

SELF-DUAL YANG-MILLS EQUATIONS AND TAUBES' THEOREM

By

Mitsuhiro ITOH

1. Introduction and statement of results.

Let M be a compact connected, oriented Riemannian 4-manifold with a metric g and G be a compact simply connected, simple Lie group. Let P be a G -principal bundle over M . The adjoint representation on its algebra \mathfrak{g} induces an associated vector bundle $\hat{g} = \hat{g}_P = P \times_{A_0 \mathfrak{g}}$, called the adjoint bundle of P .

A functional is defined over the set $\mathcal{E}(P)$ of all connections on P by $A \mapsto 1/2 \cdot \int |F(A)|^2 \sqrt{|g|} dx$, where $F(A) = dA + A \wedge A$ is curvature of A . A Yang-Mills connection which is a connection giving a critical point of this functional is a solution of the Yang-Mills equation $\delta_A F(A) = -{}^* \circ d_A \circ {}^* F(A) = 0$, that is, the Euler-Lagrange equation of the functional. A connection is said to be self-dual if $F(A)$ satisfies $*F = F$. From Bianchi's identity every self-dual connection gives automatically a Yang-Mills connection.

The functional takes the absolute minimum given by the first Pontrjagin number of bundle P when a connection is self-dual.

A connection A is said to be irreducible if the covariant derivative $\nabla_A; \Gamma(\hat{g}) \rightarrow \Gamma(\hat{g} \otimes A^1)$ has trivial kernel and a connection is reducible if it is not irreducible. A reducible connection reduces the structure group of P to the holonomy group by holonomy reduction theorem.

A differential operator $D_A; \Gamma(\hat{g} \otimes A^1) \rightarrow \Gamma(\hat{g} \otimes A^2)$ is defined by $D_A = P_- \circ d_A$, where $P_-; A^2 \rightarrow A^2$ is the orthogonal projection to the anti-self-dual part.

Since the base space is four dimensional and G satisfies $\pi_3(G) \cong \mathbf{Z}$, G -principal bundles P and P' over M are equivalent if and only if they have the same index $k \in \pi_3(G)$, given essentially by the first Pontrjagin number of the adjoint bundle \hat{g} . Index of an $SU(2)$ -principal bundle P is especially $-c_2(P \times_{\rho} \mathbf{C}^2)$, where ρ denotes the standard representation. For a G -principal bundle \bar{P} over S^4 of index 1 each smooth map $\phi; M \rightarrow S^4$ with degree k therefore induces a G -principal bundle $\phi^* \bar{P}$ of index k over M .

On S^4 with the standard metric an $SU(2)$ -principal bundle of index 1 carries an irreducible self-dual connection, written in explicit form and called Belavin-Polyakov-Schwartz-Tyupkin instanton solution ([1]).

The aim of this article is to establish existence theorem of a self-dual connection on a given $SU(2)$ -principal bundle over M .

THEOREM 1.1. Let P be an $SU(2)$ -principal bundle over M of index $k \geq 0$. If P admits a self-dual connection A satisfying $\text{Ker } D_A^* = 0$, then an arbitrary $SU(2)$ -principal bundle of index $k+1$ does also carry a self-dual connection which is irreducible and satisfies $\text{Ker } D_A^* = 0$.

We call a connection to be generic when it is irreducible and satisfies $\text{Ker } D_A^* = 0$.

Reversing the orientation of M , we obtain

COROLLARY 1.2. Let P be an $SU(2)$ -principal bundle over M of index $k \leq 0$. If P admits an anti-self-dual connection A satisfying $\text{Ker } D_A^* = 0$, then there exists also a generic anti-self-dual connection on an $SU(2)$ -principal bundle of index $k-1$.

The condition that $\text{Ker } D_A^* = 0$ for a flat (i.e., self-dual) connection on a product bundle reduces to a topological restriction on M that $H^2_-(M) = \{P_\theta; \theta \in H^2(M)\}$ vanishes. Thus the following is immediately obtained.

COROLLARY 1.3 (Taubes [11]). Let M be a compact connected, oriented Riemannian 4-manifold satisfying $H^2_-(M) = 0$ (or $H^2_+(M) = 0$). Then for all $k > 0$ (or $k < 0$) each $SU(2)$ -principal bundle of index k carries a generic self-dual connection (or anti-self-dual connection).

The condition $\text{Ker } D_A^* = 0$ for A on a bundle of index k is crucial, because $H^2_+(M)$ does not vanish for each compact Kähler surface with canonical orientation and a Kähler metric, and an $SU(2)$ -principal bundle of index -1 over a 2-dimensional complex projective space does not admit an anti-self-dual connection ([5], [7] and [10]).

The theorems can be applied to the case of anti-self-dual connections over a Kähler surface which are tightly related to the stability of holomorphic structures of a smooth vector bundle.

Over a compact Kähler surface (M, g) with canonical orientation each anti-self-dual connection A on a G -principal bundle induces a holomorphic structure J on any associated complex vector bundle E such that the $(0, 1)$ -part V_A'' of V_A coincides

with the $\bar{\partial}$ -operator with respect to J ([2], [5]). Because the connection A is Hermitian-Einstein in the sense of Kobayashi ([7]), this holomorphic structure is g -semistable in the sense of Mumford and Takemoto. Further when A is irreducible, (E, J) is g -stable.

Thus we can discuss the existence of irreducible anti-self-dual $SU(2)$ -connection and the stability of rank two holomorphic vector bundle of $c_1=0$.

COROLLARY 1.4. Let (M, g) be a compact Kähler surface. If a rank two holomorphic vector bundle of $c_1=0$ and $c_2=k (>0)$ carries a generic anti-self-dual connection, then every rank two smooth complex vector bundle of $c_1=0$ and $c_2>k$ does also, and hence admits a g -stable holomorphic structure.

NOTATION. Denote by A^p the vector bundle over M consisting of p -forms. Let A_+^2 and A_-^2 be the subbundles of A^2 , given by self-dual 2-forms and anti-self-dual 2-forms. We denote by $\Gamma(\hat{g} \otimes A^p)$ the space of smooth \hat{g} -valued p -forms over M . The metric g and the Killing form define an inner product on $\hat{g} \otimes A^p$, the L_2 -inner product and L_k -norms on $\Gamma(\hat{g} \otimes A^p)$ by

$$(1.1) \quad \|\Phi\|_{L_k} = \left(\int_M (\Phi, \Phi)^{k/2} \sqrt{|g|} dx \right)^{1/k}.$$

With respect to the L_2 -inner product the formal adjoint ∇_A^* and D_A^* are defined.

Let \mathcal{G}_P be the group of automorphisms of P which descend to the identity map of M . The quotient space {self-dual connections on P } / \mathcal{G}_P is called the moduli space \mathcal{M} of self-dual connections on P .

The remaining part of this article is devoted to verification of Theorem 1.1. We use for this in principle the idea of Taubes given in [11] except several parts. To make these parts complete we need quite different methods. Along the following course we show the theorem. For a given $SU(2)$ -principal bundle P of index $k (\geq 0)$ over M we construct a smooth map ϕ^λ with degree $k+1$ from M to S^4 , parametrized with λ and glue well a self-dual connection A on P and BPST-solution over S^4 to obtain bundle P^λ of index $k+1$ and also connection A^λ on P^λ , parametrized with $\lambda > 0$ (Definitions 4.1 and 4.2).

If we let λ be sufficiently small so that A^λ becomes "almost" self-dual, that is, the L_k -norms of the anti-self-dual part of $F(A^\lambda)$ are small (Proposition 4.3), and the first eigenvalue $\mu(A^\lambda)$ of the elliptic operator $D_A D_A^*$ has a positive lower bound (Proposition 4.4), then we can apply to A^λ Theorem 2.1, an existence theorem obtained by an iterated method due to Taubes. We state in section 2 reliably the quantities $\zeta(A)$ and $\delta(A)$ appeared in Theorem 2.1 which must be estimated to establish an

existence theorem.

To show $\mu(A') > 0$ we utilize the basic properties of the BPST-solution that the solution is of Hodge gauge and of exponential gauge in the sense of Uhlenbeck, and also it is rotational invariant (Proposition 3.1) together with Sturm's type comparison theorem related to first zero points of the ordinary differential equations associated to $\mathcal{F}_A^* \mathcal{F}_A \phi - \nu \phi = 0$ (Proposition 5.1). A lower bound estimation of $\mu(A')$ may cause difficulty in the case of general structure group G .

A self-dual connection A' on P^λ obtained by the above procedure must be irreducible when so is the given A . If A is not assumed irreducible, then we can not necessarily conclude that A' is irreducible. However the structure of the moduli space around a reducible self-dual connection can be precisely investigated (Lemmas 6.4 and 6.5). In fact the moduli space is a product of the subset of reducible connections which has a form of a $b_1(M)$ dimensional open ball and the subset of irreducible ones, written as a cone over a certain complex projective space $P_n(\mathbf{C})$. Therefore an $SU(2)$ -principal bundle of index $k+1$ admits a generic self-dual connection.

2. The self-dual equation.

Let $P \rightarrow M$ be a G -principal bundle over a compact connected, oriented Riemannian 4-manifold M with a compact simple Lie group G . Let A_0 be a fixed smooth connection on P . Since the set $\mathcal{E}(P)$ of all connections is an affine space, any connection A can be written uniquely as

$$(2.1) \quad A = A_0 + a$$

with $a \in \Gamma(\hat{\mathfrak{g}} \otimes A^1)$. If A has self-dual curvature, then

$$(2.2) \quad P_- F(A_0) + D_{A_0} a + a \# a = 0$$

where

$$(2.3) \quad a \# b = 1/2 P_-(a \wedge b + b \wedge a).$$

Conversely, if $a \in \Gamma(\hat{\mathfrak{g}} \otimes A^1)$ satisfies (2.2), then $A = A_0 + a$ is a self-dual connection. Thus in order to find a self-dual connection on P it suffices to obtain $A_0 \in \mathcal{E}(P)$ such that (2.2) has a solution.

Set $a = D_{A_0}^* u$ for $u \in \Gamma(\hat{\mathfrak{g}} \otimes A^2)$. Then (2.2) reduces to

$$(2.4) \quad D_{A_0} D_{A_0}^* u + D_{A_0}^* u \# D_{A_0}^* u = -P_- F(A_0).$$

This equation is properly elliptic, but non-linear.

For $A \in \mathcal{E}(P)$ we denote by $\mu(A)$ the first eigenvalue of $D_A D_A^*$. Define $\zeta(A)$

and $\delta(A)$ for A with $\mu(A) > 0$ by

$$(2.5) \quad \zeta(A) = \mu(A)^{-1/2} \{1 + \mu(A) + \|P_-F(A)\|_{L_4^2}\}^{1/2},$$

$$(2.6) \quad \delta(A) = \|P_-F(A)\|_{L_2} + \zeta(A) \|P_-F(A)\|_{L_{4/3}} \{1 + \|F(A)\|_{L_4} + \zeta(A) \|P_-F(A)\|_{L_4}\}.$$

THEOREM 2.1. There exists a constant $\varepsilon > 0$ which depends only on the Riemannian structure such that if $A_0 \in \mathcal{E}(P)$ satisfies $\delta(A_0)\zeta(A_0) < \varepsilon$, then there is a solution a in $\Gamma(\hat{g} \otimes A^1)$ to (2.2). Moreover there exists a constant $c > 0$, which is independent of $A_0 \in \mathcal{E}(P)$ and P such that

$$(2.7) \quad \|\nabla_{A_0} a\|_{L_2^2} + \|a\|_{L_2^2} \leq c^2 \delta(A_0)^2.$$

DEFINITION 2.2. For $u, v \in \Gamma(\hat{g} \otimes A^p)$, $p=1, 2$, we define

$$(2.8) \quad \begin{aligned} \langle u, v \rangle_H &= \langle u, v \rangle_{L_2} + \langle \nabla_{A_0} u, \nabla_{A_0} v \rangle_{L_2}, \\ \|u\|_H &= \langle u, u \rangle_H^{1/2}. \end{aligned}$$

The Hilbert spaces $\mathcal{H} = \mathcal{H}(A_0)$ and $\mathcal{K} = \mathcal{K}(A_0)$ are defined by the completions of $\Gamma(\hat{g} \otimes A^2)$ and $\Gamma(\hat{g} \otimes A^1)$ respectively, with respect to the norm $\|\cdot\|_H$.

For a fixed connection A_0 we consider the equation

$$(2.9) \quad D_{A_0} D_{A_0}^* u = q,$$

$q \in \Gamma(\hat{g} \otimes A^2)$. A solution u of (2.9) is formally a critical point of the functional

$$(2.10) \quad \begin{aligned} S_q[u] &= 1/4 \langle \nabla_{A_0} u, \nabla_{A_0} u \rangle_{L_2} + \sqrt{2}/4 \cdot \langle u, P_-F(A_0)(u) \rangle_{L_2} \\ &\quad + 1/6 \langle u, su \rangle_{L_2} - 1/2 \langle u, W_-(u) \rangle_{L_2} - \langle u, q \rangle_{L_2}, \end{aligned}$$

where s and W_- denote the scalar curvature and the anti-self-dual part of the Weyl conformal curvature W .

$S_q[u]$ is finite for $u \in \Gamma(\hat{g} \otimes A^2)$ and for such u we have

$$(2.11) \quad S_q[u] = 1/2 \langle D_{A_0}^* u, D_{A_0}^* u \rangle_{L_2} - \langle q, u \rangle_{L_2}.$$

Now we shall show the following.

LEMMA 2.3. There is a constant $z_1 > 0$ which depends only on the Riemannian structure of M with the following property; if $\mu(A_0) > 0$, then for all $u \in \mathcal{H}$ and $q \in L_{4/3}$,

$$(2.12) \quad S_q[u] \geq [z_1 \zeta(A_0)]^{-2} \|u\|_H^2 - [z_1 \zeta(A_0)]^2 \|q\|_{L_{4/3}}.$$

PROOF. It suffices to show (2.12) for $u \in \Gamma(\hat{g} \otimes A^2)$. From (2.10) and (2.11)

we obtain the estimates

$$(2.13) \quad S_q[\mathbf{u}] \geq \mu(A_0) \|\mathbf{u}\|_{L_2^2}^2 - \langle q, \mathbf{u} \rangle_{L_2},$$

$$(2.14) \quad S_q[\mathbf{u}] \geq 1/4 \|\mathcal{F}_{A_0} \mathbf{u}\|_{L_2^2}^2 - 8 \|P_- F(A_0)\|_{L_4} \|\mathbf{u}\|_{L_2} \|\mathbf{u}\|_{L_4} \\ - c(M) \|\mathbf{u}\|_{L_2^2}^2 - \langle q, \mathbf{u} \rangle_{L_2}.$$

In (2.14) we have used Hölder's inequality. The constant $c(M) > 0$ depends only on the Riemannian structure of M .

Since $\|\mathbf{u}\|_H^2 = \|\mathbf{u}\|_{L_2^2}^2 + \|\mathcal{F}_{A_0} \mathbf{u}\|_{L_2^2}^2$, we get for an arbitrary constant $c > 0$

$$(2.15) \quad S_q[\mathbf{u}] \geq 1/4 \|\mathbf{u}\|_H^2 - 1/4 \|\mathbf{u}\|_{L_2^2}^2 \\ - 8(c \|P_- F(A_0)\|_{L_4} \|\mathbf{u}\|_{L_2}^2 + 1/c \|\mathbf{u}\|_{L_4}^2) \\ - c(M) \|\mathbf{u}\|_{L_2^2}^2 - \langle q, \mathbf{u} \rangle_{L_2}.$$

By using Sobolev inequality

$$(2.16) \quad \|\mathbf{u}\|_{L_4} \leq z_2 \|\mathbf{u}\|_H$$

for a constant z_2 , which is independent of A_0 , we have

$$(2.17) \quad S_q[\mathbf{u}] \geq (1/4 - 8z_2/c) \|\mathbf{u}\|_H^2 \\ - (1/4 + c(M) + 8c \|P_- F(A_0)\|_{L_4}^2) \|\mathbf{u}\|_{L_2}^2 \\ - \langle q, \mathbf{u} \rangle_{L_2},$$

which induces the following for an appropriate value of c

$$(2.18) \quad S_q[\mathbf{u}] \geq 1/8 \|\mathbf{u}\|_H^2 - c(M) (1 + \|P_- F(A_0)\|_{L_4}^2) \|\mathbf{u}\|_{L_2}^2 \\ - \langle q, \mathbf{u} \rangle_{L_2}.$$

We apply (2.13) to (2.18) to obtain

$$(2.19) \quad \left\{ 1 + \frac{c(M)}{\mu(A_0)} (1 + \|P_- F(A_0)\|_{L_4}^2) \right\} \{S_q[\mathbf{u}] + \langle q, \mathbf{u} \rangle_{L_2}\} \geq \frac{1}{8} \|\mathbf{u}\|_H^2,$$

that is,

$$(2.20) \quad S_q[\mathbf{u}] + \langle q, \mathbf{u} \rangle_{L_2} \geq \left\{ 1 + \frac{c(M)}{\mu(A_0)} (1 + \|P_- F(A_0)\|_{L_4}^2) \right\}^{-1} \times \frac{1}{8} \|\mathbf{u}\|_H^2.$$

We have now the estimate

$$(2.21) \quad \langle q, \mathbf{u} \rangle_{L_2} \leq \alpha z_2^2 \|q\|_{L_4^{4/3}} + 1/\alpha \|\mathbf{u}\|_H^2,$$

for an arbitrary $\alpha > 0$, where we have used the Hölder's inequality and the Sobolev inequality. Then

$$(2.22) \quad S_q[\mathbf{u}] \geq \left\{ 1 + \frac{c(M)}{\mu(A_0)} (1 + \|P_- F(A_0)\|_{L_4}^2) \right\}^{-1} \cdot \frac{1}{8} \|\mathbf{u}\|_H^2 \\ - 1/\alpha \|\mathbf{u}\|_H^2 - \alpha z_2^2 \|q\|_{L_4^{4/3}}^2.$$

If we let a equal $16 \left\{ 1 + \frac{c(M)}{\mu(A_0)} (1 + \|P_- F(A_0)\|_{L_4^2}) \right\}$, then

$$(2.23) \quad S_q[u] \cong \frac{1}{16} \left\{ 1 + \frac{c(M)}{\mu(A_0)} (1 + \|P_- F(A_0)\|_{L_4^2}) \right\}^{-1} \|u\|_H^2 - 16z_2^2 \left\{ 1 + \frac{c(M)}{\mu(A_0)} (1 + \|P_- F(A_0)\|_{L_4^2}) \right\} \|q\|_{L_{4/3}^2},$$

from which (2.12) follows.

LEMMA 2.4. There is a constant $z_3 > 0$, which is independent of $A_0 \in \mathcal{E}(P)$ and P such that if $\mu(A_0) > 0$, then

$$(2.24) \quad \frac{1}{z_3 \zeta(A_0)} \|v\|_H \leq \|D_{A_0}^* v\|_{L_2} \leq z_3 \|v\|_H$$

for all $v \in \mathcal{H}$.

PROOF. We can use the estimate for $S_q[u]$ with $q=0$. From (2.12) we have $\|v\|_H / z_3 \zeta(A_0) \leq \|D_{A_0}^* v\|_{L_2}$. The inequality $\|D_{A_0}^* v\|_{L_2} \leq z_3 \|v\|_H$ follows from the definition of $D_{A_0}^*$ and the norm $\|\cdot\|_H$.

PROPOSITION 2.5. Let A_0 be a connection on P with $\mu(A_0) > 0$. Let $u \in \Gamma(\hat{g} \otimes \Lambda^2)$ be the unique solution to $D_{A_0} D_{A_0}^* u = q$ for $q \in \Gamma(\hat{g} \otimes \Lambda^2)$. Then

$$(2.25) \quad \|D_{A_0}^* u\|_{L_2} \leq z_4 \zeta(A_0) \|q\|_{L_{4/3}},$$

$$(2.26) \quad \|D_{A_0}^* u\|_H \leq z_4 (\|q\|_{L_2} + \zeta(A_0) \|q\|_{L_{4/3}} (1 + \|F(A_0)\|_{L_4} + \zeta(A_0) \|P_- F(A_0)\|_{L_4}))$$

and

$$(2.27) \quad \|D_{A_0}^* u\|_{L_4} \leq z_4 (\|q\|_{L_2} + \zeta(A_0) \|q\|_{L_{4/3}} (1 + \|F(A_0)\|_{L_4} + \zeta(A_0) \|P_- F(A_0)\|_{L_4})),$$

where z_4 depends only on the Riemannian structure.

PROOF. From the Sobolev inequality (2.27) immediately follows (2.26).

We show first (2.25). Since u satisfies $1/2 \|D_{A_0}^* u\|_{L_2}^2 = \langle q, u \rangle_{L_2}$, $\|D_{A_0}^* u\|_{L_2}^2$ is estimated by $\|D_{A_0}^* u\|_{L_2}^2 \leq 2 \|u\|_{L_4} \|q\|_{L_{4/3}} \leq 2z_2 \|u\|_H \|q\|_{L_{4/3}}$. From (2.24) we have

$$\|D_{A_0}^* u\|_{L_2}^2 \leq 2z_2 z_3 \zeta(A_0) \|D_{A_0}^* u\|_{L_2} \|q\|_{L_{4/3}}.$$

Hence we obtain (2.25).

The proof of (2.26) is as follows. Set $b = D_{A_0}^* u$. Then b satisfies

$$(2.28) \quad D_{A_0} b = q,$$

$$(2.29) \quad \nabla_{A_0}^* b = -*[u \wedge P_- F(A_0)].$$

Here we get (2.29) from the following; for all $\phi \in \Gamma(\hat{g})$

$$\langle \nabla_{A_0}^* b, \phi \rangle_{L_2} = \langle \nabla_{A_0}^* D_{A_0}^* u, \phi \rangle_{L_2} = \langle u, P_- d_{A_0} \nabla_{A_0} \phi \rangle_{L_2} = \langle u, [P_- F(A_0), \phi] \rangle_{L_2}.$$

The norm $\|b\|_{L_2} = \|D_{A_0}^* u\|_{L_2}$ is estimated by (2.25). Since $b \in \Gamma(\hat{g} \otimes A^1)$, we can apply to $\|\nabla_{A_0} b\|_{L_2}^2 = \langle \nabla_{A_0}^* \nabla_{A_0} b, b \rangle_{L_2}$ the Bochner-Weitzenböck formula, given by Bourguignon and Lawson [3], that is, for $a \in \Gamma(\hat{g} \otimes A^1)$

$$(2.30) \quad \begin{aligned} \nabla_{A_0}^* \nabla_{A_0} a &= 2D_{A_0}^* D_{A_0} a + \nabla_{A_0} \nabla_{A_0}^* a \\ &\quad + 2[P_+ F(A_0), a] - \mathcal{R}(a), \end{aligned}$$

where \mathcal{R} is an endomorphism of $\hat{g} \otimes A^1$ defined by the curvature tensor of the Riemannian structure. Then we have

$$(2.31) \quad \begin{aligned} \|\nabla_{A_0} b\|_{L_2}^2 &= 2\langle D_{A_0}^* D_{A_0} b, b \rangle_{L_2} + \langle \nabla_{A_0} \nabla_{A_0}^* b, b \rangle_{L_2} \\ &\quad + 2\langle [P_+ F(A_0), b], b \rangle_{L_2} - \langle \mathcal{R}(b), b \rangle_{L_2} \\ &= 2\|q\|_{L_2}^2 + \|[\mathcal{U} \wedge P_- F(A_0)]\|_{L_2}^2 \\ &\quad + 2\langle [P_+ F(A_0), b], b \rangle_{L_2} - \langle \mathcal{R}(b), b \rangle_{L_2}. \end{aligned}$$

By using the Hölder's inequality together with the Sobolev inequality, we get

$$(2.32) \quad \begin{aligned} \|\nabla_{A_0} b\|_{L_2}^2 &\leq 2\|q\|_{L_2}^2 + 8z_2 \|P_- F(A_0)\|_{L_4^2} \|\mathcal{U}\|_H^2 \\ &\quad + cz_2^2 \|b\|_H^2 + 64/c \|P_+ F(A_0)\|_{L_4^2} \|b\|_{L_2}^2 + 2\|\mathcal{R}\|_{L_\infty} \|b\|_{L_2}^2, \end{aligned}$$

for an arbitrary constant $c > 0$. Hence

$$(2.33) \quad \begin{aligned} \|b\|_H^2 &\leq 2\|q\|_{L_2}^2 + 8z_2 \|P_- F(A_0)\|_{L_4^2} \|\mathcal{U}\|_H^2 \\ &\quad + \|b\|_{L_2}^2 (1 + 2\|\mathcal{R}\|_{L_\infty} + 64/c \|P_+ F(A_0)\|_{L_4^2}^2) + cz_2^2 \|b\|_H^2, \end{aligned}$$

that is,

$$(2.34) \quad \begin{aligned} (1 - cz_2^2) \|b\|_H^2 &\leq 2\|q\|_{L_2}^2 + 8z_2 \|P_- F(A_0)\|_{L_4^2} \|\mathcal{U}\|_H^2 \\ &\quad + \|b\|_{L_2}^2 (1 + 2\|\mathcal{R}\|_{L_\infty} + 64/c \|P_+ F(A_0)\|_{L_4^2}^2). \end{aligned}$$

Since $\|\mathcal{U}\|_H / z_3 \zeta(A_0) \leq \|b\|_{L_2}$, we obtain

$$(2.35) \quad \begin{aligned} (1 - cz_2^2) \|b\|_H^2 &\leq 2\|q\|_{L_2}^2 + \{1 + 2\|\mathcal{R}\|_{L_\infty} + 64/c \|P_+ F(A_0)\|_{L_4^2}^2 \\ &\quad + 8z_2 z_3^2 \zeta(A_0)^2 \|P_- F(A_0)\|_{L_4^2}\} \|b\|_{L_2}^2. \end{aligned}$$

If we let $c = 1/2z_2^2$, then we derive from (2.25)

$$(2.36) \quad \begin{aligned} 1/2 \|b\|_H^2 &\leq 2\|q\|_{L_2}^2 + \{1 + 2\|\mathcal{R}\|_{L_\infty} + 128z_2^2 \|P_+ F(A_0)\|_{L_4^2}^2 \\ &\quad + 8z_2 z_3^2 \zeta(A_0)^2 \|P_- F(A_0)\|_{L_4^2}\} z_3^2 \zeta(A_0)^2 \|q\|_{L_4^2}^2. \end{aligned}$$

Therefore there is a constant $z_6 > 0$, which depends only on the Riemannian structure such that

$$(2.37) \quad \|b\|_H \leq z_6 \{\|q\|_{L_2} + \zeta(A_0) \|q\|_{L_4} + \|P_+ F(A_0)\|_{L_4} + \zeta(A_0) \|P_- F(A_0)\|_{L_4}\}$$

from which (2.26) follows, since $\|P_+ F(A_0)\|_{L_4} \leq \|F(A_0)\|_{L_4}$.

In the following we need the Hölder's inequality

$$(2.38) \quad \|q\|_{L_4/3} \leq \|q\|_{L_1}^{1/2} \cdot \|q\|_{L_2}^{1/2}.$$

From Proposition 2.5 we have

$$(2.39) \quad \|D_{A_0}^* u\|_{L_2} \leq z_4 \zeta(A_0) \|q\|_{L_4/3} \leq z_4 \delta(A_0) / \psi(A_0),$$

$$(2.40) \quad \max \{ \|D_{A_0}^* u\|_{L_4}, \|D_0^* u\|_H \} \leq z_4 \delta(q),$$

where

$$(2.41) \quad \begin{aligned} \psi(A_0) &= 1 + \|F(A_0)\|_{L_4} + \zeta(A_0) \|P_- F(A_0)\|_{L_4}, \\ \delta(q) &= \delta(q, A_0) = \|q\|_{L_2} + \zeta(A_0) \|q\|_{L_4/3} \cdot \psi(A_0). \end{aligned}$$

Now consider a sequence of solutions $\{u_k\}$ to the linear equations $D_{A_0} D_{A_0}^* u_k = q_k$ for a given sequence $\{q_k\}$ in $I'(\hat{g} \otimes A^2)$. Define q_k and u_k inductively by

$$(2.42) \quad q_1 = -P_- F(A_0),$$

and for $k \geq 2$

$$(2.43) \quad \begin{aligned} q_k &= -2 \sum_{j=1}^{k-2} D_{A_0}^* u_j \# D_{A_0}^* u_{k-1} \\ &\quad - D_{A_0}^* u_{k-1} \# D_{A_0}^* u_{k-1}. \end{aligned}$$

If all u_k exist, then the partial sum $s_m = \sum_{k=1}^m u_k$ satisfies

$$(2.44) \quad D_{A_0} D_{A_0}^* s_m + D_{A_0}^* s_{m-1} \# D_{A_0}^* s_{m-1} = -P_- F(A_0).$$

PROPOSITION 2.6. Let A_0 be a connection on P satisfying $32 z_4^2 \delta(A_0) \zeta(A_0) < 1$. Then each u_k and q_k exist and are smooth, and moreover satisfy

$$(2.45) \quad \|D_{A_0}^* u_k\|_{L_2} \leq \frac{1}{16z_4} (16z_4^2 \delta(A_0))^k \zeta(A_0)^{k-1} / \psi(A_0),$$

$$(2.46) \quad \begin{aligned} &\max \{ \|D_{A_0}^* u_k\|_{L_4}, \|D_{A_0}^* u_k\|_H \} \\ &\leq \frac{1}{16z_4} (16z_4^2 \delta(A_0))^k \zeta(A_0)^{k-1}. \end{aligned}$$

PROOF. The proposition is verified inductively on k .

By the definition of q_k

$$(2.47) \quad \|q_k\|_{L_1} \leq 4 \left(\sum_{j=1}^{k-1} \|D_{A_0}^* u_j\|_{L_2} \right) \|D_{A_0}^* u_{k-1}\|_{L_2}$$

and

$$(2.48) \quad \|q_k\|_{L_2} \leq 4 \left(\sum_{j=1}^{k-1} \|D_{A_0}^* u_j\|_{L_4} \right) \|D_{A_0}^* u_{k-1}\|_{L_4}.$$

The inequalities (2.45) and (2.46) for $k=1$ are just (2.39) and (2.40). By induction on $j < k$, we have

$$\begin{aligned} \|q_k\|_{L_1} &\leq 4 \left(\frac{1}{16z_4} \right)^2 (16z_4^2 \delta)^{k-1} \zeta^{k-2} \cdot \psi^{-1} \sum_{j=1}^{k-1} \psi^{-1} (16z_4^2 \delta)^j \zeta^{j-1} \\ &= 4 \left(\frac{1}{16z_4} \right)^2 \psi^{-2} (16z_4^2 \delta)^k \zeta^{k-2} \{1 + 16z_4^2 \delta \zeta + \cdots + (16z_4^2 \delta)^{k-2} \zeta^{k-2}\}. \end{aligned}$$

Since $16z_4^2 \delta \zeta < 1/2$, which is the hypothesis of the proposition, we have $1 + 16z_4^2 \delta \zeta + \cdots + (16z_4^2 \delta \zeta)^{k-2} < 2$. Thus

$$(2.50) \quad \|q_k\|_{L_1} \leq 8 \left(\frac{1}{16z_4} \right)^2 \psi^{-2} (16z_4^2 \delta)^k \zeta^{k-2}.$$

Similarly we obtain

$$(2.51) \quad \|q_k\|_{L_2} \leq 8 \left(\frac{1}{16z_4} \right)^2 (16z_4^2 \delta)^k \zeta^{k-2}.$$

Hence from (2.38) $\|q_k\|_{L_{4/3}}$ is estimated by

$$(2.52) \quad \|q_k\|_{L_{4/3}} \leq 8 \left(\frac{1}{16z_4} \right)^2 \psi^{-1} (16z_4^2 \delta)^k \zeta^{k-2}.$$

Since $\delta(q_k)$ is given by $\delta(q_k) = \|q_k\|_{L_2} + \zeta \cdot \|q_k\|_{L_{4/3}} \cdot \psi$, $\delta(q_k)$ is estimated by

$$(2.53) \quad \delta(q_k) \leq 16 \left(\frac{1}{16z_4} \right)^2 (16z_4^2 \delta)^k \zeta^{k-1}.$$

From (2.39) we get

$$\begin{aligned} \|D_{A_0}^* \mathbf{u}_k\| &\leq z_4 \delta(q_k) \psi^{-1} \leq 16z_4 \left(\frac{1}{16z_4} \right)^2 (16z_4^2 \delta)^k \zeta^{k-1} \psi^{-1} \\ &\leq \frac{1}{16z_4} (16z_4^2 \delta)^k \zeta^{k-1} \psi^{-1}, \end{aligned}$$

which is just (2.45). The inequality (2.46) is also obtained in the similar manner.

PROPOSITION 2.7. Let A_0 be a connection on P satisfying $\mu(A_0) > 0$ and $32z_4^2 \delta(A_0) \zeta(A_0) < 1$. Then $\{s_m\}$ converges to \mathbf{u} in \mathcal{H} , and $\{D_{A_0}^* s_m\}$ also converges to \mathbf{a} in \mathcal{K} which satisfies

$$(2.54) \quad D_{A_0}^* \mathbf{u} = \mathbf{a}$$

and

$$(2.55) \quad \|\mathbf{a}\|_{\mathcal{H}} \leq 2z_4 \delta(A_0).$$

PROOF. We show that $\{s_m\}$ and $\{D_{A_0}^* s_m\}$ are Cauchy. For $n \geq m \geq N$ we obtain from Lemma 2.4 together with (2.45)

$$\|s_n - s_m\|_{\mathcal{H}} \leq z_4 \zeta(A_0) \sum_{k=m+1}^n \|D_{A_0}^* \mathbf{u}_k\|_{L_2}$$

$$\begin{aligned}
&\leq \frac{z_3 \zeta(A_0)}{16z_4 \psi(A_0)} \sum \{16z_4^2 \delta(A_0)\}^k \zeta(A_0)^{k-1} \\
&\leq \frac{z_3}{16z_4 \psi(A_0)} \sum (16z_4^2 \delta \zeta)^k \\
&\leq \frac{z_3}{16z_4 \psi(A_0)} \cdot 2^{-N},
\end{aligned}$$

hence

$$(2.56) \quad \|s_n - s_m\|_H \leq \frac{z_3}{16z_4} \cdot 2^{-N},$$

since $\psi(A_0) \geq 1$.

Similarly we get from (2.46)

$$(2.57) \quad \|D_{A_0}^* s_n - D_{A_0}^* s_m\|_H \leq \sum_{k=m+1}^n \|D_{A_0}^* u_k\|_H \leq \frac{1}{16z_4} \cdot 2^{-N}.$$

(2.54) is a standard result.

To show (2.55) we must estimate $\|\sum_{k=1}^{\infty} D_{A_0}^* u_k\|_H$.

By (2.46) $\|\sum D_{A_0}^* u_k\|$ is estimated by

$$\begin{aligned}
\|\sum D_{A_0}^* u_k\|_H &\leq \sum \|D_{A_0}^* u_k\|_H \leq \frac{1}{16z_4} \sum (16z_4^2 \delta)^k \zeta^{k-1} \\
&= z_4 \delta \sum (16z_4^2 \delta \zeta)^k \leq 2z_4 \delta.
\end{aligned}$$

LEMMA 2.8. The sequence $\{v_m\}$ given by

$$(2.58) \quad v_m = D_{A_0} D_{A_0}^* s_m + D_{A_0}^* s_m \# D_{A_0}^* s_m + P_- F(A_0)$$

converges to zero in L_2 .

PROOF. Let $n \geq m \geq N$. Since $\#$ is symmetric,

$$\|v_n - v_m\|_{L_2} \leq 8 \|D_{A_0}^*(s_n - s_m)\|_H + \|D_{A_0}^*(s_n - s_m) \# D_{A_0}^*(s_n + s_m)\|_{L_2},$$

where we used the fact $\|D_{A_0} b\|_{L_2} \leq 8 \|b\|_H$ for each $b \in \Gamma(\hat{g} \otimes A^1)$. By the Hölder's inequality and Proposition 2.6 we see easily that $\{v_m\}$ is Cauchy and converges from (2.44) to zero in the sense of L_2 -norm.

PROOF OF THEOREM 2.1. Since v_m converges to zero in L_2 , the limit $u = \lim s_m$ is a weak solution to (2.4), hence u satisfies

$$(2.59) \quad \langle D_{A_0} D_{A_0}^* u + D_{A_0}^* u \# D_{A_0}^* u + P_- F(A_0), v \rangle_{L_2} = 0$$

for all $v \in \Gamma(\hat{g} \otimes A^2)$. Since A_0 is smooth, it is claimed from a regularity theorem of elliptic equations that u belongs to $\Gamma(\hat{g} \otimes A^2)$.

3. Self-dual connections associated to BPST-solutions.

In this section we give precise definition of BPST-solutions and deal with some properties of them.

The 4-space \mathbf{R}^4 canonically has the structure of quaternion numbers $\mathbf{H} = \{x = x^1 + x^2i + x^3j + x^4k\}$. We identify $G = SU(2)$ with $S\mathfrak{p}(1) = \{x \in \mathbf{H}; |x| = 1\}$ and its algebra $\mathfrak{su}(2)$ with the subspace of purely imaginary numbers by the aid of the cross product.

We define an $SU(2)$ -connection W over \mathbf{R}^4 , called a Belavin-Polyakov-Schwartz-Tyupkin (BPST)-solution by

$$(3.1) \quad W(x) = \text{Im} \left\{ \frac{x d\bar{x}}{1 + |x|^2} \right\}, \quad x \in \mathbf{H} = \mathbf{R}^4.$$

The curvature form F_W is then given by

$$(3.2) \quad F_W(x) = \frac{dx \wedge d\bar{x}}{(1 + |x|^2)^2}.$$

Since $dx \wedge d\bar{x}$ is an $\mathfrak{su}(2)$ -valued self-dual 2-form with respect to the standard metric of \mathbf{R}^4 , W defines a self-dual connection. From the form of F_W it is easily seen that W is irreducible.

By simple computations we obtain the following

PROPOSITION 3.1. The BPST-solution $W = \sum W_\mu(x) dx^\mu$ satisfies the following properties

$$(1) \quad W(\partial/\partial r) = 0 \quad \text{at } x \neq 0,$$

here $r(x) = |x|$,

$$(2) \quad \sum_\mu \partial/\partial x^\mu W_\mu = 0,$$

$$(3) \quad L_\alpha^* W = W \quad \text{for all } \alpha \in SU(2),$$

here L_α is an \mathbf{R} -linear mapping from \mathbf{H} to \mathbf{H} given by $L_\alpha(x) = \alpha x$ and

$$(4) \quad \text{for any mapping } \phi; \mathbf{H} \longrightarrow \mathfrak{su}(2)$$

$$(3.3) \quad \sum_\mu [W_\mu, [W_\mu, \phi]] = -2 \frac{|x|^2}{(1 + |x|^2)^2} \phi.$$

The BPST-solution is a rotation-invariant connection such that the fixed gauge is a Hodge gauge which is moreover exponential ([12]).

Let p denote the north pole of a 4-sphere S^4 and \bar{p} the south pole. Open

subsets $U_1 = S^4 \setminus \bar{p}$ and $U_2 = S^4 \setminus p$ give a trivializing covering for any bundle over S^4 . Then a bundle over S^4 is determined by its transition function $h; U_1 \cap U_2 \rightarrow G$. Let $s; \mathbf{R}^4 \rightarrow U_1$ and $\bar{s}; \mathbf{R}^4 \rightarrow U_2$ respectively be the stereographic projections from \bar{p} and p . The maps s and \bar{s} also define local coordinates of S^4 .

Define a transition function h over $U_1 \cap U_2$ by $h(x) = x/|x|$. Then we have an $SU(2)$ -principal bundle \bar{P} of index 1 and a connection on \bar{P} satisfying the cocycle condition

$$(3.4) \quad h(x)^{-1} \cdot dh(x) + h(x)^{-1} \cdot W^1(x) \cdot h(x) = W^2(y)$$

where W^1 and W^2 are the same BPST-solution over $\mathbf{R}^4 = \mathbf{H}$, and $y = (\bar{s}^{-1} \circ s)(x) = x^{-1}$ ($x \in \mathbf{R}^4 \setminus 0$) ([1]).

If we put directly $y = x^{-1}$, $y \in \mathbf{R}^4 \setminus 0$ in the right hand side of the above, then we get

$$(3.5) \quad ((\bar{s}^{-1} \circ s)^* W)(x) = -\text{Im} \left\{ \frac{dx \cdot x^{-1}}{1 + |x|^2} \right\}$$

which describes the BPST-solution in a singular form.

For $\lambda > 0$ the scale transformation $\lambda; \mathbf{R}^4 \rightarrow \mathbf{R}^4$ is defined by $\lambda(x) = x/\lambda$. Then it is easily seen that self-dual connections are mapped by the scale transformation into self-dual connections.

PROPOSITION 3.2. Let $\lambda > 0$. Then the following hold

$$(3.6) \quad (\lambda^* W)(x) = \text{Im} \left\{ \frac{x d\bar{x}}{\lambda^2 + |x|^2} \right\},$$

$$(3.7) \quad (\lambda^* F_W)(x) = \frac{\lambda^2 dx \wedge d\bar{x}}{(\lambda^2 + |x|^2)^2},$$

and

$$(3.8) \quad (\lambda^*(\bar{s}^{-1} \circ s)^* W)(x) = -\text{Im} \left\{ \frac{\lambda^2 dx \cdot x^{-1}}{\lambda^2 + |x|^2} \right\},$$

$x \in \mathbf{R}^4 \setminus 0$.

These are shown by straight computation.

4. The existence of generical self-dual connections.

Let M be a compact connected, oriented Riemannian 4-manifold. Let $(V_1, \phi_1), \dots, (V_k, \phi_k)$ be disjoint local coordinates of M such that each $\phi_i; V_i \rightarrow \mathbf{R}^4$ can be extended to a smooth map to S^4 . Then $\{V_0, V_1, \dots, V_k\}$ gives an open

covering of M where $V_0 = M \setminus \{m_1, \dots, m_k\}$ ($m_i = \phi_i^{-1}(0)$, $1 \leq i \leq k$). Thus we have an onto mapping $\phi; M \rightarrow S^1$ satisfying $\phi|_{V_i} = \phi_i$, $1 \leq i \leq k$ and ϕ maps $M \setminus \bigcup_{i=1}^k V_i$ into the north pole p . Since $\deg(\phi) = k$, the pulled back $\phi^* \bar{P}$ of the $SU(2)$ -principal bundle \bar{P} over S^1 of index 1 defines an $SU(2)$ -principal bundle of index k .

The transition function g_{0i} of $\phi^* \bar{P}$ over $V_0 \cap V_i$, $1 \leq i \leq k$ is given by $g_{0i}(m) = h(\phi_i(m))$, $m \in V_0 \cap V_i$.

Let A be a generical self-dual connection on $\phi^* \bar{P}$. Then there is a system $\{A_i\}_{i=0,1,\dots,k}$ where each A_i is a smooth $\mathfrak{su}(2)$ -valued 1-form over V_i satisfying the cocycle condition over $V_0 \cap V_i$

$$(4.1) \quad A_i = g^{-1} \cdot dg + g^{-1} \cdot A_0 \cdot g, \quad g = g_{0i}.$$

Choose a point $m = m_{k+1}$ in $M \setminus \bigcup_{i=1}^k V_i$. We define a local coordinate neighborhood with parameter λ contained in an open ball $B \subset M \setminus \bigcup_{i=1}^k V_i$.

Let U be a Gaussian normal coordinate neighborhood in B around the m and $\phi; U \rightarrow \phi(U) \subset \mathbf{R}^4$ a coordinate chart. Set $B_r = \{m' \in U; |\phi(m')| < r\}$ for $r > 0$. For a sufficiently small $R > 0$ the metric g satisfies

$$(4.2) \quad |g^{\mu\nu}(m') - \delta^{\mu\nu}| < \zeta$$

for all $m' \in B_R$ where ζ is a small constant which depends only on R and the Riemannian curvature at $m = m_{k+1}$.

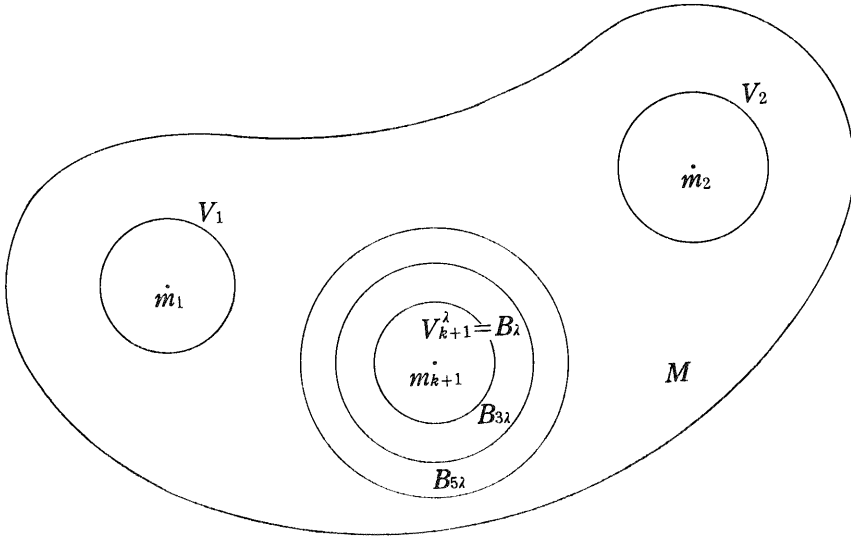


FIGURE 1.

For technical convenience we let λ be in $(0, \min(1, R/10))$. Set $V_{k+1}^i = B_i$ and cover M by $V_0 = M \setminus \{m_1, \dots, m_{k+1}\}$, V_1, \dots, V_k and $V_{k+1} = V_{k+1}^i$. We define a local chart $\phi_{k+1} = \phi_{k+1}^i$ over V_{k+1} by $\phi_{k+1}(m') = \phi(m')/\lambda$.

DEFINITION 4.1. Let P^λ be an $SU(2)$ -principal bundle over M with parameter λ defined by transition functions related to the covering $\{V_i\}_{0 \leq i \leq k+1}$ as follows. The transition function g_{0i} over $V_0 \cap V_i$ is just the transition function of $\phi^* \bar{P}$ for $1 \leq i \leq k$ and $g_{0, k+1}$ is defined by

$$(4.3) \quad g_{0, k+1}(m') = h(\phi_{k+1}(m')), \quad m' \in V_0 \cap V_{k+1}$$

by the aid of the chart ϕ_{k+1} .

Since the chart ϕ_{k+1} together with ϕ_1, \dots, ϕ_k can be extended to a smooth mapping of M onto S^4 with degree $k+1$, the bundle P^λ , thus constructed, has index $k+1$.

We introduce a smooth function $\beta; \mathbf{R}^+ \rightarrow \mathbf{R}^+$ with the following properties

$$(4.4) \quad \begin{aligned} \beta(x) &= 1 & x \leq 1 \\ &= 0 & x \geq 3/2 \end{aligned}$$

and we set β_r for $r > 0$ by $\beta_r(x) = \beta(x/r)$.

We define an $SU(2)$ -connection A^λ on P^λ .

DEFINITION 4.2. A connection A^λ is a system of $\mathfrak{su}(2)$ -valued 1-forms A_i^λ 's over V_i 's $0 \leq i \leq k+1$ satisfying in V_0

$$(4.5) \quad A_0^\lambda = (1 - \beta_{3\lambda})A_0 + \phi^*(\beta_\lambda \cdot \lambda^*(\bar{W}^2)),$$

in V_i

$$(4.6) \quad A_i^\lambda = A_i$$

and in V_{k+1}

$$(4.7) \quad A_{k+1}^\lambda = \phi^*(\lambda^*(W^1)).$$

Here we denote by \bar{W}^2 the form $(\bar{s}^{-1} \circ s)^* W^2$.

Because A is a connection on $\phi^* \bar{P}$, A^λ satisfies the cocycle condition in $V_0 \cap V_i$, $1 \leq i \leq k$. Since in $V_0 \cap V_{k+1}$ A_0^λ and A_{k+1}^λ respectively reduce to $\phi^*(\lambda^* \bar{W}^2)$ and $\phi^*(\lambda^* W^1)$, we see from the definition of $g_{0, k+1}$ that A^λ satisfies the cocycle condition also in $V_0 \cap V_{k+1}$.

PROPOSITION 4.3. There is a constant $k_1 > 0$ which is independent of λ such that

$$(4.8) \quad \|F(A^\lambda)\|_{L_4} \leq k_1 \lambda^{-1}$$

and

$$(4.9) \quad \|P_- F(A^\lambda)\|_{L_p} \leq k_1 \lambda^{(4/p-1)}, \quad p > 0.$$

PROOF. For simplicity we write A^λ in $M \setminus \{m\}$ as $A^\lambda = \phi^*(\beta_\lambda \cdot \lambda^* \bar{W}^2) + (1 - \beta_{3\lambda})A$.

Then $F^\lambda = F(A^\lambda) = dA^\lambda + A^\lambda \wedge A^\lambda$ and $F_-^\lambda = P_- F^\lambda$ are given by

$$(4.10) \quad \begin{aligned} F^\lambda &= \phi^*(\beta_\lambda \cdot \lambda^* F_W) + \phi^*(d\beta_\lambda \wedge \lambda^* \bar{W}^2) \\ &\quad - \phi^*\{(\beta_\lambda - \beta_\lambda^2) \lambda^*(\bar{W}^2 \wedge \bar{W}^2)\} + (1 - \beta_{3\lambda})F(A) \\ &\quad - d\beta_{3\lambda} \wedge A - \{(1 - \beta_{3\lambda}) - (1 - \beta_{3\lambda})^2\}A \wedge A \end{aligned}$$

and

$$(4.11) \quad \begin{aligned} F_-^\lambda &= \phi^*\{\beta_\lambda \cdot P_- \lambda^* F_W + P_-(d\beta_\lambda \wedge \lambda^* \bar{W}^2) \\ &\quad - (\beta_\lambda - \beta_\lambda^2) P_- \lambda^*(\bar{W}^2 \wedge \bar{W}^2)\} - P_-(d\beta_{3\lambda} \wedge A) \\ &\quad - \{(1 - \beta_{3\lambda}) - (1 - \beta_{3\lambda})^2\} P_-(A \wedge A). \end{aligned}$$

We divide M into four pieces $M \setminus B_{3\lambda}, B_{5\lambda} \setminus B_{3\lambda}, B_{3\lambda} \setminus B_\lambda$ and B_λ . For brevity we denote $m' \in B_{5\lambda}$ and $x = \phi(m')$ by the same x . From (4.2) we see that the norm $|\cdot|_\rho$ and the Euclidean norm $|\cdot|$ is equivalent in the ball B_R and the volume element $\sqrt{|g|} dx$ is also equivalent to the standard volume element dx .

First we show (4.8). Since $\text{supp}(\beta_\lambda - \beta_\lambda^2) \subset B_{3\lambda} \setminus B_\lambda$ and $\text{supp}\{(1 - \beta_{3\lambda}) - (1 - \beta_{3\lambda})^2\} \subset B_{5\lambda} \setminus B_{3\lambda}$ we have in $M \setminus B_{5\lambda}$

$$(4.12) \quad F^\lambda = F(A),$$

in $B_{5\lambda} \setminus B_{3\lambda}$

$$(4.13) \quad F^\lambda = -(d\beta_{3\lambda} \wedge A) - \{(1 - \beta_{3\lambda}) - (1 - \beta_{3\lambda})^2\}A \wedge A$$

and in $B_{3\lambda} \setminus B_\lambda$

$$(4.14) \quad F = \phi^*\{(\beta_\lambda \lambda^* F_W) + (d\beta_\lambda \wedge \lambda^* \bar{W}^2) - (\beta_\lambda - \beta_\lambda^2) \lambda^*(\bar{W}^2 \wedge \bar{W}^2)\}.$$

Moreover in B_λ F^λ reduces to $\phi^*(\lambda^* F_W)$.

We have then

$$(4.15) \quad \left(\int_{M \setminus B_{5\lambda}} |F^\lambda|_\rho^4 \sqrt{|g|} dx \right)^{1/4} \leq \|F(A)\|_{L_4}.$$

Because $\lambda \leq 1$ and $|d\beta_{3\lambda}|(x) = 1/3\lambda |d\beta|(|x|/3\lambda)$, we get in $B_{5\lambda} \setminus B_{3\lambda}$

$$(4.16) \quad |F^\lambda|_\rho(x) \leq k_2/\lambda$$

where k_2 is a finite constant, independent of λ , hence

$$(4.17) \quad \left(\int_{B_{5\lambda} \setminus B_{3\lambda}} |F^\lambda|_g^4 \sqrt{|g|} dx \right)^{1/4} \leq k_3$$

for some constant k_3 which is independent of λ . Moreover we have from (3.7) and (3.8)

$$(4.18) \quad |F^\lambda|_g(x) \leq k_4 \left\{ \frac{\lambda^2}{(\lambda^2 + |x|^2)^2} + \frac{\lambda^{3/2}}{|x|(\lambda^2 + |x|^2)} + \frac{\lambda^4}{|x|^2(\lambda^2 + |x|^2)^2} \right\}$$

for $x \in B_{3\lambda} \setminus B_\lambda$ where k_4 is a constant, independent of λ . Because $\lambda \leq 1$ and $\lambda \leq |x|$ it follows that

$$(4.19) \quad |F^\lambda|_g(x) \leq k_5$$

for $\lambda \leq |x| \leq 3\lambda$, from which we obtain

$$(4.20) \quad \left(\int_{B_{3\lambda} \setminus B_\lambda} |F^\lambda|_g^4 \sqrt{|g|} dx \right)^{1/4} \leq k_6 \lambda$$

for some constant k_6 . In the ball B_λ we see from (3.7) that

$$(4.21) \quad |F^\lambda|_g(x) \leq k_7 \frac{\lambda^2}{(\lambda^2 + |x|^2)^2},$$

hence by a simple computation

$$(4.22) \quad \left(\int_{B_\lambda} |F^\lambda|_g^4 \sqrt{|g|} dx \right)^{1/4} \leq k_8 \lambda^{-1}$$

for a constant k_8 . By these estimates we obtain

$$(4.23) \quad \|F^\lambda\|_{L_4} \leq k_9 \lambda^{-1}$$

where k_9 is a constant which is independent of λ .

We shall now estimate the L_p -norms of the anti-self-dual part F_-^λ of F^λ .

As in the case of the full curvature F^λ we have in $M \setminus B_{5\lambda}$

$$(4.24) \quad F_-^\lambda = 0,$$

in $B_{5\lambda} \setminus B_{3\lambda}$

$$(4.25) \quad F_-^\lambda = P_-[-(d\beta_{3\lambda} \wedge A) - \{(1 - \beta_{3\lambda}) - (1 - \beta_{3\lambda})^2\} A \wedge A]$$

and in $B_{3\lambda} \setminus B_\lambda$

$$(4.26) \quad F_-^\lambda = P_-[\phi^* \{\beta_\lambda \cdot \lambda^* F_W + (d\beta_\lambda \wedge \lambda^* \bar{W}^2) - (\beta_\lambda - \beta_\lambda^2) \lambda^* \bar{W}^2 \wedge \bar{W}^2\}].$$

Moreover in the ball B_λ $F_-^\lambda = P_-(\phi^* \lambda^* F_W)$.

Because $|F_-^\lambda|_g \leq |F^\lambda|_g$ we have

$$(4.27) \quad |F_-^\lambda|_g(x) \leq k_{10} \lambda^{-1}$$

in $B_{5\lambda} \setminus B_{3\lambda}$, hence

$$(4.28) \quad \left[\int_{B_{5\lambda} \setminus B_{3\lambda}} (|F^\lambda|_g)^p \sqrt{|g|} dx \right]^{1/p} \leq k_{11} \lambda^{(4/p-1)},$$

where k_{11} is independent of λ . Similarly we get in $B_{3\lambda} \setminus B_\lambda$ $|F^\lambda|_g(x) \leq |F^\lambda|_g(x) \leq k_{12}$, from which

$$(4.29) \quad \left[\int_{B_{3\lambda} \setminus B_\lambda} (|F^\lambda|_g)^p \sqrt{|g|} dx \right]^{1/p} \leq k_{12} \lambda^{4/p}.$$

In B_λ F^λ is self-dual with respect to the flat metric. Then

$$(4.30) \quad F^\lambda = (* - *_g) F^\lambda$$

where $*$ denotes the Hodge operator related to the flat metric. By using (4.2) (see also (8.20) in [11])

$$(4.31) \quad |F^\lambda|_g(x) \leq k_{13} |x|^2 |F^\lambda|(x)$$

for $x \in B_\lambda$. We have from (3.7)

$$(4.32) \quad \left[\int_{B_\lambda} (|F^\lambda|_g)^p \sqrt{|g|} dx \right]^{1/p} \leq k_{14} \lambda^{4/p}.$$

Thus (4.9) is derived.

PROPOSITION 4.4. Let A be a self-dual connection on the $SU(2)$ -bundle $\phi^* \bar{P}$ satisfying $\text{Ker } D_A^* = 0$. Then there exists a constant $\bar{\mu} > 0$ which is independent of λ such that $\mu(A^\lambda) \geq \bar{\mu}$ for sufficiently small $\lambda > 0$.

PROOF. By the definition of A^λ we have that

$$(4.33) \quad \hat{g} \otimes A^p|_{M \setminus B_{5\lambda}} = \hat{g}_{\phi^* \bar{P}} \otimes A^p|_{M \setminus B_{5\lambda}}$$

and the covariant derivative ∇_i with respect to A^λ coincides with ∇_A over $M \setminus B_{5\lambda}$.

For $\phi \in \Gamma(\hat{g} \otimes A^2)$ we define ϕ_1 and ϕ_2 respectively by $\phi_1 = \beta_{5\lambda} \phi$ and $\phi_2 = (1 - \beta_{5\lambda}) \phi$. Then $\phi = \phi_1 + \phi_2$ and

$$(4.34) \quad \|D_i^* \phi\|_{L_2^2} = \|D_i^* \phi_1\|_{L_2^2} + 2 \langle D_i^* \phi_1, D_A^* \phi_2 \rangle_{L_2} + \|D_A^* \phi_2\|_{L_2^2}$$

where $D_i = D_{A^\lambda}$ and $D_i^* = D_{A^\lambda}^*$.

Suppose that ϕ is a normalized eigensection of $D_i D_i^*$ with eigenvalue μ . Now we derive a lower bound for μ . To estimate the first term of (4.34) we use the Bochner-Weitzenböck formula

$$(4.35) \quad \|D_i^* \phi\|_{L_2^2}^2 = \|\nabla_i \phi\|_{L_2^2}^2 + \langle \mathcal{R}(\phi), \phi \rangle_{L_2} + \sqrt{2} \langle F^\lambda(\phi), \phi \rangle_{L_2}$$

and also the Hölder's inequality and the Sobolev inequalities to obtain

$$(4.36) \quad \begin{aligned} \|D_i^* \phi_1\|_{L_2^2}^2 &\geq \|\nabla_i \phi_1\|_{L_2^2}^2 \{1 - c_1 (\|\mathcal{R}\|_{L_2; B_{5\lambda}} + \|F^\lambda\|_{L_2; B_{5\lambda}})\} \\ &\quad - c_1 (\|\mathcal{R}\|_{L_2; B_{5\lambda}} + \|F^\lambda\|_{L_2; B_{5\lambda}}) \|\phi_1\|_{L_2^2} \end{aligned}$$

where c_1 is a constant which is independent of λ . Since $A^2 = \phi^*(\beta_\lambda \cdot \lambda^* \bar{W}^2) + (1 - \beta_{3\lambda})A$, we have

$$(4.37) \quad \nabla_\lambda \phi_1 = \nabla_\lambda \phi_1 + (1 - \beta_{3\lambda})[A, \phi_1]$$

hence

$$(4.38) \quad 2\|\nabla_\lambda \phi_1\|_{L_2^2} \geq \|\nabla_\lambda \phi_1\|_{L_2^2} - 2\|A\|_{L_\infty; B_R} \|\phi_1\|_{L_2^2}$$

where ∇_λ denotes the covariant derivative with respect to $\beta_\lambda \cdot \lambda^* \bar{W}^2$. Thus we obtain

$$(4.39) \quad \begin{aligned} \|D_\lambda^* \phi_1\|_{L_2^2} &\geq 1/2 \cdot \|\nabla_\lambda \phi_1\|_{L_2^2} \{1 - c_1(\|\mathcal{R}\|_{L_2; B_{5\lambda}} + \|F^2\|_{L_2; B_{5\lambda}})\} \\ &\quad - \{\|A\|_{L_\infty; B_R} + c_1(1 - \|A\|_{L_\infty; B_R})(\|\mathcal{R}\|_{L_2; B_{5\lambda}} + \|F^2\|_{L_2; B_{5\lambda}})\} \|\phi_1\|_{L_2^2}. \end{aligned}$$

A lower estimate of $\|\nabla_\lambda \phi_1\|_{L_2^2}$ is obtained as follows. Since $\text{supp}(\phi_1) \subset B_{10\lambda}$ and $\hat{\mathfrak{g}}_{P^2} \otimes A^2$ is over $B_{10\lambda}$ a direct sum of three copies of $\mathfrak{g} = \mathfrak{su}(2)$, it follows from Proposition 5.1 in the next section that

$$(4.40) \quad \|\nabla_\lambda \phi_1\|_{L_2^2} \geq c_2/\lambda^2 \|\phi_1\|_{L_2^2},$$

where c_2 is a finite constant, independent of λ . Because $\|\mathcal{R}\|_{L_2; B_{5\lambda}} \leq c_3\lambda^2$ and $\|F^2\|_{L_2; B_{5\lambda}} \leq \|F^2\|_{L_2; M}$ has a bound from (4.9), there is a constant c_4 which is independent of λ such that for sufficiently small $\lambda > 0$

$$(4.41) \quad \|D_\lambda^* \phi_1\|_{L_2^2} \geq c_4/\lambda^2 \|\phi_1\|_{L_2^2}.$$

For the second term of (4.34) we use formulas

$$(4.42) \quad D_\lambda^* \phi_1 = *(d\beta_{5\lambda} \wedge \phi) + \beta_{5\lambda} \cdot D_\lambda^* \phi$$

and

$$(4.43) \quad \|d\beta_{5\lambda} \wedge \phi\|_{L_2} \leq c_5\lambda \|\phi\|_{L_\infty}$$

to obtain

$$(4.44) \quad |\langle D_\lambda^* \phi_1, D_A^* \phi_2 \rangle| \leq \|D_A^* \phi_2\|_{L_2} (\|D_\lambda^* \phi_1\|_{L_2} + c_5\lambda \|\phi_1\|_{L_\infty}).$$

Here the L_∞ -norm of ϕ can be taken over $B_{10\lambda} \setminus B_{5\lambda}$, since $\text{supp}(\phi_1) \subset B_{10\lambda}$ and $\text{supp}(\phi_2) \subset M \setminus B_{5\lambda}$. The estimation of $\|\phi\|_{L_\infty}$ is done by using the eigenvalue equation (see also the proof of Proposition 8.8. in [11])

$$(4.45) \quad \langle D_\lambda^* \eta, D_\lambda^* \phi \rangle - \mu \langle \eta, \phi \rangle = 0$$

for all $\eta \in \mathcal{K}$, which is equivalent to

$$(4.46) \quad \langle \nabla_\lambda \eta, \nabla_\lambda \phi \rangle_{L_2} + \langle \mathcal{R}(\eta), \phi \rangle_{L_2} + \sqrt{2} \langle \eta, F^2(\phi) \rangle_{L_2} - \mu \langle \eta, \phi \rangle_{L_2} = 0.$$

Set $v = \sqrt{1 + |\phi|_\theta^2}$ and $\eta = f \cdot v^{-1} \cdot \phi$ where $f \in C^\infty(M)$, $f \geq 0$. Since

$$(4.47) \quad \nabla_\lambda \eta = \nabla f v^{-1} \phi - f v^{-2} \nabla v \cdot \phi + f v^{-1} \nabla_\lambda \phi$$

we get

$$(4.48) \quad \langle \nabla v, \nabla f \rangle_{L_2} - \sqrt{2} \langle v, |F^{\perp}| f \rangle_{L_2} - (c_6 + \mu) \langle v, f \rangle_{L_2} \leq 0$$

for all $f \in C^\infty(M)$, $f \geq 0$, where c_6 depends only on the metric g .

Because $F^{\perp} = 0$ in $B_{10i} \setminus B_{5i}$ and we have normalized ϕ , we can apply Theorem 5.3.1 in [9] to obtain the following bound on $\|v\|_{L_\infty}$.

$$(4.49) \quad \|\phi\|_{L_\infty} \leq \|v\|_{L_\infty} \leq c_7(1 + \|\phi\|_{L_2}) \leq 2c_7\|\phi\|_{L_2}.$$

Then the second term of (4.34) is estimated by

$$(4.50) \quad \begin{aligned} 2|\langle D_i^* \phi_1, D_A^* \phi_2 \rangle| &\leq 2\|D_A^* \phi_2\|_{L_2} (\|D_i^* \phi\|_{L_2} + c_8 \lambda \|\phi\|_{L_2}) \\ &\leq 1/2 \cdot \|D_A^* \phi_2\|_{L_2^2} + 4\|D_i^* \phi\|_{L_2^2} + 4c_8^2 \lambda^2 \|\phi\|_{L_2^2}. \end{aligned}$$

Hence we see

$$(4.51) \quad \begin{aligned} \|D_i^* \phi\|_{L_2^2} &\geq \|D_i^* \phi_1\|_{L_2^2} + \|D_A^* \phi_2\|_{L_2^2} \\ &\quad - 1/2 \|D_A^* \phi_2\|_{L_2^2} - 4\|D_i^* \phi\|_{L_2^2} - 4c_8^2 \lambda^2 \|\phi\|_{L_2^2}, \end{aligned}$$

that is,

$$(4.52) \quad 5\|D_i^* \phi\|_{L_2^2} \geq \|D_i^* \phi\|_{L_2^2} + 1/2 \cdot \|D_A^* \phi_2\|_{L_2^2} - 4c_8^2 \lambda^2 \|\phi\|_{L_2^2}.$$

Thus the following estimation is established from (4.41) together with the condition that $\text{Ker } D_A^* = 0$

$$(4.53) \quad \|D_i^* \phi\|_{L_2^2} \geq c_9 / \lambda^2 \|\phi_1\|_{L_2^2} + 1/2 \cdot \bar{\mu}_2 \|\phi_2\|_{L_2^2} - 4c_8^2 \lambda^2 \|\phi\|_{L_2^2} \geq \bar{\mu} \|\phi\|_{L_2^2}$$

for sufficiently small $\lambda > 0$, where $\bar{\mu}$ is a constant which is independent of λ .

PROPOSITION 4.5. Let A be a self-dual connection on the bundle $\phi^* \bar{P}$ satisfying $\text{Ker } D_A^* = 0$. Then for sufficiently small $\lambda > 0$ there exists on the bundle P^λ a self-dual connection $A' = A^\lambda + a$, $a \in I'(\hat{g} \otimes A^\lambda)$ satisfying $\text{Ker } D_{A'}^* = 0$. Moreover, if A is irreducible, then so is A' .

PROOF. From Proposition 4.4 we have $\mu(A^\lambda) \geq \bar{\mu}$. Then for small $\lambda > 0$ $\zeta(A^\lambda)$, introduced at §2 has a uniform bound by Proposition 4.3

$$(4.54) \quad \zeta(A^\lambda) \leq d_1.$$

Moreover we see from Proposition 4.3 that $\delta(A^\lambda)$ is estimated as

$$(4.55) \quad \delta(A^\lambda) \leq d_2 \lambda,$$

where d_2 does not depend on λ . Then for λ sufficiently small we find from Theorem 2.1 a solution $a = a^\lambda$ in $I'(\hat{g} \otimes A^\lambda)$ to (2.2). Hence $A' = A^\lambda + a$ gives a self-dual connection on P^λ .

To prove $\text{Ker } D_{A'}^* = 0$ we need to verify that $\mu(A') > 0$.

Since $D_{A'}^* \phi = D_i^* \phi + a(\phi)$, $\phi \in I'(\mathfrak{g} \otimes L^2)$, where a map $\phi \longrightarrow a(\phi)$ is represented in terms of a , we have by using the Hölder's inequality together with the Sobolev inequality and also (2.7) that

$$(4.56) \quad \|D_i^* \phi\|_{L^2} \leq 2 \|D_{A'}^* \phi\|_{L^2} + 2d_3 \lambda^2 (\|\phi\|_{L^2} + \|\mathcal{F}_i \phi\|_{L^2}).$$

From the Bochner-Weitzenböck formula the last term of (4.56) is estimated by

$$(4.57) \quad \|\mathcal{F}_i \phi\|_{L^2} \leq 2 \|D_i^* \phi\|_{L^2} + 2 \|\mathcal{R}\|_{L^\infty} \|\phi\|_{L^2} + 4 \|F^{\lambda}\|_{L^2} \|\phi\|_{L^2}.$$

Since we have from (4.9)

$$(4.58) \quad \|\mathcal{F}_i \phi\|_{L^2} \leq d_4 \|D_i^* \phi\|_{L^2} + d_5 \|\phi\|_{L^2}$$

for small λ , $\|D_{A'}^* \phi\|_{L^2}$ has a bound

$$(4.59) \quad 2 \|D_{A'}^* \phi\|_{L^2} \geq (1 - 2d_3 d_4 \lambda^2) \|D_i^* \phi\|_{L^2} - 2d_3(1 + d_5) \|\phi\|_{L^2}.$$

Hence it follows from Proposition 4.4 that $\|D_{A'}^* \phi\|_{L^2} \geq \bar{\mu} \|\phi\|_{L^2}$.

We now show that A' is irreducible if so is A . It suffices for this to verify that

$$\inf \{ \|\mathcal{F}_{A'} \phi\|_{L^2}; \phi \in I'(\mathfrak{g}), \|\phi\|_{L^2} = 1 \} > 0,$$

because any reducible connection has a non-trivial parallel section of \mathfrak{g} .

Since $\mathcal{F}_i \phi = \mathcal{F}_{A'} \phi - [a, \phi]$,

$$\|\mathcal{F}_i \phi\|_{L^2} \leq 2 \|\mathcal{F}_{A'} \phi\|_{L^2} + 4 \|\phi\|_{L^4} \|a\|_{L^4}$$

where we used an inequality $\|[a, \phi]\| \leq \sqrt{2} |a| |\phi|$ and Hölder's inequality. Because

$$(4.60) \quad \|a\|_{L^4} \leq d_6 \|a\|_{\mathcal{H}} \leq d_7 \lambda^2,$$

we obtain by using the Sobolev inequality and the lower estimation of $\|\mathcal{F}_i \phi\|_{L^2}$, given easily in the similar manner as in the case of $\|D_i^* \phi\|_{L^2}$

$$(4.61) \quad \|\mathcal{F}_{A'} \phi\|_{L^2} \geq 1/2 \{(1 - d_8 \lambda^2) \bar{\mu} - d_8 \lambda^2\} \|\phi\|_{L^2}.$$

Hence A' is irreducible for small λ .

5. The first eigenvalue of the rough Laplacian related to BPS T-solutions.

Let B_R be the Gaussian normal coordinate neighborhood of M centered at the fixed point m , which we gave in § 4. Let $\lambda \in (0, \min(1, R/10))$. Denote by $\Gamma_0 = \Gamma_0(B_{10\lambda}; \mathfrak{g})$ the set $\{\phi; B_{10\lambda} \longrightarrow \mathfrak{g} \text{ smooth, } \phi|_{\partial B_{10\lambda}} = 0\}$.

We define in $B_{10\lambda}$

$$(5.1) \quad \nu(\lambda) = \inf \{I_\lambda[\phi]; \phi \in \Gamma_0, \neq 0\}$$

where
$$I_\lambda[\phi] = \|\nabla_\lambda \phi\|_{L^2; B_{10\lambda}} / \|\phi\|_{L^2; B_{10\lambda}}$$

with respect to the covariant derivative $\nabla_\lambda \cdot = d \cdot + [\beta_\lambda \cdot \lambda^* W^1, \cdot]$.

The aim of this section is to verify the following

PROPOSITION 5.1. There is a constant $c > 0$ such that

$$(5.2) \quad \nu(\lambda) \geq c/\lambda^2.$$

LEMMA 5.2. (1) There exists $\phi_0 \in \Gamma_0$ which attains the infimum of $I_\lambda[\cdot]$.

(2) This ϕ_0 satisfies that

$$(5.3) \quad \int_{B_{10\lambda}} \langle \nabla_\lambda \phi_0, \nabla_\lambda \psi \rangle dx = \nu(\lambda) \int_{B_{10\lambda}} \langle \phi_0, \psi \rangle dx$$

for all $\psi \in \Gamma_0$ and also that in $B_{10\lambda}$

$$(5.4) \quad \nabla_\lambda^* \nabla_\lambda \phi_0 = \nu(\lambda) \phi_0.$$

PROOF. (1) is a consequence of theorems in Ch VI of [6]. To show (2) we set $\phi_t = \phi_0 + t\psi$, $\psi \in \Gamma_0$ and differentiate $I_\lambda[\phi_t]$ with respect to t and put $t=0$. Then we obtain (5.3). (5.4) follows immediately from (5.3) by Stokes' theorem.

For any \mathfrak{g} -valued function ψ and each $a \in SU(2)$ define a new \mathfrak{g} -valued function ψ^a by $\psi^a(x) = \psi(ax)$. Then we have from Proposition 3.1

$$(5.5) \quad (L_a^* \nabla_\lambda \psi)(x) = (\nabla_\lambda \psi^a)(x), \quad x \in B_R.$$

We notice that $SU(2)$ acts effectively on $\mathbf{R}^4 = \mathbf{H}$ and the normalized invariant measure da on $SU(2)$ coincides with the canonical measure on the unit 3-sphere $S^3 \approx SU(2)$.

For any $\psi \in \Gamma_0$ define ψ^* by integration

$$(5.6) \quad \psi^*(x) = \int_{SU(2)} \psi^a(x) da.$$

Of course ψ^* is L_a -invariant.

LEMMA 5.3. For each $\phi, \psi \in \Gamma_0$ and each $a \in SU(2)$

$$(5.7) \quad \langle \nabla_\lambda \phi^a, \nabla_\lambda \psi^a \rangle(x) = \langle \nabla_\lambda \phi, \nabla_\lambda \psi \rangle(ax)$$

and further

$$(5.8) \quad (\nabla_i \phi^*)(x) = \int_{SU(2)} (\nabla_i \phi^a)(x) da.$$

The proof of this lemma is easily done, because (5.7) is given by a simple computation and (5.8) follows from the commutability of the integration with respect to $SU(2)$ and differentiation with respect to the coordinate.

By using this lemma we have

LEMMA 5.4. If $\phi \in \Gamma_0$ satisfies $I_i[\phi] = \inf I_i[\psi]$, then the L_a -invariant ϕ^* also attains the infimum.

PROOF. If ϕ in (5.7) is L_a -invariant, then each $\phi \in \Gamma_0$ satisfies from (5.8)

$$(5.9) \quad \langle \nabla_i \phi, \nabla_i \psi \rangle(x) = \int_{SU(2)} \langle \nabla_i \phi, \nabla_i \psi \rangle(ax) da.$$

Then we have from (5.9) for $\psi = \phi^*$

$$(5.10) \quad |\nabla_i \phi^*|^2(x) = \int_{SU(2)} \langle \nabla_i \phi, \nabla_i \phi^* \rangle(ax) da.$$

Since the right hand side depends only on $|x|$,
 $|\nabla_i \phi^*|^2(x) = |\nabla_i \phi^*|^2(ax)$ for all $a \in SU(2)$.

We assume that ϕ attains the infimum. Set $\psi = \phi^*$ in (5.3). Then we obtain

$$(5.11) \quad \int_{B_{10\lambda}} \langle \nabla_i \phi, \nabla_i \phi^* \rangle dx = \nu(\lambda) \int_{B_{10\lambda}} \langle \phi, \phi^* \rangle dx$$

Since $\partial B_r = \{ra; a \in SU(2)\}$, the left hand side reduces to

$$\int_0^{10\lambda} K_3 t^3 dt \int_{SU(2)} \langle \nabla_i \phi, \nabla_i \phi^* \rangle(ta) da,$$

where K_3 is the volume of S^3 . From (5.10) this is given by

$$\int_0^{10\lambda} K_3 t^3 dt |\nabla_i \phi^*|^2(t) = \int_0^{10\lambda} K_3 t^3 dt \int_{SU(2)} |\nabla_i \phi^*|^2(ta) da = \int_{B_{10\lambda}} |\nabla_i \phi^*|^2 dx.$$

In the similar manner we can also reduce the right hand side of (5.11) to

$$\nu(\lambda) \int_{B_{10\lambda}} |\phi^*|^2 dx. \quad \text{It follows then that } \phi^* \text{ attains the infimum.}$$

Since $\mathfrak{g} = \mathfrak{su}(2)$ is identified with the space of pure imaginary numbers, every \mathfrak{g} -valued function ϕ is written by $\phi_1 i + \phi_2 j + \phi_3 k$, where ϕ_l is a real valued function, $1 \leq l \leq 3$.

LEMMA 5.5. If $\phi \in \Gamma_0$ attains the infimum and is L_a -invariant, then each component of ϕ does also.

This is easily verified from (5.3) together with Proposition 3.1.

The rough Laplacian $\nabla^* \nabla = \nabla_i^* \nabla_i$ operates on an L_a -invariant ϕ as

$$(5.12) \quad (\nabla^* \nabla \phi)(x) = -(\Delta \phi)(x) + \beta_i^2 \cdot \frac{2|x|^2}{(\lambda^2 + |x|^2)^2} \cdot \phi(x),$$

where $\Delta = \sum_{a=1}^4 (\partial/\partial x^a)^2$.

We suppose that $\phi \in \Gamma_0$ is L_a -invariant and attains the infimum. Then ϕ satisfies that

$$(5.13) \quad \Delta \phi + \left\{ \nu(\lambda) - \frac{2|x|^2 \beta_i^2}{(\lambda^2 + |x|^2)^2} \right\} \phi = 0.$$

Since ϕ is a function of $t=|x|$, this reduces to

$$(5.14) \quad \frac{d^2 \phi}{dt^2} + \frac{3}{t} \frac{d\phi}{dt} + \{\nu(\lambda) - f(t)\} \phi = 0.$$

where $f(t) = 2t^2 \beta_i^2(t)/(\lambda^2 + t^2)^2$. Each component of ϕ also satisfies this equation. For simplicity we denote by the same symbol ϕ one of components of ϕ .

Since $\phi = \phi(x)$ is smooth and depends only on $t=|x|$, $d\phi/dt \rightarrow 0$, if $t \rightarrow 0$. Because the equation is linear, we can extend the solution ϕ over $t > 10\lambda$.

Since the BPST-solution is analytic, there exists for this ϕ a value λ_0 in $(0, 10\lambda)$ such that $\phi(\lambda_0) = 0$ and $\phi(t) \neq 0$ for $0 < t < \lambda_0$.

Now we shall estimate the first zero point λ_0 of ϕ by comparing (5.14) with so-called Bessel equation.

We may assume that $\phi > 0$ in $(0, \lambda_0)$. Compare (5.14) with the following equation

$$(5.15) \quad \frac{d^2 y}{dt^2} + \frac{3}{t} \frac{dy}{dt} + \nu(\lambda) y = 0.$$

This equation reduces to the following

$$(5.16) \quad \frac{d^2 z}{dt^2} + \frac{1}{t} \frac{dz}{dt} + \left\{ \nu(\lambda) - \frac{1}{t^2} \right\} z = 0,$$

if we set $z = t \cdot y(t)$ ([13]). In terms of Bessel functions each solution $z(t)$ of (5.16) is represented by

$$(5.17) \quad z(t) = a J_1(\sqrt{\nu(\lambda)} t) + b Y_1(\sqrt{\nu(\lambda)} t),$$

where J_1 and Y_1 respectively are the Bessel functions of the first kind and of the second kind with $n=1$. Assume that $\lim_{t \rightarrow 0} y(t) < \infty$. Then $y(t)$ is given by $y(t) = aJ_1(\sqrt{\nu(\bar{\lambda})}t)/t$. We have further $\lim_{t \rightarrow 0} (dy/dt) = 0$. We can normalize y as $\lim_{t \rightarrow 0} y(t) = 1$.

Define a new function ϕ by $\phi(t) = t^{3/2} \cdot y(t)$ for the solution y of (5.14). Then ϕ satisfies

$$(5.18) \quad \frac{d^2\phi}{dt^2} + \left(\nu(\lambda) - f(t) - \frac{3}{4t^2} \right) \phi = 0.$$

Of course $\lim_{t \rightarrow 0} \frac{d\phi}{dt} = 0$.

Define a new function $\bar{\phi}$ similarly $\bar{\phi}(t) = t^{3/2} \cdot y(t)$ for the solution $y(t)$ of (5.15). Then $\bar{\phi}$ satisfies

$$(5.19) \quad \frac{d^2\bar{\phi}}{dt^2} + \left(\nu(\lambda) - \frac{3}{4t^2} \right) \bar{\phi} = 0 \quad \text{and} \quad \lim_{t \rightarrow 0} \frac{d\bar{\phi}}{dt} = 0.$$

If we set $h(t)$ and $\bar{h}(t)$ respectively by $h(t) = \nu(\lambda) - f(t) - 3/4t^2$ and $\bar{h}(t) = \nu(\lambda) - 3/4t^2$, then we see that $h(t) < \bar{h}(t)$ for $t > 0$.

Denote by $\bar{\lambda}_0$ the first zero point of $\bar{\phi}$, in other words, the first zero point of $J_1(\sqrt{\nu(\bar{\lambda})}t)$. Then we have

LEMMA 5.6. $\bar{\lambda}_0 \leq \lambda_0$.

PROOF. Assume that $\bar{\lambda}_0 > \lambda_0$. Then $\phi > 0$ and $y > 0$ in $(0, \lambda_0)$.

Fix ε in $(0, \lambda_0)$ and apply Sturm's technique to (5.18) and (5.19). Then

$$(5.20) \quad \begin{aligned} 0 &= \int_{\varepsilon}^{\lambda_0} \left\{ \bar{\phi} \left(\frac{d^2\phi}{dt^2} + h\phi \right) - \phi \left(\frac{d^2\bar{\phi}}{dt^2} + \bar{h}\bar{\phi} \right) \right\} dt \\ &= \bar{\phi}(\lambda_0) \frac{d\phi}{dt}(\lambda_0) - \left\{ \bar{\phi}(\varepsilon) \frac{d\phi}{dt}(\varepsilon) - \phi(\varepsilon) \frac{d\bar{\phi}}{dt}(\varepsilon) \right\} \\ &\quad + \int_{\varepsilon}^{\lambda_0} (h - \bar{h}) \phi \bar{\phi} dt. \end{aligned}$$

Since $h(t) < \bar{h}(t)$, the right hand side of (5.20) is smaller than

$$-\bar{\phi}(\varepsilon) \frac{d\phi}{dt}(\varepsilon) + \frac{d\bar{\phi}}{dt}(\varepsilon) \phi(\varepsilon) + \bar{\phi}(\lambda_0) \frac{d\phi}{dt}(\lambda_0).$$

Because $\phi > 0$ in $(0, \lambda_0)$ and $\phi(\lambda_0) = 0$, we have $d\phi/dt < 0$ at λ_0 . If we let ε tend to 0, then the above has the limit $\bar{\phi}(\lambda_0) \cdot d\phi/dt(\lambda_0)$ which must be negative. This leads a contradiction. Thus we obtain $\bar{\lambda}_0 \leq \lambda_0$.

PROOF of PROPOSITION 5.1. Denote by t_0 the first zero point (>0) of the Bessel function $J_1(t)$. Then $\bar{\lambda}_0 = t_0/\sqrt{\nu(\bar{\lambda})}$. Therefore from the above lemma we conclude that

$$t_0/\sqrt{\nu(\bar{\lambda})} \leq \lambda_0 \leq 10\lambda,$$

that is, $t_0^2/100\lambda^2 \leq \nu(\lambda)$.

6. Reducible self-dual connections.

We introduce in this section a topological condition on $SU(2)$ -principal bundles which carry reducible connections and investigate structure of the moduli space of self-dual connections around a reducible self-dual connection. As a consequence we obtain Theorem 1.1.

PROPOSITION 6.1 (see also [4]). Let P be an $SU(2)$ -principal bundle of index k . It admits a reducible connection if and only if there exists a complex line bundle L with $c_1(L)^2 = k$.

PROOF. Assume that a complex line bundle L satisfies that $c_1(L)^2 = k$. Since $S(U(1) \times U(1)) \subset SU(2)$ and L carries a $U(1)$ -structure with a $U(1)$ -connection, a bundle $L \oplus L^{-1}$ is associated to an $SU(2)$ -principal bundle P_0 . The connection on $L \oplus L^{-1}$ induced from L defines a connection A_0 on P_0 which is indeed reducible. Index of P_0 equals to $-c_2(L \oplus L^{-1}) = c_1(L)^2$.

The inverse implication is shown by the following lemma.

LEMMA 6.2. Let A be a reducible connection on an $SU(2)$ -principal bundle P . Then there is a complex line bundle L with a $U(1)$ -connection α such that P reduces to a bundle associated to an $S(U(1) \times U(1))$ bundle $L \oplus L^{-1}$ and $\hat{\mathfrak{g}}_P$ splits into $L^2 \oplus \mathbb{1}$, and moreover A reduces to $\begin{pmatrix} \alpha & \\ & -\alpha \end{pmatrix}$. Further $\text{Ker } d_A \subset I(\hat{\mathfrak{g}})$ and the isotropy group \mathcal{G}_A of A are respectively given by

$$\left\{ \begin{pmatrix} \sqrt{-1}c & \\ & -\sqrt{-1}c \end{pmatrix}; c \in \mathbf{R} \right\} \quad \text{and} \quad \left\{ \begin{pmatrix} e^{\sqrt{-1}c} & \\ & e^{-\sqrt{-1}c} \end{pmatrix}; c \in \mathbf{R} \right\}.$$

PROOF. Since A has a nontrivial parallel section ψ of $\hat{\mathfrak{g}}$, $g_t = \exp t\psi = \sum (t^n/n!) \psi^n$ defines a nontrivial circle subgroup in \mathcal{G}_P satisfying $g_t(A) = A$. Then for a fixed u in P we obtain a circle subgroup $\{b_t\}$ in $SU(2)$ by $g_t(u) = u \cdot b_t$. If $u_s(u_0 = u)$ is a horizontal lift of a curve in M , then $g_t(u_s)$ is also horizontal and satisfies $g_t(u_s) = u_s \cdot b_t$. Hence the holonomy group of A is contained in the centralizer of the circle $\{b_t\}$. Then the holonomy group is conjugate with $\left\{ \begin{pmatrix} e^{\sqrt{-1}\theta} & \\ & e^{-\sqrt{-1}\theta} \end{pmatrix}; \theta \in \mathbf{R} \right\}$. From a reduc-

tion theorem ([8]) P is equivalent to an $S(U(1) \times U(1))$ -bundle Q and A does also reduce to an $S(U(1) \times U(1))$ -connection on Q . The vector bundle canonically associated to Q is written as $L \oplus L^{-1}$ for some $U(1)$ -vector bundle L . The index of P is certainly $-c_2(L \oplus L^{-1}) = c_1(L)^2$.

The rest of Lemma follows from [4].

REMARK. The simply-connectedness of M is not necessarily assumed for this proposition.

LEMMA 6.3 ([2]). Let A be self-dual connection. Then we have an elliptic complex associated to A

$$0 \longrightarrow \Gamma(\hat{\mathfrak{g}}) \xrightarrow{d_A} \Gamma(\hat{\mathfrak{g}} \otimes A^1) \xrightarrow{D_A} \Gamma(\hat{\mathfrak{g}} \otimes A^2) \longrightarrow 0$$

By using this lemma together with Kuranishi's method we obtain

LEMMA 6.4 ([4]). Let A be a reducible self-dual $SU(2)$ -connection satisfying $\text{Ker } D_A^* = 0$. Then the moduli space \mathcal{M} around $[A]$ has a form of an S^1 -quotient of a slice neighborhood as $\{\phi \in \Gamma(\hat{\mathfrak{g}} \otimes A^1); \|\phi\| < \varepsilon, d_A^* \phi = 0, D_A \phi = -\phi \# \phi\} / \mathcal{G}_A$, which is homeomorphic to $\{\phi \in H^1(\hat{\mathfrak{g}}); \|\phi\| < \varepsilon\} / \mathcal{G}_A$ where $H^1(\hat{\mathfrak{g}}) = \text{Ker } d_A^* \cap \text{Ker } D_A$.

Since $D_A; \Gamma(\hat{\mathfrak{g}} \otimes A^1) \longrightarrow \Gamma(\hat{\mathfrak{g}} \otimes A^2)$ decomposes into

$$\Gamma(A^1) \oplus \Gamma(L^2 \otimes A^1) \xrightarrow{D \oplus D_\alpha} \Gamma(A^2) \oplus \Gamma(L^2 \otimes A^2), \quad \text{Ker } D_A^* = \text{Ker } D^* \oplus \text{Ker } D_\alpha^*.$$

Because \mathcal{G}_A and $H^0(\hat{\mathfrak{g}}) = \text{Ker } d_A$ are one dimensional from Lemma 6.2 and $b_-^2(M) = \dim \text{Ker } D^* = 0$, we have from the Atiyah-Singer index theorem ([2])

$$(6.1) \quad \begin{aligned} \dim H^1(\hat{\mathfrak{g}}) &= 8k - \dim SU(2)/2 \cdot (\chi(M) - \tau(M)) + 1 \\ &= 8k - 2 - 3b_-^2(M) + 3b^1(M) \\ &= 8k - 2 + 3b^1(M) \end{aligned}$$

which is equal or greater than $6 + 3b^1(M)$.

LEMMA 6.5. Let $\{A_t\}$ be a one-parameter family of reducible self-dual connections on an $SU(2)$ -bundle P which is non-trivial with respect to gauge transformations. Then $\{A_t\}$ induces canonically a harmonic 1-form α . Conversely each harmonic 1-form yields a one-parameter family of non-trivial self-dual connections which are all reducible.

PROOF. It is seen that for a reducible self-dual connection A and a harmonic

1-form $a \left\{ A + t \begin{pmatrix} \sqrt{-1} a & \\ & -\sqrt{-1} a \end{pmatrix}; t \in \mathbf{R} \right\}$ defines a one-parameter family of reducible connections on P .

Conversely, let $\{A_t\}$ be a one-parameter family of reducible connections which are not equivalent to $A=A_0$. From Lemma 6.2 P reduces to an $S(U(1) \times U(1))$ -bundle such that A reduces to $\begin{pmatrix} \alpha & \\ & -\alpha \end{pmatrix}$ with respect to $\mathfrak{g}_P = \mathbf{1} \oplus L^2$. Then A_t has for each t a form of $\begin{pmatrix} \alpha & \\ & -\alpha \end{pmatrix} + \begin{pmatrix} \sqrt{-1} a_t & b_t \\ -\bar{b}_t & -\sqrt{-1} a_t \end{pmatrix}$ for a real 1-form a_t and an L^2 -valued 1-form b_t . By choosing suitable gauge transformations we can assume that $d_A^* \dot{A} = 0$ for $\dot{A} = d/dt A_t|_{t=0}$, which implies $d^* \dot{a} = 0$ and $d_a^* \dot{b} = 0$, where $\dot{a} = d/dt a_t|_{t=0}$ and $\dot{b} = d/dt b_t|_{t=0}$.

Since each $\text{Ker } d_{A_t}$ is one dimensional from Lemma 6.2, we can choose a parallel section ϕ_t of $L^2(\mathfrak{g})$, smoothly parametrized with t . Differentiate $d_{A_t} \phi_t = 0$ with respect to t and put $t=0$. Then we have

$$(6.2) \quad d_A \dot{\phi} + [\dot{A}, \phi_0] = 0,$$

where $\dot{\phi} = d/dt \phi_t|_{t=0}$. Since $\phi_0 = \begin{pmatrix} \sqrt{-1} c & \\ & -\sqrt{-1} c \end{pmatrix}$ for constant c , (6.2) is equivalent to

$$(6.3) \quad \begin{cases} d\dot{\phi}_1 = 0, \\ d_a \dot{\phi}_2 + \sqrt{-1} c \dot{b} = 0 \end{cases}$$

for $\dot{\phi} = \begin{pmatrix} \dot{\phi}_1 & \dot{\phi}_2 \\ -\dot{\phi}_2 & -\dot{\phi}_1 \end{pmatrix}$. Then it follows from $d_a^* \dot{b} = 0$ that \dot{b} must vanish.

Since each A_t is self-dual, \dot{a} is closed, and hence is a harmonic 1-form,

PROOF OF THEOREM 1. From Lemma 6.5 (reducible self-dual connections on $P\}/\mathcal{G}_P$, denoted by \mathcal{R} is $b^1(M)$ dimensional. Since $\dim \mathcal{M} = \dim H^1(\mathfrak{g}) - 1 > \dim \mathcal{R}$, \mathcal{M} is packed full with irreducible self-dual connections. Hence we obtain Theorem 1.1.

References

- [1] Atiyah, M. F., Geometry of Yang-Mills fields, Accademia Nazionale dei lincei scuola superiore, Pisa 1979.
- [2] Atiyah, M. F., Hitchin, N. J. & Singer, I. M., Self-duality in four-dimensional Riemannian geometry, Proc. R. Soc. Lond., A. **362** (1978), 425-461.
- [3] Bourguignon, J. P. & Lawson, H. B., Stability and isolation phenomena for Yang-Mills fields, Com. Math. Phys., **79** (1981), 189-230.
- [4] Donaldson, S. K., An application of gauge theory to four dimensional topology, J. Differential Geometry, **18** (1983), 279-315.
- [5] Itoh, M., On the moduli space of anti-self-dual Yang-Mills connections on Kähler surfaces, Publi. R.I.M.S. (Kyoto), **19** (1983), 15-32.

- [6] Jaffe, A. & Taubes, C. H., *Vortices and Monopoles*, Birkhäuser, Boston 1980.
- [7] Kobayashi, S., *Curvature and stability of vector bundles*, Proc. Japan Acad., **58 A** (1982), 158-162.
- [8] Kobayashi, S. & Nomizu, K., *Foundations of differential geometry*, vol I. Interscience, New York 1963.
- [9] Morrey, C.B., *Multiple integrals in the calculus of variations*, Springer, New York 1966.
- [10] Okonek, C., Schneider, M. & Spindler, H., *Vector bundles on complex projective spaces*, Birkhäuser, Boston 1980.
- [11] Taubes, C. H., *Self-dual Yang-Mills connections on non-self-dual 4-manifolds*, J. Differential Geometry, **17** (1982), 139-170.
- [12] Uhlenbeck, K. K., *Removable singularities in Yang-Mills fields*, Com. Math. Phys., **83** (1982), 11-29.
- [13] Whittaker, E. T. & Watson, G. N., *A course of modern analysis* (third ed.), Cambridge Univ. Press, London 1920.

Institute of Mathematics
University of Tsukuba
Ibaraki 305 JAPAN