TSUKUBA J. MATH.
Vol. 16 No. 2 (1992), 439—442

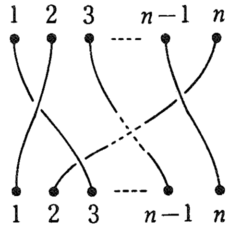
A COMBINATORIAL PROOF FOR ARTIN'S PRESENTATION OF THE BRAID GROUP B_n AND SOME CYCLIC ANALOGUE

By

Jun MORITA

1. Artin's presentation.

For each $n \geqq 1$, let S_n be the symmetric group on n letters $\{1, 2, \cdots, n\}$, and B_n the geometric braid group with n strings.



There is a natural homomorphism, called χ_n, of B_n onto S_n. As usual, S_{n-1} and B_{n-1} are regarded as subgroups of S_n and B_n respectively, and then the restriction of χ_n to B_{n-1} coinsides with χ_{n-1}. Put $B_n^0 = \chi_n^{-1}(S_{n-1})$. Then B_{n-1} is a subgroup of B_n^0.

Let $\tilde{B}_n$ be the group presented by the generators:

$$\sigma_1, \sigma_2, \cdots, \sigma_{n-1}$$

and the defining relations:

$$\begin{cases} \sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j & \text{if } |i-j|=1\,; \\ \sigma_i \sigma_j = \sigma_j \sigma_i & \text{if } |i-j| \neq 0, 1\,. \end{cases}$$

Put

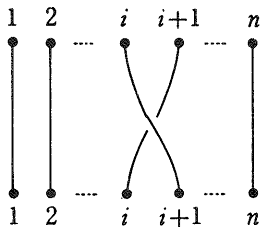
$$\tau_i = \sigma_{n-1}^{-1} \cdots \sigma_{i+1}^{-1} \sigma_i^2 \sigma_{i+1} \cdots \sigma_{n-1} \qquad \text{for } 1 \leqq i \leqq n-2\,,$$

$$\tau_{n-1} = \sigma_{n-1}^2\,.$$

Let $\tilde{B}_n^0$ be the subgroup of $\tilde{B}_n$ generated by $\sigma_1, \cdots, \sigma_{n-2}, \tau_1, \cdots, \tau_{n-1}$. Then there is a natural homomorphism of $\tilde{B}_{n-1}$ into $\tilde{B}_n^0$.

Received November 12, 1991.

Taking σ_i to the i-th fundamental braid:



for $1 \leqq i \leqq n-1$, we obtain a homomorphism, called ϕ_n, of $\tilde{B}_n$ onto B_n. Then the following result is well-known.

ARTIN'S THEOREM. *ϕ_n is an isomorphism.*

PROOF. We proceed by induction on n. The result is trivial if $n=1, 2$. Suppose $n \geqq 3$, and that ϕ_{n-1} is an isomorphism. Forgetting the n-th string, we obtain a homomorphism, called θ, of B_n^0 onto B_{n-1}. Hence, $B_n^0 = B_{n-1} \ltimes \operatorname{Ker}\theta$, and $\operatorname{Ker}\theta \cong F_{n-1}$, where F_{n-1} is the free group of rank $n-1$. This fact implies that $\tilde{B}_n^0$ is isomorphic to B_n^0 under ϕ_n. Let $\rho = \sigma_1\sigma_2 \cdots \sigma_{n-1}$, and put

$$\tilde{X} = \tilde{B}_n^0 \cup \tilde{B}_n^0 \rho \cup \cdots \cup \tilde{B}_n^0 \rho^{n-1} .$$

Then $\tilde{B}_n = \langle \tilde{B}_n^0, \rho \rangle$, and $\tilde{X}$ is a subgroup since

$$\rho\sigma_i = \sigma_{i+1}\rho \qquad (1 \leqq i \leqq n-2),$$

$$\rho\sigma_{n-2} = \sigma_{n-2}^{-1} \cdots \sigma_2^{-1}\sigma_1^{-1}\rho^2 , \qquad \rho^2\sigma_{n-2} = \sigma_1\sigma_2 \cdots \sigma_{n-2}\tau_{n-1}\rho ,$$

$$\rho\tau_i = \sigma_1\sigma_2 \cdots \sigma_{i-1}\sigma_i^2\sigma_{i-1}^{-1} \cdots \sigma_2^{-1}\sigma_1^{-1}\rho \qquad (1 \leqq i \leqq n-2),$$

$$\rho\tau_{n-1} = \tau_1\rho ,$$

$$\rho^n = (\sigma_1\sigma_2 \cdots \sigma_{n-2})^{n-1}\tau_{n-1}\tau_{n-2} \cdots \tau_2\tau_1 ,$$

Therefore, $\tilde{X} = \tilde{B}_n$, and the group index $[\tilde{B}_n : \tilde{B}_n^0]$ is at most n, which implies $[\tilde{B}_n : \tilde{B}_n^0] = n$. Hence, ϕ_n is an isomorphism. ■

2. Some cyclic analogue.

Here we consider the braid group $B_{n+1} = \langle \sigma_1, \sigma_2, \cdots, \sigma_n \rangle$ with $n \geqq 3$ and a certain subgroup. Put

$$\delta = \sigma_n^{-2}\sigma_1\sigma_2 \cdots \sigma_{n-2}\sigma_{n-1}\sigma_{n-2}^{-1} \cdots \sigma_2^{-1}\sigma_1^{-1}\sigma_n^2 ,$$

$$\pi = \sigma_1\sigma_2 \cdots \sigma_{n-1}\sigma_n^2$$

and set $C_{n+1}^0 = \langle \sigma_1, \cdots, \sigma_{n-1}, \delta \rangle \subset B_{n+1}$. Then $B_{n+1}^0 = \langle C_{n+1}^0, \pi \rangle$. Let C_{n+1}^* be the group presented by the generators:

$$\beta_1, \beta_2, \cdots, \beta_n$$

and the defining relations:

$$\begin{cases} \beta_i\beta_j\beta_i=\beta_j\beta_i\beta_j & \text{if } |i-j|=1,\ n-1 \\ \beta_i\beta_j=\beta_j\beta_i & \text{if } |i-j|\neq 0,\ 1,\ n-1 \end{cases}$$

and Z the infinite cyclic group generated by ζ. We construct the semi-direct product, called $B^*_{n+1}=Z\ltimes C^*_{n+1}$ of Z and C^*_{n+1} with $\zeta\beta_i\zeta^{-1}=\beta_{i+1}$ $(1\leqq i\leqq n-1)$ and $\zeta\beta_n\zeta^{-1}=\beta_1$. Then there is a homomorphism ϕ_1 of B^*_{n+1} onto B^0_{n+1} with

$$\phi_1:\begin{cases} \beta_i \longmapsto \sigma_i & (1\leqq i\leqq n-1); \\ \beta_n \longmapsto \delta; \\ \zeta \longmapsto \pi. \end{cases}$$

On the other hand, there is a homomorphism ϕ_2 of B^0_{n+1} onto B^*_{n+1} with

$$\phi_2:\begin{cases} \sigma_i \longmapsto \beta_i & (1\leqq i\leqq n-1); \\ \tau_i \longmapsto \gamma_i & (1\leqq i\leqq n), \end{cases}$$

where $B^0_{n+1}=\langle\sigma_1, \cdots, \sigma_{n-1}, \tau_1, \cdots, \tau_n\rangle\cong B_n\ltimes F_n$ and

$$\gamma_i=\beta_{i-1}^{-1}\cdots\beta_2^{-1}\beta_1^{-1}\zeta\beta_{n-1}^{-1}\cdots\beta_{i+1}^{-1}\beta_i^{-1}.$$

Then one can see both $\phi_1\phi_2=id.$ and $\phi_2\phi_1=id.$ Hence we obtain the following.

THEOREM. *$B^0_{n+1}\cong B^*_{n+1}$ and $C^0_{n+1}\cong C^*_{n+1}$.*

Therefore, the group C^0_{n+1} may be called a braid covering of the affine Weyl group $W_a(S_n)$ associated with S_n. We can describe this fact more precisely as follows. Let f_n be the canonical gradation homomorphism of $F_n\cong\langle\tau_1, \cdots, \tau_n\rangle$ onto $\boldsymbol{Z}$, and put $E_n=\operatorname{Ker} f_n$. Then $C^0_{n+1}\cong B_n\ltimes E_n$ and E_n is the normal subgroup of F_n generated by

$$\tau_1\tau_2^{-1}, \tau_2\tau_3^{-1}, \cdots, \tau_{n-1}\tau_n^{-1}.$$

Hence, we obtain a homomorphism ν_{n+1} of $C^0_{n+1}\cong B_n\ltimes E_n$ onto

$$S_n\ltimes E_n/[F_n, F_n]\cong S_n\ltimes \boldsymbol{Z}^{n-1}\cong W_a(S_n).$$

The (C^0_{n+1}, ν_{n+1}) gives the above braid covering of $W_a(S_n)$. Put $Q_{n+1}=\operatorname{Ker}\nu_{n+1}$. Then $Q_{n+1}\cong P_n\ltimes[F_n, F_n]$, where P_n is the kernel of χ_n and called the pure braid group with n strings.

We refer to [1], [2] for braid groups, and [3] for affine Weyl groups.

References

[1] J.S. Birman, "Braids, links, and mapping class groups," Ann. Math. Studies 82, Princeton Univ. Press, Princeton, 1975.

[2] V.L. Hansen, "Braids and coverings," London Math. Soc. Student Texts 18, Cambridge Univ. Press, Cambridge, New York, Port Chester, Melbourne, Sydney, 1989.

[3] J.E. Humphreys, "Reflection groups and Coxeter groups," Cambridge Studies in Advanced Math. 29, Cambridge Univ. Press, Cambridge, New York, Port Chester, Melbourne, Sydney, 1990.

Institute of Mathematics
University of Tsukuba
Tsukuba, Ibaraki, 305
Japan