that $B = \{x : \sup A_x \leq \gamma\} \in I^*$. For each $x \in B$, $1 \vdash \underline{f}(x) \leq \gamma$. Clearly every condition forces " $B \subset \{x : \underline{f}(x) \leq \gamma\}$ " and B is a member of J^* . \square

We only had to require $\{x : cf(\sup x) \geq \delta\} \in I^+$, since $I \upharpoonright X$ is also weakly normal for any $X \in I^+$.

We can now show that the inaccessibility of κ is necessary in Theorem 1.4.

THEOREM 2.5. It is consistent that there is a weakly normal ideal on $P_\kappa\lambda$ with $\omega < cf\lambda < \kappa$ and κ is not inaccessible.

PROOF. Assume that κ is inaccessible, $\omega < cf\lambda < \kappa$, and I is a weakly normal ideal on $P_\kappa\lambda$ in the ground model. We blow up 2^ω to κ by c.c.c. forcing. All the cardinals and cofinalities are preserved and I generates a weakly normal ideal in the generic extension by 2.3 and 2.4. \square

Two problems remain: Is it possible that $P_\kappa\lambda$ carries a weakly normal ideal and κ is inaccessible (weakly compact) and nevertheless κ is not λ -compact? Is it consistent that κ is inaccessible, not weakly compact and there is a weakly normal ideal on $P_\kappa\lambda$?

For this purpose We state $P_\kappa\lambda$ generalization of what Kunen and Paris [7] did for saturated ideals on κ that is measurable in the ground model.

Let G be Q -generic over V and U is a V - κ -complete V -weakly normal filter on $P(P_\kappa\lambda)\cap V$ which lies in $V[G]$ i.e., We have in $V[G]$

- (1) $U \subset P(P_\kappa\lambda)\cap V$ and $0 \notin U$
- (2) $X \in U \wedge X \subset Y \in P(P_\kappa\lambda)\cap V \rightarrow Y \in U$
- (3) $\{X_\alpha : \alpha < \delta\} \in V \cap P(U) \wedge \delta < \kappa \rightarrow \bigcap \{X_\alpha : \alpha < \delta\} \in U$
- (4) For every $\alpha < \lambda$, $\{x \in P_\kappa\lambda \cap V : \alpha \in x\} \in U$
- (5) For each $f \in V$ which is regressive on $P_\kappa\lambda \cap V$, there is a $\gamma < \lambda$ such that $\{x \in P_\kappa\lambda \cap V : f(x) \leq \gamma\} \in U$.

In addition, assume that Q satisfies the $cf\lambda$ -c.c. and \underline{U} is a name such that $\llbracket \underline{U} \text{ is a } V\text{-}\kappa\text{-complete } V\text{-weakly normal filter on } P(P_\kappa\lambda)\cap V \rrbracket^{B(Q)} = 1$. In V , let $F = \{X \subset P_\kappa\lambda : \llbracket X \in \underline{U} \rrbracket^{B(Q)} = 1\}$.

LEMMA 2.6. F is a weakly normal filter on $P_\kappa\lambda$.

PROOF. F is clearly a filter on $P_\kappa\lambda$. For its weak normality, let $f : P_\kappa\lambda \rightarrow \lambda$ regressive. Then, $\llbracket \{x \in P_\kappa\lambda : f(x) \leq \gamma\} \in U \text{ for some } \gamma < \lambda \rrbracket^{B(Q)} = 1$.

For $p \in Q$, $p(\gamma)$ is the ordinal such that p forces “ $p(\gamma)$ is the least ordinal γ so that $\{x \in P_\kappa\lambda : f(x) \leq \gamma\} \in \underline{U}$ ”.

Set $A = \{\gamma < \lambda : \gamma = p(\gamma) \text{ for some } p\}$.

Since Q is a $cf\lambda$ -c.c. notion, p and p' are incompatible if $p(\gamma) \neq p'(\gamma)$, $|A| < cf\lambda$. So, $\beta = \sup A < \lambda$ and $\llbracket \{x : f(x) \leq \beta\} \in \underline{U} \rrbracket^{B(Q)} = 1$. Hence $\{x \in P_\kappa\lambda : f(x) \leq \beta\} \in F$. \square

The situation of lemma is familiar in the large cardinal theory.

Suppose that $cf\lambda \geq \kappa$, $R = P * Q$, G is P -generic over V and H is Q -generic over $V[G]$, $\llbracket Q \text{ is } cf\lambda\text{-c.c.} \rrbracket^{B(Q)} = 1$, and there exists a $V[G]$ - κ -complete $V[G]$ -weakly normal filter on $P(P_\kappa\lambda)\cap V[G]$ in $V[G][H]$. Then we can find a weakly normal filter on $P_\kappa\lambda$ in $V[G]$.

Assume that $V \models GCH + \kappa$ is supercompact, and R is a usual backward Easton forcing adding α^{++} subsets to every inaccessible $\alpha \leq \kappa$. Let G' be R -generic over V . Then $V[G'] \models \kappa$ is supercompact + there is a normal ultrafilter on $P_\kappa \lambda$. $R = P * Q$ where $\llbracket Q \text{ adds } \kappa^{++} \text{ subsets to } \kappa \rrbracket^{B(P)} = 1$ and $V[G] = V[G_1][H]$, G_1 is P -generic and H is Q -generic over $V[G]$. Since $V[G_1] \models 2^\alpha = \kappa^+ + 2^\alpha = \alpha^{++}$ for any inaccessible $\alpha < \kappa$, κ is an inaccessible cardinal that is not measurable in $V[G_1]$. Moreover Q satisfies κ^+ -c.c. in $V[G_1]$. Hence we get a weakly normal filter on $P_\kappa \lambda$ in $V[G_1]$ if $cf \lambda \geq \kappa^+$.

However, it is not clear whether κ is weakly compact or not in this construction. So, we follow Kunen's argument in [6].

Again we start from the universe V_0 where κ is supercompact. Without loss of generality, supercompactness of κ is indestructible under κ -directed closed forcing (Ref. [8]). Let $V_1 = V_0[G]$, G is V_0 -generic over P adding a Cohen subset of κ . Since P is κ -directed closed, κ is supercompact in V_1 . $V_1 = V_0[H][K]$, H is Q -generic over V_0 , K is R -generic over $V_1[H]$, where Q is a notion of forcing which adds a κ -Suslin tree T , and R is T itself.

$$V_1 \models \exists U : V_0[H]\text{-weakly normal filter on } P(P_\kappa \lambda) \cap V_0[H]$$

and

$$V_0[H] \models Q \text{ is the } \kappa\text{-c.c.} + cf \lambda \geq \kappa$$

We have a weakly normal filter in $V_0[H]$ and κ is inaccessible but not weakly compact since there is a κ -Suslin tree. We have proved;

THEOREM 2.7. *Con ($\exists \kappa$ (κ is supercompact)) implies Con ($\exists \kappa$ (κ is not weakly compact + κ is inaccessible + $P_\kappa \lambda$ carries a weakly normal filter + $cf \lambda \geq \kappa$)).*

Next we try the case where κ is weakly compact but not measurable. It is impossible that $cf \lambda \leq \kappa$ by Theorem 1.4, 1.6. The following lemma is standard.

LEMMA 2.8. *Let $j : V \rightarrow M$ be a λ -supercompact embedding with the critical point κ , $P \subset V_\kappa$, $j(P) = P \oplus Q$, G be $j(P)$ -generic. If U is defined as;*

$$p \Vdash_{j(P)} \dot{X} \in \dot{U} \text{ iff for some } P\text{-term } \dot{X}' \ p \Vdash_P \dot{X} = \dot{X}' \subset P_\kappa \lambda$$

and

$$M \models (p \Vdash_{j(P)} j'' \lambda \in j(X')),$$

then $V[G] \models U$ is a $V[G_1]$ - κ -complete $V[G_1]$ -normal ultrafilter on $P_\kappa \lambda \cap V[G_1]$, where G_1 is V -generic on P . (Note that $j(p) = p$ for all $p \in P$.)

PROOF. U is trivially a proper fine filter. Let $\tau \in V[G_1]$ and $\tau(\alpha) \in U$ for all $\alpha < \delta < \kappa$. For some $p \in G$, $p \Vdash \forall \alpha < \delta (\tau'(\alpha) = \tau(\alpha) \in \dot{U})$. So, p forces in M that

$j''\lambda \in j(\tau'(\alpha))$. (τ' is a P -term that every condition forces that $\tau' = \bigcap_{\alpha < \delta} \tau(\alpha)$.) Then $j(P) \Vdash j(\tau') = \bigcap_{\alpha < \delta} j(\tau(\alpha))$. Hence $p \Vdash j''\lambda \in j(\tau')$ in M , which says that U is $V[G_1]$ - κ -complete.

To show that U is ultra, pick a P -terms \underline{X} and \underline{X}^c which always denotes X and $P_\kappa\lambda - X$ respectively for given $X \in P_\kappa\lambda \cap V[G_1]$. If X is not in U , then we can find a $p \in G$ such that $p \Vdash \neg j''\lambda \in j(\underline{X})$ since $\{p : p \text{ decides } "j''\lambda \in j(\underline{X})"\}$ is dense in $j(P)$. So, $p \Vdash j''\lambda \in j(\underline{X}^c)$ and $P_\kappa\lambda - X \in U$.

For the normality of U , let $f : P_\kappa\lambda \rightarrow \lambda$ be a regressive function in $V[G_1]$ and \underline{f} be its P -name. Then $j(P) \Vdash j(\underline{f})(j''\lambda) \in j''\lambda$. $V[G] \supset M[G]$ and $j(f)(j''\lambda) = j(\alpha)$ for some $\alpha < \lambda$ in $M[G]$. So, " $j(\underline{f})(j''\lambda) = j(\alpha)$ " is forced by some $p \in G$. In $V[G_1]$, let $X = \{x \in P_\kappa\lambda : f(x) = \alpha\}$ and \underline{X} be its P -name. $j(P) \Vdash j(\underline{X}) = \{x \in P_{j(\kappa)}j(\lambda) : j(\underline{f})(x) = j(\alpha)\}$. Hence p forces " $j''\lambda \in j(\underline{X})$ " and $X \in U$. \square

THEOREM 2.9. *If the existence of a supercompact cardinal is consistent, it is also consistent that κ is weakly compact, not measurable, and there is a weakly normal filter on $P_\kappa\lambda$ with $cf\lambda \geq \kappa^+$.*

PROOF. We follow the proof of Theorem 4.4 in [7]. Let P be a forcing notion adding one generic subset to each inaccessible cardinal less than κ . $j : V \rightarrow M$ is an elementary embedding with the critical point κ and M is closed under 2^λ -sequences. Note that $cf\lambda \geq \kappa^+$. We can further assume that $2^\kappa \geq \lambda$.

$j(P) = P \oplus Q \oplus R$, where Q adds a subset to κ and R treats inaccessibles between κ and $j(\kappa)$ in M . So, Q satisfies the κ^+ -c.c. and R is λ^+ -closed. Let G_1, G_2, G_3 be V -generic over P, Q, R respectively. By the lemma, we have a $V[G_1]$ - κ -complete $V[G_1]$ -normal ultrafilter on $P_\kappa\lambda \cap V[G_1]$ in $V[G_1 \times G_2 \times G_3]$.

$P(P_\kappa\lambda) \cap V[G_1 \times G_3] = P(P_\kappa\lambda) \cap V[G_1]$ since R is λ^+ -closed and $\lambda = (\lambda^{<\kappa})^V = (\lambda^{<\kappa})^{V[G_1]} = (\lambda^{<\kappa})^{V[G_1 \times G_3]}$.

Now, U is a $V[G_1 \times G_3]$ - κ -complete $V[G_1 \times G_3]$ -normal ultrafilter on $P(P_\kappa\lambda) \cap V[G_1 \times G_3]$ in $V[G_1 \times G_2 \times G_3]$.

Since Q is κ^+ -c.c., we can find a weakly normal filter on $P_\kappa\lambda$ in $V[G_1 \times G_2]$ using Lemma 2.6. It is known that κ is weakly compact but not measurable in $V[G_1 \times G_3]$. \square

Recent work of Abe and Matsubara show that weakly normal filters on $P_\kappa\lambda$ exist if there is a precipitous ideal I with no pairwise disjoint family of $cf\lambda$ -many sets in I^+ .

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