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3-DIMENSIONAL SUBMANIFOLDS OF SPHERES WITH PARALLEL MEAN CURVATURE VECTOR*

By

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Abstract. In this paper, for a 3-dimensional complete submanifold M with parallel mean curvature vector in $S^{3+p}(c)$, we give a pinching condition of the Ricci curvature under which M is a 3-dimensional small sphere.

1. Introduction

Let M be an n-dimensional complete submanifold immersed in a sphere $S^{n+p}(c)$. It is well-known that properties of M can be described by a pinching condition of some curvatures. When M is a minimal submanifold or a submanifold with parallel mean curvature vector, many authors studied the pinching problem with respect to the sectional curvature or the scalar curvature of M and a lot of beautiful results were obtained. It is natural to consider whether we can describe the properties of M by a pinching condition of the Ricci curvature. When M is minimal, Ejiri [2] and Shen [5] studied the pinching problem. Shun [6] researched compact submanifolds of a sphere with parallel mean curvature vector for $n>3$. He gave a pinching condition of the Ricci curvature under which M is totally umbilic.

In this paper, for $n=3$, we consider same problem. That is, we prove the following:

THEOREM 1. *Let M be a 3-dimensional complete submanifold of $S^{3+p}(c)(p\leqq 2)$ with parallel mean curvature vector $\boldsymbol{h}$. If*

$$Ric(M)\geqq\frac{3}{4}c+\frac{39}{64}H^2+\frac{1}{8}\sqrt{\frac{1521}{64}H^4+\frac{45}{2}H^2c},$$

then M is totally umbilic. Hence M is a 3-dimensional small sphere, where $Ric(M)$ and $H=|\boldsymbol{h}|$ denote the Ricci curvature and the norm of the mean cur-

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vature vector $\boldsymbol{h}$ *respectively.*

THEOREM 2. *Let* M *be a 3-dimensional complete submanifold with parallel mean curvature vector of* $S^{3+p}(c)(p>2)$. *If*

$$Ric(M)\geqq\delta\,,$$

then M *is totally umbilic. Hence* M *is a 3-dimensional small sphere, where*

$$\delta=Max\left\{\frac{3}{4}c+\frac{39}{64}H^2+\frac{1}{8}\sqrt{\frac{1521}{64}H^4+\frac{45}{2}H^2c},\ \frac{5p-p}{2(2p-3)}(c+H^2)\right\}.$$

2. Preliminaries.

Throughout this paper all manifolds are assumed to be smooth, connected without boundary. We discuss in smooth category.

Let M be a 3-dimensional submanifold of a $(3+p)$-dimensional sphere $S^{3+p}(c)$. We choose a local field of orthonormal frame $e_1, \cdots, e_{3+p}$ in $S^{3+p}(c)$ and the dual coframe $\omega_1, \cdots, \omega_{3+p}$ in such a way that e_1, e_2 and e_3 are tangent to M. In the sequel, the following convention on the range of indices is used, unless otherwised stated:

$$1\leqq A, B, \cdots\leqq 3+p\,;\qquad 1\leqq i, j, \cdots\leqq 3\,;$$

$$4\leqq\alpha, \beta, \cdots\leqq 3+p\,.$$

And we agree that the repeated indices under a summation sign without indication are summed over the respective range. The connection forms $\{\omega_{AB}\}$ of $S^{3+p}(c)$ are characterized by the structure equations

$$\text{(2.1)}\qquad \begin{cases} d\omega_A-\sum\omega_{AB}\wedge\omega_B=0\,, \qquad \omega_{AB}+\omega_{AB}=0\,,\\ d\omega_{AB}-\sum\omega_{AC}\wedge\omega_{CB}=\Omega_{AB}\,,\\ \Omega_{AB}=-\dfrac{1}{2}\sum R'_{ABCD}\omega_C\wedge\omega_D\,, \end{cases}$$

$$\text{(2.2)}\qquad R'_{BBCD}=c(\delta_{AC}\delta_{BD}-\delta_{AD}\delta_{BC})\,,$$

where Ω_{AB} (resp. R'_{ABCD}) denotes the Riemannian curvature form (resp. the components of the Riemannian curvature tensor) of $S^{3+p}(c)$. Therefore the components of Ricci curvature tensor Ric' and the scalar curvature r' are given as

$$R'_{AB}=c(n+p-1)\delta_{AB}\,,\qquad r'=(n+p)(n+p-1)c\,.$$

Restricting these forms to M, we have

$$\text{(2.3)}\qquad \omega_\alpha=0 \qquad \text{for } \alpha=4, \cdots, 3+p\,.$$

We see that e_1, e_2 and e_3 is a local field of orthonormal frames on M and ω_1, ω_2 and ω_3 is a local field of its dual coframes on M. It follows from (2.1), (2.3) and Cartan's Lemma that

$$\omega_{\alpha i}=\sum h_{ij}^{\alpha}\omega_j\,, \qquad h_{ij}^{\alpha}=h_{ji}^{\alpha}\,. \tag{2.4}$$

The second fundamental form α and the mean curvature vector $\boldsymbol{h}$ of M are defined by

$$\alpha=\sum h_{ij}^{\alpha}\omega_i\omega_j e_\alpha\,, \qquad \boldsymbol{h}=\frac{1}{3}\sum(\sum_i h_{ii}^{\alpha})e_\alpha\,. \tag{2.5}$$

The mean curvature H is given by

$$H=|\boldsymbol{h}|=\frac{1}{3}\sqrt{\sum(\sum_i h_{ii}^{\alpha})^2}\,. \tag{2.6}$$

Let $S=\sum(h_{ij}^{\alpha})^2$ denote the squared norm of the second fundamental form of M. The connection forms $\{\omega_{ij}\}$ of M are characterized by the structure equations

$$\begin{cases} d\omega_i-\sum\omega_{ij}\wedge\omega_j=0\,, \qquad \omega_{ij}+\omega_{ji}=0\,, \\ d\omega_{ij}-\sum\omega_{ik}\wedge\omega_{kj}=\Omega_{ij}\,, \\ \Omega_{ij}=-\dfrac{1}{2}\sum R_{ijkl}\omega_k\wedge\omega_l\,, \end{cases} \tag{2.7}$$

where Ω_{ij} (resp. R_{ijkl}) denotes the Riemannian curvature form (resp. the components of the Riemannian curvature tensor) of M. Therefore the Gauss equation is given by, from (2.1) and (2.7),

$$R_{ijkl}=c(\delta_{ik}\delta_{jl}-\delta_{il}\delta_{jk})+\sum(h_{ik}^{\alpha}h_{jl}^{\alpha}-h_{il}^{\alpha}h_{jk}^{\alpha})\,. \tag{2.8}$$

The components of the Ricci curvature Ric and the scalar curvature r are given by

$$R_{jk}=2c\delta_{jk}+\sum h_{ii}^{\alpha}h_{jk}^{\alpha}-\sum h_{ik}^{\alpha}h_{ij}^{\alpha}\,, \tag{2.9}$$

$$r=6c+9H^2-\sum(h_{ij}^{\alpha})^2\,. \tag{2.10}$$

We also have

$$d\omega_{\alpha\beta}-\sum\omega_{\alpha\gamma}\wedge\omega_{\gamma\beta}=-\frac{1}{2}\sum R_{\alpha\beta ij}\omega_i\wedge\omega_j\,,$$

where

$$R_{\alpha\beta ij}=\sum(h_{il}^{\alpha}h_{jl}^{\beta}-h_{jl}^{\alpha}h_{il}^{\beta})\,. \tag{2.11}$$

Define h_{ijk}^{α} and h_{ijkl}^{α} by

$$\sum h_{ijk}^{\alpha}\omega_k=dh_{ij}^{\alpha}+\sum h_{ik}^{\alpha}\omega_{kj}+\sum h_{jk}^{\alpha}\omega_{ki}-\sum h_{ij}^{\beta}\omega_{\alpha\beta}\,, \tag{2.12}$$

$$\sum h^\alpha_{ijkl}\omega_l = dh^\alpha_{ijk} + \sum h^\alpha_{ilk}\omega_{lj} + \sum h^\alpha_{ijl}\omega_{lk} + \sum h^\alpha_{ljk}\omega_{li} - \sum h^\beta_{ijk}\omega_{\alpha\beta} .$$

The Codazzi equation and the Ricci formula for the second fundamental form are given by

$$h^\alpha_{ijk} - h^\alpha_{ikj} = 0 ,$$

$$h^\alpha_{ijkl} - h^\alpha_{ijlk} = \sum h^\alpha_{im} R_{mjkl} + \sum h^\alpha_{mj} R_{mikl} + \sum h^\beta_{ij} R_{\beta\alpha kl} , \tag{2.13}$$

The Laplacian Δh^α_{ij} of the components h^α_{ij} of the second fundamental form α is given by

$$\Delta h^\alpha_{ij} = \sum_k h^\alpha_{ijkk} .$$

From (2.13) we get

$$\Delta h^\alpha_{ij} = \sum_k h^\alpha_{kkij} + \sum h^\alpha_{km} R_{mijk} + \sum h^\alpha_{mi} R_{mkjk} + \sum h^\beta_{ki} R_{\beta\alpha jk} . \tag{2.14}$$

In this paper, we assume that the mean curvature vector $\boldsymbol{h}$ of M is parallel. Hence the mean curvature H is constant. We choose e_4 such that $\boldsymbol{h} = He_4$, then

$$\sum_i h^4_{ii} = 3H , \qquad \sum_i h^\alpha_{ii} = 0 \qquad \text{for any } \alpha \neq 4 , \tag{2.15}$$

$$H_\alpha H_4 = H_4 H_\alpha \qquad \text{for any } \alpha , \tag{2.16}$$

where H_α denotes 3×3-matrix (h^α_{ij}). From (2.14), we have

$$\sum_{\alpha\neq 4} h^\alpha_{ij}\Delta h^\alpha_{ij} = \sum_{\alpha\neq 4} h^\alpha_{ij} h^\alpha_{km} R_{mijk} + \sum_{\alpha\neq 4} h^\alpha_{ij} h^\alpha_{mi} R_{mkjk} + \sum_{\alpha\neq 4} h^\alpha_{ij} h^\beta_{ki} R_{\beta\alpha jk} , \tag{2.17}$$

$$\sum h^4_{ij}\Delta h^4_{ij} = \sum h^4_{ij} h^4_{km} R_{mijk} + \sum h^4_{ij} h^4_{mi} R_{mkjk} . \tag{2.18}$$

Define $|\tau|^2 = \sum_{\alpha\neq 4} \operatorname{tr}(H_\alpha^2)$ and $|\sigma|^2 = \operatorname{tr}(H_4^2)$. Then $S = |\tau|^2 + |\sigma|^2$. A submanifold M is said to be *pseudo-umbilic* if it is umbilic with respect to the direction of the mean curvature vector $\boldsymbol{h}$, that is

$$h^4_{ij} = H\delta_{ij} .$$

3. Proofs of Theorems

In this section, we will give the proofs of Theorem 1 and Theorem 2. In order to prove Theorems, at first we give the following Propositions 1 and 2.

PROPOSITION 1. *Let M be a 3-dimensional complete pseudo-umbilical submanifold in $S^{3+p}(c)(p>1)$ with parallel mean curvature vector. If*

$$Ric(M) \geqq \frac{5p-9}{2(2p-3)}(c+H^2),$$

then M is a totally umbilical submanifold.

PROOF. Because of $Ric(M) \geqq [(5p-9)/2(2p-3)](c+H^2)>0$, we know that M is a compact submanifold from Myers' theorem (2.17) implies

(3.1) $$\frac{1}{2}\Delta|\tau|^2 = \sum_{\alpha\neq 4}(h_{ijk}^\alpha)^2 + \sum_{\alpha\neq 4} h_{ij}^\alpha \Delta h_{ij}^\alpha$$

$$= \sum_{\alpha\neq 4}(h_{ijk}^\alpha)^2 + \sum_{\alpha\neq 4}(h_{km}^\alpha R_{mijk} + h_{mi}^\alpha R_{mkjk})h_{ij}^\alpha + \sum_{\alpha,\beta\neq 4} h_{ij}^\alpha h_{ki}^\beta R_{\beta\alpha jk},$$

$$\sum_{\alpha\neq 4}(h_{km}^\alpha R_{mijk} + h_{mi}^\alpha R_{mkjk})h_{ij}^\alpha$$

$$= \sum_{\alpha,\beta\neq 4}\{\mathrm{tr}(H_\alpha H_\beta)^2 - \mathrm{tr}(H_\beta^2 H_\alpha^2)\} - \sum_{\alpha,\beta\neq 4}\{\mathrm{tr}(H_\alpha H_\beta)\}^2$$

$$+3c|\tau|^2 + 3H\sum_{\alpha\neq 4}\mathrm{tr}(H_\alpha H_4 H_\alpha) - \sum_{\alpha\neq 4}\{\mathrm{tr}(H_\alpha H_4)\}^2$$

$$+\sum_{\alpha\neq 4}\mathrm{tr}(H_\alpha H_4)^2 - \sum_{\alpha\neq 4}\mathrm{tr}(H_\alpha^2 H_4^2).$$

Since M is a pseudo-umbilical submanifold, we have $H_4 = HI$, where I is the identity matrix. Hence

$$\sum_{\alpha\neq 4}\mathrm{tr}(H_\alpha H_4)^2 - \sum_{\alpha\neq 4}\mathrm{tr}(H_\alpha^2 H_4^2) = 0,$$

$$\sum_{\alpha\neq 4}\mathrm{tr}(H_\alpha H_4 H_\alpha) = H|\tau|,$$

$$\sum_{\alpha\neq 4}\{\mathrm{tr}(H_\alpha H_4)\}^2 = 0 \qquad \text{(by (2.15))}.$$

Thus

(3.2) $$\sum_{\alpha\neq 4}(h_{km}^\alpha R_{mijk} + h_{mi}^\alpha R_{mkjk})h_{ij}^\alpha$$

$$= \sum_{\alpha,\beta\neq 4}\{\mathrm{tr}(H_\alpha H_\beta)^2 - \mathrm{tr}(H_\beta^2 H_\alpha^2)\} - \sum_{\alpha,\beta\neq 4}\{\mathrm{tr}(H_\alpha H_\beta)\}^2 + 3(c+H^2)|\tau|^2.$$

(3.3) $$\sum_{\alpha,\beta\neq 4} h_{ij}^\alpha h_{ki}^\beta R_{\beta\alpha jk} = \sum_{\alpha,\beta\neq 4}\{\mathrm{tr}(H_\alpha H_\beta)^2 - \mathrm{tr}(H_\beta^2 H_\alpha^2)\}.$$

According to (3.1), (3.2) and (3.3), we get

(3.4) $$\frac{1}{2}\Delta|\tau|^2 = \sum_{\alpha\neq 4}(h_{ijk}^\alpha)^2 - \sum_{\alpha,\beta\neq 4}\{\mathrm{tr}(H_\alpha H_\beta)\}^2$$

$$+3(c+H^2)|\tau|^2 + 2\sum_{\alpha,\beta\neq 4}\{\mathrm{tr}\{(H_\alpha H_\beta)^2 - \mathrm{tr}(H_\beta^2 H_\alpha^2)\}.$$

For a suitable choice of $e_5, \cdots, e_{3+p}$, we can assume $(p-1)\times(p-1)$ matrix $(\mathrm{tr}(H_\alpha H_\beta))$ is diagonal. Hence

$$\sum_{\alpha,\beta\neq 4}\{\operatorname{tr}(H_\alpha H_\beta)\}^2=\sum_{\alpha\neq 4}\{\operatorname{tr}(H_\alpha^2)\}^2 \tag{3.5}$$

From Lemma 1 in [1], we have

$$\begin{aligned}&2\{\operatorname{tr}(H_\alpha H_\beta)^2-\operatorname{tr}(H_\beta^2 H_\alpha^2)\}\\&=-\operatorname{tr}(H_\alpha H_\beta-H_\beta H_\alpha)^2\geqq -2\operatorname{tr}(H_\alpha^2)\operatorname{tr}(H_\beta^2),\end{aligned} \tag{3.6}$$

and equality holds for nonzero matrices H_α and H_β if and only if H_α and H_β can be transformed simultaneously by an orthogonal matrix into

$$H_\alpha^*=\lambda\begin{pmatrix}1&0&0\\0&-1&0\\0&0&0\end{pmatrix}\quad\text{and}\quad H_\beta^*=\mu\begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix}.$$

Moreover if $H_{\alpha_1},\cdots,H_{\alpha_s}$ satisfy

$$\operatorname{tr}(H_{\alpha_i}H_{\alpha_k}-H_{\alpha_k}H_{\alpha_i})^2+2\operatorname{tr}(H_{\alpha_i}^2)\operatorname{tr}(H_{\alpha_k}^2)\qquad\text{for } 1\leqq i,\ k\leqq s\,,$$

then at most two of the matrices H_α, are nonzero. Let

$$\begin{aligned}&(p-1)\sigma_1=|\tau|^2\,,\\&(p-1)(p-2)\sigma_2=2\sum_{\alpha<\beta,\,\alpha,\,\beta\neq 4}\operatorname{tr}(H_\alpha^2)\operatorname{tr}(H_\beta^2).\end{aligned} \tag{3.7}$$

Then

$$(p-1)^2(p-2)(\sigma_1^2-\sigma_2)=\sum_{\alpha<\beta,\,\alpha,\,\beta\neq 4}\{\operatorname{tr}(H_\alpha^2)-\operatorname{tr}(H_\beta^2)\}^2\,. \tag{3.8}$$

Hence we obtain

$$\begin{aligned}\frac{1}{2}\Delta|\tau|^2&\geqq\sum_{\alpha\neq 4}(h_{ijk}^\alpha)^2+3(c+H^2)|\tau|^2\\&\quad-2\{\sum_{\alpha\neq 4}\operatorname{tr}(H_\alpha^2)\}^2+\sum_{\alpha\neq 4}\{\operatorname{tr}(H_\alpha^2)\}^2\\&\geqq-\{2(p-1)-1\}(p-1)\sigma_1^2+(p-1)(p-2)(\sigma_1^2-\sigma_2)+3(c+H^2)|\tau|^2\\&\geqq-(p-1)(2p-3)\sigma_1^2+3(c+H^2)|\tau|^2\\&=-\Big(2-\frac{1}{p-1}\Big)|\tau|^4+3(c+H^2)|\tau|^2\,.\end{aligned} \tag{3.9}$$

On the other hand, for each fixed $\alpha\neq 4$, we can choose a local field of orthonormal frames e_1, e_2 and e_3 such that, from (2.16),

$$h_{ij}^4=H\delta_{ij}\quad\text{and}\quad h_{ij}^\alpha=\lambda_i^\alpha\delta_{ij}\,.$$

Since $\operatorname{tr} H_\alpha=\sum\lambda_i^\alpha=0$, we have

$$\sum_i (\lambda_i^\alpha)^4 = \frac{1}{2}\{\sum_i (\lambda_i^\alpha)^2\}^2,$$

that is,

$$\mathrm{tr}\, H_\alpha^4 = \frac{1}{2}\{\mathrm{tr}\, H_\alpha^2\}^2.$$

Hence

$$\sum_{\alpha\neq 4} \mathrm{tr}\, H_\alpha^4 = \frac{1}{2}\sum_{\alpha\neq 4}\{\mathrm{tr}\, H_\alpha^2\}^2. \tag{3.10}$$

For any $\alpha\neq 4$,

$$\begin{aligned} &2\sum_{\beta=5}^{3+p}\{\mathrm{tr}(H_\alpha^2 H_\beta^2)-\mathrm{tr}(H_\alpha H_\beta)^2\} \\ &=\{\sum_{\beta=5}^{3+p}\sum_{ij}(h_{ij}^\beta)^2(\lambda_i^\alpha-\lambda_j^\alpha)^2 \\ &\leqq 4\sum_{\beta\neq 4,\,\beta\neq\alpha}\sum_{ij}(h_{ij}^\beta)^2(\lambda_i^\alpha)^2. \end{aligned} \tag{3.11}$$

According to (2.9), we get

$$R_{ii}=2(c+H^2)-(\lambda_i^\alpha)^2-\sum_{\beta\neq 4,\,\beta\neq\alpha}\sum_j (h_{ij}^\beta)^2, \tag{3.12}$$

$$\begin{aligned} &\sum_{\beta\neq 4,\,\beta\neq\alpha}\sum_{ij}(h_{ij}^\beta)^2(\lambda_i^\alpha)^2 \\ &=2(c+H^2)\sum_i(\lambda_i^\alpha)^2-\sum_i(\lambda_i^\alpha)^4-\sum_i R_{ii}(\lambda_i^\alpha)^2 \\ &\leqq 2(c+H^2)\,\mathrm{tr}(H_\alpha^2)-\frac{1}{2}\{\mathrm{tr}(H_\alpha^2)\}^2-\delta_1\,\mathrm{tr}(H_\alpha^2), \end{aligned} \tag{3.13}$$

where δ_1 is the infimum of the Ricci curvature of M. Hence

$$\begin{aligned} &2\sum_{\beta=5}^{3+p}\{\mathrm{tr}(H_\alpha^2 H_\beta^2)-\mathrm{tr}(H_\alpha H_\beta)^2\} \\ &\leqq\{8(c+H^2)-4\delta_1\}\,\mathrm{tr}(H_\alpha^2)-2\{\mathrm{tr}(H_\alpha^2)\}^2. \end{aligned} \tag{3.14}$$

The terms at the both ends of the inequality above do not depend on the choice of the frame fields. Hence

$$\begin{aligned} &2\sum_{\alpha,\,\beta\neq 4}\{\mathrm{tr}(H_\alpha^2 H_\beta^2)-\mathrm{tr}(H_\alpha H_\beta)^2\} \\ &\leqq\{8(c+H^2)-4\delta_1\}\,|\tau|^2-2\sum_{\alpha\neq 4}\{\mathrm{tr}(H_\alpha^2)\}^2. \end{aligned} \tag{3.15}$$

(3.4), (3.5) and (3.15) yield

(3.16) $$\frac{1}{2}\Delta|\tau|^2 \geqq -\{8(c+H^2)-4\delta_1\}|\tau|^2+\sum_{\alpha\neq 4}\{\mathrm{tr}(H_\alpha^2)\}^2+3(c+H^2)|\tau|^2$$

$$\geqq\{-5(c+H^2)+4\delta_1\}|\tau|^2+\frac{1}{p-1}|\tau|^4 .$$

(3.9)×1/(2p−3)+(3.16) implies

(3.17) $$\frac{1}{2}\left[1+\frac{1}{2p-3}\right]\Delta|\tau|^2 \geqq \left\{4\delta_1-\left(5-\frac{3}{2p-3}\right)(c+H^2)\right\}|\tau|^2 .$$

Since δ_1 is the infimum of the Ricci curvature, we have

$$\delta_1 \geqq \frac{5p-9}{2(2p-3)}(c+H^2) .$$

If $\delta_1>((5p-3)/2(2p-3))(c+H^2)$, from (3.17) and Hopf's maximum principle, we obtain $|\tau|^2=0$. If $\delta_1=((5p-3)/2(2p-3))(c+H^2)$, (3.16) and Hopf's maximum principle yield $|\tau|^2$=constant and all inequalities above become actually equalites.

If $|\tau|^2=0$, then M is totally umbilic. If $|\tau|^2\neq 0$, from (3.6) and (3.9), we have

(3.18) $$h_{ijk}^\alpha=0 ,$$

(3.19) $$|\tau|^2=\frac{3}{2-\frac{1}{p-1}}(c+H^2) ,$$

$$\mathrm{tr}(H_\alpha H_\beta-H_\beta H_\alpha)^2=2\,\mathrm{tr}(H_\alpha^2)\,\mathrm{tr}(H_\beta^2) \qquad \text{for } \alpha\neq\beta ,$$

(3.20) $$(p-1)(p-2)(\sigma_1^2-\sigma_2)=0 .$$

From Lemma 1 in [1], we know that at most two of the matrices H_α are non-zero, say H_{α_1} and H_{β_1}, and we can suppose

$$H_{\alpha_1}=\lambda\begin{pmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 0\end{pmatrix} \quad \text{and} \quad H_{\beta_1}=\mu\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix} .$$

From (2.16), we have

(3.21) $$H_{\alpha_1}H_4=H_4H_{\alpha_1}, \qquad H_{\beta_1}H_4=H_4H_{\beta_1}, \qquad \mathrm{tr}\,H_4=3H .$$

Hence under this local field of orthonormal frames, we also have

$$h_{ij}^4=H\delta_{ij} .$$

a) Case $p=2$. (2.16) implies for a suitable choice of the orthonormal frame field

$$h_{ij}^4=H\delta_{ij} ,$$

$$h^{\alpha}_{ij}=\lambda^{\alpha}_{i}\delta_{ij}\,,$$

$$\sum_{i}\lambda^{\alpha}_{i}=0\,. \tag{3.22}$$

If $\lambda^{\alpha}_{i}\neq 0$, from (3.12) and (3.13), we have

$$R_{ii}=\delta_1=\frac{1}{2}(c+H^2) \qquad \text{for } i=1,\,2,\,3,$$

$$3\delta_1=\frac{3}{2}(c+H^2)=6(c+H^2)-|\tau|^2=3(c+H^2) \qquad (\text{from (3.19)})\,.$$

This is a contradiction. Hence at least one of λ^{α}_{i} is zero, say $\lambda^{\alpha}_{3}=0$. Thus $\lambda^{\alpha}_{1}=-\lambda^{\alpha}_{2}$ from (3.22).

$$|\tau|^2=(\lambda^{\alpha}_{1})^2+(\lambda^{\alpha}_{2})^2=3(c+H^2)\,,$$

$$(\lambda^{\alpha}_{1})^2=(\lambda^{\alpha}_{2})^2=\frac{3}{2}(c+H^2)\,,$$

$$R_{ii}=\frac{1}{2}(c+H^2)=\text{constant}>0, \qquad i=1,\,2,$$

$$R_{33}=2(c+H^2)=\text{constant}>0\,,$$

$$r=\sum_{i}R_{ii}=3(c+H^2)>0\,.$$

$$\sum_{ij}R^2_{ij}=\frac{9}{2}(c+H^2)^2=\text{constant}.$$

Hence $\nabla_k R_{ij}=0$. Thus M is a 3-dimensional conformally flat submanifold with positive definite Ricci curvature. From Theorem 2 due to Goldberg [3], we know that M is a space form. Hence M is totally umbilic. This is a contradiction.

b) Case $p\geqq 3$. In this cases, (3.20) implies

$$\sigma_1^2=\sigma_2\,.$$

We obtain that at most two of H_α, $\alpha=5,\cdots,3+p$, are different from zero. Suppose that only one of them, say H_{α_1}, is different from zero. Then we have $\sigma_1^2=(1/p-1)|\tau|^2$ and $\sigma_2=0$, which is a contradiction. Therefore we can suppose that

$$H_5=\lambda\begin{pmatrix}1&0&0\\0&-1&0\\0&0&0\end{pmatrix} \quad\text{and}\quad H_6=\mu\begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix},$$

$$H_\alpha=0 \qquad \text{for } \alpha\geqq 7\,.$$

In this case,

$$H_4=HI,\qquad \operatorname{tr} H_5^2=2\lambda^2,\qquad \operatorname{tr} H_6^2=2\mu^2\,,$$

(3.23) $$2\lambda^2+2\mu^2=|\tau|^2=3(c+H^2)\,.$$

(2.3) implies

$$\omega_{4i}=H\omega_i,\qquad \omega_{51}=\lambda\omega_1,\qquad \omega_{52}=-\lambda\omega_2,\qquad \omega_{53}=0,$$

$$\omega_{61}=\mu\omega_2,\qquad \omega_{62}=\mu\omega_1,\qquad \omega_{63}=0,\qquad \omega_{\alpha i}=0\qquad \text{for } \alpha=2,\cdots,3+p\,.$$

Since $h_{ijk}^\alpha=0$ from (3.9), we have, for $\alpha=5,\cdots,3+p$,

$$-dh_{ij}^\alpha=\sum h_{ik}^\alpha\omega_{kj}+\sum h_{kj}^\alpha\omega_{ki}+\sum h_{ij}^\beta\omega_{\beta\alpha}\,.$$

Setting $\beta=6$, $i=1$ and $j=2$, we have

$$d\mu=dh_{12}^6=0\,.$$

Hence μ is constant. Thus λ is also constant from (3.23).

$$R_{11}=R_{22}=2(c+H^2)-\lambda^2-\mu^2=\frac{1}{2}(c+H^2)=\text{constant}>0\,.$$

$$R_{33}=2(c+H^2)=\text{constant}>0\,.$$

Making use of the same proof as in case $p=2$, we obtain $|\tau|^2=0$. This is a contradiction. Thus we complete the proof of Proposition 1.

COROLLARY. *Let M be a 3-dimensional minimal submanifold in a sphere $S^{3+p}(c)$. If*

$$Ric(M)\geqq\frac{5p-4}{2(2p-1)}c\,,$$

then M is totally geodesic.

PROOF. Since M is a minimal submanifold in $S^{3+p}(c)$ and $S^{3+p}(c)$ is a totally umbilical hypersurface in $S^{3+p+1}(c-H^2)$, then M can be seen as a submanifold in $S^{3+p+1}(c-H^2)$. It is a pseudo-umbilical submanifold with parallel mean curvature vector $\boldsymbol{h}$. According to Proposition 1, we know that Corollary is true.

REMARK. The result in Corollary is better than one due to Shen [5].

PROPOSITION 2. *Let M be a 3-dimensional complete submanifold in $S^{3+p}(c)$ with parallel mean curvature vector. If*

$$Ric(M)\geqq\frac{3}{4}c+\frac{39}{64}H^2+\frac{1}{8}\sqrt{\frac{1521}{64}H^4+\frac{45}{2}H^2c}\,,$$

then M is a pseudo-umblical submanifold.

PROOF. Because of

$$Ric(M)\geqq \frac{3}{4}c+\frac{39}{64}H^2+\frac{1}{8}\sqrt{\frac{1521}{64}H^4+\frac{45}{2}H^2c}>0\,,$$

we conclude that M is compact from Myers' theorem. We choose a frame field in such a way that

$$h^4_{ij}=\lambda_i\delta_{ij}\,.$$

Let $\mu_i=\lambda_i-H$, we have

$$\sum_i \mu_i=0\,,\qquad \sum \mu_i^2=|\sigma|^2-3H^2\,,$$

$$\sum_i \mu_i^3=6H^3-3H|\sigma|^2+\sum \lambda_i^3\,,$$

$$-\frac{1}{\sqrt{6}}\sqrt{(|\sigma|^2-3H^2)^3}\leqq \sum \mu_i^3\leqq \frac{1}{\sqrt{6}}\sqrt{(|\sigma|^2-3H^2)^3}\,, \tag{3.26}$$

and equality holds if and only if two of μ_i are equal (cf. [4]). Because of

$$\begin{aligned} \sum_{\alpha\neq 4}\Big(\sum_i \lambda_i h^\alpha_{ii}\Big)^2 &= \sum_{\alpha\neq 4}\Big\{\sum_i(\lambda_i-H)h^\alpha_{ii}\Big\}^2 \\ &\leqq (|\sigma|^2-3H^2)|\tau|^2\,, \end{aligned} \tag{3.27}$$

from (2.18), (3.26) and (3.27), we obtain

$$\begin{aligned} \frac{1}{2}\Delta|\sigma|^2 &= \sum (h^4_{ijk})^2+\sum h^4_{ij}\Delta h^4_{ij} \\ &= \sum (h^4_{ijk})^2+\sum (h^4_{km}R_{mijk}+h^4_{mi}R_{mkjk})h^4_{ij} \\ &= \sum (h^4_{ijk})^2+3c(|\sigma|^2-3H^2)-|\sigma|^4+3H\sum \lambda_i^3-\sum_{\alpha\neq 4}\Big(\sum_i \lambda_i h^\alpha_{ii}\Big)^2 \\ &\geqq \sum (h^4_{ijk})^2+3c(|\sigma|^2-3H^2)+9H^2(|\sigma|^2-2H^2) \\ &\quad -\frac{3H}{\sqrt{6}}\sqrt{(|\sigma|^2-3H^2)^3}-(|\sigma|^2-3H^2)|\tau|^2-|\sigma|^4 \\ &= \sum (h^4_{ijk})^2+(|\sigma|^2-3H^2) \\ &\quad \times\Big\{3(c+H^2)-\frac{3H}{\sqrt{6}}\sqrt{(|\sigma|^2-3H^2)}-|\sigma|^2+3H^2-|\tau|^2\Big\}\,. \end{aligned} \tag{3.28}$$

On the other hand, since M is a 3-dimensional submanifold, its Weyl conformally curvature tensor vanishes, i.e.,

$$R_{ijkm}=R_{ik}\delta_{jm}-R_{im}\delta_{jk}+R_{jm}\delta_{ik}-R_{jk}\delta_{im}-\frac{1}{2}r(\delta_{ik}\delta_{jm}-\delta_{im}\delta_{jk})\,,$$

$$\begin{aligned} &\sum (h^4_{km}R_{mijk}+h^4_{mi}R_{mkjk})h^4_{ij} \\ &\quad =\frac{1}{2}\sum_{i\neq j}(\lambda_i-\lambda_j)^2R_{ijij} \end{aligned}$$

$$=\frac{1}{2}\sum_{i\neq j}(\lambda_i-\lambda_j)^2\left(R_{ii}+R_{jj}-\frac{r}{2}\right)$$

$$\geqq 3\left(2\delta_1-\frac{r}{2}\right)(|\sigma|^2-3H^2).$$

From (2.18), we have

$$\frac{1}{2}\Delta|\sigma|^2\geqq\sum(h^4_{ijk})^2+3\left(2\delta_1-\frac{r}{2}\right)(|\sigma|^2-3H^2) \tag{3.29}$$

$$\geqq 3\left(2\delta_1-3c-\frac{9}{2}H^2+\frac{1}{2}|\tau|^2+\frac{1}{2}|\sigma|^2\right)(|\sigma|^2-3H^2).$$

(3.28)×3/2+(3.29) implies

$$\frac{5}{2}\Delta|\sigma|^2\geqq\left\{6\delta_1-\frac{9}{2}(c+H^2)-\frac{3\sqrt{6}}{4}H\sqrt{(|\sigma|^2-3H^2)}\right\}(|\sigma|^2-3H^2).$$

Because of

$$3\delta_1\leqq\sum R_{ii}=r=6c+9H^2-|\sigma|^2-|\tau|^2,$$

we have

$$|\sigma|^2-3H^2\leqq 6c+6H^2-3\delta_1.$$

Hence

$$\frac{5}{4}\Delta(|\sigma|^2-3H^2)=\frac{5}{4}\Delta|\sigma|^2 \tag{3.30}$$

$$\geqq\left\{6\delta_1-\frac{9}{2}(c+H^2)-\frac{9\sqrt{2}}{4}H\sqrt{2(c+H^2)-\delta_1}\right\}(|\sigma|^2-3H^2).$$

By a straightforward calculation, we can easily verify that if

$$\delta_1>\frac{3}{4}c+\frac{39}{64}H^2+\frac{1}{8}\sqrt{\frac{1521}{64}H^4+\frac{45}{2}H^2c},$$

we have

$$\left\{6\delta_1-\frac{9}{2}(c+H^2)-\frac{9\sqrt{2}}{4}H\sqrt{2(c+H^2)-\delta_1}\right\}(|\sigma|^2-3H^2)>0.$$

According to (3.30) and Hopf's maximum principle, we conclude

$$|\sigma|^2-3H^2=0.$$

Hence, M is pseudo-umbilic. If

$$\delta_1=\frac{3}{4}c+\frac{39}{64}H^2+\frac{1}{8}\sqrt{\frac{1521}{64}H^4+\frac{45}{2}H^2c},$$

then,

$$\left\{6\delta_1-\frac{9}{2}(c+H^2)-\frac{9\sqrt{2}}{4}H\sqrt{2(c+H^2)-\delta_1}\right\}(|\sigma|^2-3H^2)=0.$$

Therefore from (3.30), we obtain that $|\sigma|^2-3H^2$=constant and all inequalities above are equalities. If $|\sigma|^2-3H^2=0$, then M is pseudo-umbilic. If $|\sigma|^2-3H^2>0$, we get that two of μ_i are equal. Without loss of generality, we can suppose $\mu_1=\mu_2$, then $\mu_3=-2h_1$. From (2.18), we have

$$0=\sum(h^4_{km}R_{mijk}+h^4_{mi}R_{mkjk})h^4_{ij}$$
$$=\frac{1}{2}\sum_{i\neq j}(\lambda_i-\lambda_j)^2\left(R_{ii}+R_{jj}-\frac{r}{2}\right)$$
$$=9\mu_1^2R_{33}.$$

Therefore $R_{33}=0$. On the other hand,

$$R_{33}\geqq\delta_1=\frac{3}{4}c+\frac{39}{64}H^2+\frac{1}{8}\sqrt{\frac{1521}{64}H^4+\frac{45}{2}H^2c}>0.$$

This is a contradiction. Hence M is pseudo-umbilic.

PROOF OF THEOREM 1. When $p=2$, $((5p-9)/2(2p-3))=1/2$. Hence

$$\frac{3}{4}c+H^2+\frac{1}{8}\sqrt{\frac{1521}{64}H^4+\frac{45}{2}H^2c}>\frac{1}{2}(c+H^2).$$

According to Propositions 1 and 2, we conclude easily that M is a 3-dimensional small sphere. When $p=1$, Proposition 1 implies that Theorem 1 is true.

PROOF OF THEOREM 2. According to Propositions 1 and 2, Theorem 2 holds good obviously.

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