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CONSTRUCTIONS OF COMPLETE METRIC SPACES AND COMPACT HAUSDORFF SPACES WHICH ARE NOT BAIRE SPACES

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The standard proofs of the Baire category theorem for complete metric spaces (for compact Hausdorff spaces) use the Principle of Dependent Choice. Goldblatt [1] showed the Baire Category Theorem for complete metric spaces is equivalent to the Principle of Dependent Choice. So there are models of set theory in which the Baire Category Theorem for complete metric spaces fails. On the other hand, in case of the Baire category theorem for compact Hausdorff spaces the existence of such models has not been known.

The purpose of this paper is to construct compact Hausdorff spaces which are not Baire spaces in models of set theory. We also show explicitly complete metric spaces which are not Baire spaces.

§1. Preliminaries.

The Baire Category Theorem. A subset S of a topological space X is *meager* (or *of first category*) if S is a countable union of nowhere dense sets. A topological space X is a *Baire space* if every nonempty open set of X is not meager. The *Baire Category Theorem* for complete metric spaces (resp. for compact Hausdorff spaces) is the assertion every complete metric space (resp. compact Hausdorff space) is a Baire space.

Dependent Choice. The *principle of Dependent Choice* (**DC**) is a weakened form of the Axiom of Choice (**AC**):

(**DC**) *If R is a binary relation on a set S such that for all $s\in S$ there exists a $t\in S$ with sRt, then for any $s\in S$ there exists a sequence $f:\omega\rightarrow S$ with $f(0)=s$ and $f(n)Rf(n+1)$ for all $n<\omega$.*

Set theory with atoms, ZFA. The *set theory with atoms*, **ZFA**, is a modified version of Zermelo-Fraenkel set theory (**ZF**), and it admits objects other than

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sets, *atoms*. Atoms are objects which have no elements. The language of **ZFA** consists of $=$ and $\in$ and of two constant symbols 0(the empty set) and A(the set of all atoms). The axioms of **ZFA** are as follows:

O. *Empty set* $\neg\exists x(x\in 0)$.

A. *Atoms* $\forall x[x\in A\leftrightarrow\neg x=0\wedge\neg\exists y(y\in x)]$.

Atoms are the elements of A; *sets* are all objects which are not atoms.

A1. *Extensionality* $(\forall \text{ sets } X, Y)[\forall z(z\in X\leftrightarrow z\in Y)\leftrightarrow X=Y]$.

A2. *Pairing*; A3. *Comprehension*; A4. *Union*; A5. *Power-Set*;

A6. *Replacement*; A7. *Infinity* are like the axioms of **ZF**.

A8. *Regularity* $(\forall \text{ set } X)[X=0\vee\exists y\in X(X\cap y=0)]$.

For any set S let $\mathcal{P}^{\alpha}(S)$ be defined as follows:

$$\mathcal{P}^{0}(S)=S\,,$$

$$\mathcal{P}^{\alpha+1}(S)=\mathcal{P}^{\alpha}(S)\cup\mathcal{P}(\mathcal{P}^{\alpha}(S))\,,$$

$$\mathcal{P}^{\alpha}(S)=\bigcup_{\beta<\alpha}\mathcal{P}^{\beta}(S)\qquad\text{for limit }\alpha,$$

and let

$$\mathcal{P}^{\infty}(S)=\bigcup_{\alpha\in On}\mathcal{P}^{\alpha}(S)\,.$$

Then we have $V=\mathcal{P}^{\infty}(A)$. So we can define the *rank* of x, $\rho(x)$, for every set x:

$$\rho(x)=\text{the }\alpha\text{ such that }x\in\mathcal{P}^{\alpha+1}(A)-\mathcal{P}^{\alpha}(A)\,.$$

(For an atom x, let $\rho(x)=-1$.) If we add to **ZFA** the axiom $A=0$, we get **ZF**. **ZFA+AC** is consistent iff **ZF** is consistent.

Permutation Models. Let $\mathcal{M}$ be a transitive model of **ZFA** and let A be the set of atoms of $\mathcal{M}$. For each permutation π of A, we can define $\pi(x)$ for every set x by recursion on the rank of x:

$$\pi(0)=0,\qquad \pi(x)=\{\pi(y)\mid y\in x\}\,.$$

Then π becomes an automorphism of $\mathcal{M}$. Let $\mathcal{G}$ be a group of permutations of A. A set $\mathcal{F}$ of subgroups of $\mathcal{G}$ is a *normal filter* on $\mathcal{G}$ if for all subgroups H, K of $\mathcal{G}$:

(1) $\mathcal{G}\in\mathcal{F}$;

(2) if $H\in\mathcal{F}$ and $K\in\mathcal{F}$, then $H\cap K\in\mathcal{F}$;

(3) if $H\in\mathcal{F}$ and $H\subset K$, then $K\in\mathcal{F}$;

(4) if $\pi\in\mathcal{G}$ and $H\in\mathcal{F}$, then $\pi H\pi^{-1}\in\mathcal{F}$;

(5) for each $a \in A$, $\{\pi \in \mathcal{G} \mid \pi a = a\} \in \mathcal{F}$.

Let $\mathcal{G}$ and $\mathcal{F}$ be fixed. $x \in \mathcal{M}$ is called *symmetric* if $\{\pi \in \mathcal{G} \mid \pi x = x\} \in \mathcal{F}$. The *permutation model* determined by $\mathcal{G}$ and $\mathcal{F}$ is the class

$$\mathcal{N} = \{x \mid x \subset \mathcal{N} \wedge \{\pi \in \mathcal{G} \mid \pi x = x\} \in \mathcal{F}\}$$

which consists of all *hereditarily symmetric* objects. Then

$\mathcal{N}$ is a transitive model of **ZFA**; *$\mathcal{P}^{\infty}(0) \subset \mathcal{N}$ and $A \in \mathcal{N}$.*

Let $\mathcal{G}$ be a group of permutations of A. An ideal I on A is *normal* if

(1) if $\pi \in \mathcal{G}$ and $E \in I$, then $\{\pi e \mid e \in E\} \in I$;

(2) for each $a \in A$, $\{a\} \in I$.

For each x, let fix(x) be the subgroup of $\mathcal{G}$ defined by:

$$\text{fix}(x) = \{\pi \in \mathcal{G} \mid \pi y = y \text{ for all } y \in x\}.$$

Let $\mathcal{F}$ be the filter on $\mathcal{G}$ generated by the subgroups fix(E), $E \in I$. Then $\mathcal{F}$ is normal. In this case, x is symmetric iff there exists an $E \in I$ such that

$$\text{fix}(E) \subset \{\pi \in \mathcal{G} \mid \pi x = x\}.$$

E is called a *support* of x. Normal filters used in this paper will be of this type.

For more detailed descriptions on **ZFA** and on permutation models refer to [2; Chapter 4].

§ 2. Complete metric spaces which are not Baire.

Let $\mathcal{M}$ be a model of **ZFA+AC** with countable atoms. Divide the set A of atoms of $\mathcal{M}$ into countably many disjoint pairs:

$$A = \bigcup_{n<\omega} A_n, \qquad A_n = \{a_n, b_n\}.$$

Let $\mathcal{G}$ be the group of all those permutations of A which preserve the pairs (i. e. $\pi A_n = A_n$ for all $n < \omega$). Let I be the ideal of all finite subsets of A. I is normal. Let $\mathcal{F}$ be the filter generated by subgroups fix(E), $E \in I$. Let $\mathcal{N}$ be the permutation model determined by $\mathcal{G}$ and $\mathcal{F}$. In [2], $\mathcal{N}$ is called the *second Fraenkel model*. Note that for $x \in \mathcal{N}$ we can take a support of x of the form $A_0 \cup A_1 \cup \cdots \cup A_k$ (as each A_n is finite).

For each $n < \omega$, let

$$A_n^* = A_n \cup \{n\}.$$

In the model $\mathfrak{N}$, consider the set

$$P=\prod_{n<\omega} A_n^*$$

and the metric on P defined by

$$d(f, g)=\{\mu n[f(n)\neq g(n)]+1\}^{-1} \qquad \text{for } f, g\in P\ (f\neq g).$$

Then we have the

THEOREM 1. *In $\mathfrak{N}$, $\langle P, d\rangle$ is a complete metric space which is not a Baire space.*

PROOF. First note that

(*) *If $A_0\cup A_1\cup\cdots\cup A_k$ is a support of $f\in P$, then $f(n)=n$ for each $n>k$.*

For if $n>k$ and $f(n)\neq n$, then $f(n)\in A_n$ and there is a $\pi\in\mathrm{fix}(A_0\cup A_1\cup\cdots\cup A_k)$ such that $f(n)\neq\pi(f(n))$, but $\pi(f(n))=(\pi f)(\pi n)=f(n)$, a contradiction.

Next we show that the metric space $\langle P, d\rangle$ is not a Baire space in $\mathfrak{N}$. For each $n<\omega$, let

$$S_n=\{f\in P\,|\,\forall m>n\quad f(m)=m\}.$$

Then, by (*), we have

(**) $$P=\bigcup_{n<\omega} S_n .$$

As every $\pi\in\mathcal{G}$ preserves S_n and the sequence $\langle S_n\,|\,n<\omega\rangle$, these are in $\mathfrak{N}$. Each S_n is nowhere dense in $\mathfrak{N}$, since S_n is closed and does not contain any non-empty open set. So (**) shows that $\langle P, d\rangle$ is not a Baire space in $\mathfrak{N}$.

Finally we show that the metric space $\langle P, d\rangle$ is complete in $\mathfrak{N}$. By (*) we have

(***) *if $f, g\in P$ have a support $A_0\cup A_1\cup\cdots\cup A_k$,*
then either $f=g$ or $d(f, g)\geqq(k+1)^{-1}$.

Let $\langle f_i\,|\,i<\omega\rangle\in\mathfrak{N}$ be a fundamental sequence of $\langle P, d\rangle$. Take a k such that $A_0\cup\cdots\cup A_k$ is a support of $\langle f_i\,|\,i<\omega\rangle$. Then for $\pi\in\mathrm{fix}(A_0\cup\cdots\cup A_k)$

$$\langle\pi(f_i)\,|\,i<\omega\rangle=\pi\langle f_i\,|\,i<\omega\rangle=\langle f_i\,|\,i<\omega\rangle .$$

So $A_0\cup\cdots\cup A_k$ is also a support of f_i for every $i<\omega$. As $\langle f_i\,|\,i<\omega\rangle$ is a fundamental sequence, by (***), there must exist an n such that

$$\forall i, j\geqq n\ \ f_i=f_j .$$

Hence

$$\lim_{i\to\infty} f_i = f_n .$$

This shows P is complete in $\mathcal{N}$. □

Goldblatt [1] showed that under $\neg$**DC** there is a complete metric space which is not Baire. We show here how to construct such a space because his construction is indirect.

CONSTRUCTION: Assume $\neg$**DC**. Then we can take a set S, a binary relation R on S, and an element s of S such that $\forall x\in S\exists y\in S(xRy)$ and such that

$$\forall f:\omega\to S\{f(0)=s\to\exists k<\omega[\forall n<k \ \ f(n)Rf(n+1)\wedge\neg f(k)Rf(k+1)]\}.$$

Let

$$P=\{p\,|\,\mathrm{Fnc}(p)\wedge 0<\mathrm{dom}(p)<\omega\wedge p(0)=s\wedge\forall n[n+1\in\mathrm{dom}(p)\to p(n)Rp(n+1)]\}.$$

Consider the metric space P^ω with the metric defined by

$$d(f,\ g)=\{\mu n[f(n)\neq g(n)]+1\}^{-1} \qquad \text{for } f,\ g\in P^\omega(f\neq g).$$

P^ω is complete, for the limit of a fundamental sequence can be defined explicitly. The subset

$$C=\{f\in P^\omega\,|\,\forall n<\omega \ \ f(n)\subset f(n+1)\}$$

consisting of increasing sequences is nonempty and closed, for, if $f(m)\not\subset f(m+1)$, then $\{g\in P^\omega\,|\,\forall n<m+2[g(n)=f(n)]\}$ is a neighbourhood of f disjoint from C. So C is a complete metric space with the metric $d\cap C^2$. Now we shall show C is not a Baire space. For each $n<\omega$, let

$$C_n=\{f\in C\,|\,\exists m<\omega \ \ n\in\mathrm{dom}(f(m))\}.$$

Then C_n is open in C. For, if $f\in C$ and $n\in\mathrm{dom}(f(m))$, then $\{g\in P^\omega\,|\,g(m)=f(m)\}\cap C$ is a neighbourhood of f in C included in C_n. To see that C_n is dense in C, take $f\in C$ and let $N=\{g\in P^\omega\,|\,\forall i<k \ \ g(i)=f(i)\}\cap C$ be a basic open neighbourhood of f in C. By applying the condition $\forall x\in S\exists y\in S(xRy)$ at most n times, $f(k)$ can be extended to a $p\in P$ with $n\in\mathrm{dom}(p)$. Define g by:

$$g=(f\,|\,k)\cup\{\langle n,\ p\rangle\,|\,k\leqq n<\omega\}.$$

Then $g\in N\cap C_n$, so C_n is dense. Now by the definitions of C and C_n's we have

$$f\in\bigcap_{n<\omega} C_n \longleftrightarrow \forall n<\omega[f(n)\subset f(n+1)]\wedge\bigcup_{n<\omega}\mathrm{dom}(f(n))=\omega .$$

So by the choice of P

$$\bigcap_{n<\omega} C_n=0, \qquad \text{i. e. } C=\bigcup_{n<\omega}(C-C_n).$$

As $C-C_n$ is nowhere dense, C is not a Baire space.

§ 3. Construction of a compact Hausdorff space which is not Baire.

Let $\mathcal{M}$ be a model of **ZFA+AC** and assume that the set A of atoms of $\mathcal{M}$ is countable in $\mathcal{M}$. Let A be divided into countably many disjoint countable sets:

$$A=\bigcup_{n<\omega} A_n, \qquad A_n=\{a_{nm} \mid m<\omega\}.$$

By **AC** in $\mathcal{M}$, we can take sets $<_n$ and maps φ_n such that each $\varphi_n:\langle A_n, <_n\rangle \to\langle \boldsymbol{Q}, <\rangle$ is an order isomorphism, where $\langle \boldsymbol{Q}, <\rangle$ is the usual order structure of rationals. Let $\mathcal{G}$ be the group of all those permutations of A which preserve each A_n and each order $<_n$, and which move on only finitely many A_n's. Let I be the ideal of all finite subsets of A. Let $\mathcal{F}$ be the normal filter on $\mathcal{G}$ generated by the subgroups $\text{fix}(E)$, $E\in I$. Let $\mathcal{N}$ be the permutation model determined $\mathcal{G}$ and $\mathcal{F}$. By the choice of $\mathcal{G}$, each $\langle A_n, <_n\rangle$ is in $\mathcal{N}$.

In $\mathcal{N}$, consider A_n as a topological space with the open interval topology induced by the ordered set $\langle A_n, <_n\rangle$. Then we have the

LEMMA. (a) *Every open set of A_n is a union of finitely many open intervals of $\langle A_n, <_n\rangle$.*

(b) *Every closed set of A_n is a union of finitely many intervals of the forms $(\leftarrow, a]$, $[a, b]$, $[a, \to)$.*

(c) *Every compact set of A_n is a union of finitely many closed intervals of $\langle A_n, <_n\rangle$, and vice versa.*

PROOF. Let $B\in\mathcal{N}$ be a subset of A_n and let E be a support of B. As $\pi(B)$ is determined by $\pi\mid A_n$, we may suppose $E\subset A_n$. Let

$$E=\{e_0, e_1, \cdots, e_k\}, \qquad e_0<_n e_1<_n \cdots <_n e_k .$$

If $a\in B$, then $\pi(a)\in B$ for all $\pi\in\text{fix}(E)$. If $a\in(\leftarrow, e_0)$, then $\{\pi(a)\mid \pi\in\text{fix}(E)\} =(\leftarrow, e_0)$. So, if $a\in B\cap(\leftarrow, e_0)$, then $(\leftarrow, e_0)\subset B$. Hence

$$\textit{either } (\leftarrow, e_0)\cap B=0 \textit{ or } (\leftarrow, e_0)\subset B .$$

In the same way, we can prove

$$\textit{either } (e_i, e_{i+1})\cap B=0 \textit{ or } (e_i, e_{i+1})\subset B \quad \text{for } i=0, 1, \cdots, k-1;$$

$$\textit{either } (e_k, \to)\cap B=0 \textit{ or } (e_k, \to)\subset B .$$

Thus B must be a union of some of sets:

$$(\leftarrow, e_0),\ \{e_0\},\ (e_0, e_1),\ \{e_1\},\ \cdots,\ (e_{k-1}, e_k),\ \{e_k\},\ (e_k, \rightarrow).$$

Therefore (a) and (b) hold.

Now we turn to proving (c). For each subset B of A_n, let $B^\wedge$ be the corresponding subset of reals in $\mathcal{M}$: for example

$$(a, b)^\wedge = \{r \mid \mathcal{M} \models [r \in \boldsymbol{R} \wedge \varphi_n(a) < r < \varphi_n(b)]\};$$

$$(\leftarrow, a)^\wedge = \{r \mid \mathcal{M} \models [r \in \boldsymbol{R} \wedge r < \varphi_n(a)]\};$$

$$[a, b]^\wedge = \{r \mid \mathcal{M} \models [r \in \boldsymbol{R} \wedge \varphi_n(a) \leqq r \leqq \varphi_n(b)]\};$$

if $B = (\leftarrow, b_0) \cup (b_1, b_2) \cup \cdots \cup (b_{2k-1}, b_{2k})$ with $b_0 <_n b_1 <_n b_2 <_n \cdots <_n b_{2k}$,

then $B^\wedge = (\leftarrow, b_0)^\wedge \cup (b_1, b_2)^\wedge \cup \cdots \cup (b_{2k-1}, b_{2k})^\wedge$;

if $B = [b_0, b_1] \cup \cdots \cup [b_{2k}, b_{2k+1}]$ with $b_0 \leqq_n b_1 <_n b_2 \leqq_n b_3 <_n \cdots <_n b_{2k} \leqq_n b_{2k+1}$,

then $B^\wedge = [b_0, b_1]^\wedge \cup \cdots \cup [b_{2k}, b_{2k+1}]^\wedge$; and so on.

Let B be a union of finitely many closed intervals of $\langle A_n, <_n\rangle$. Then the set $B^\wedge$ is a bounded closed subset of $\boldsymbol{R}$ in $\mathcal{M}$. Let $\{G_j \mid j \in J\}$ be an open covering of B in $\mathcal{N}$. Then $\{G_j^\wedge \mid j \in J\}$ is an open covering of $B^\wedge$ in $\mathcal{M}$. By the Heine-Borel theorem in $\mathcal{M}$, we can take a finite subcovering $\{G_j^\wedge \mid j \in J_0\}$ of B. Then $\{G_j \mid j \in J_0\}$ covers B. As J_0 is finite, $\{G_j \mid j \in J_0\}$ is in $\mathcal{N}$. Therefore B is compact in $\mathcal{N}$. This proves the reverse direction of (c). Next, to prove the forward direction of (c), assume that B is compact and that B is not a finite union of closed intervals of $\langle A_n, <_n\rangle$. Since A_n is a Hausdorff space, B is closed. By (b), B must be one of the forms:

$$(\leftarrow, e_i] \cup B' \text{ with } B' \subset [e, \rightarrow) \quad \text{for some } e >_n e_i,$$

$$B' \cup [e_i, \rightarrow) \text{ with } B' \subset (\leftarrow, e] \quad \text{for some } e <_n e_i,$$

$$A_n.$$

But then in either case, we can take explicitly an open covering of B which has no finite subcovering of B, a contradiction. Therefore (c) holds. □

By (c), every point of A_n has a compact neighbourhood, and hence A_n is locally compact. So let A_n^* be a one-point compactification of A_n:

$$A_n^* = A_n \cup \{n\}.$$

By (c) and (a), an open neighbourhood of the point n is of the form

$$(\leftarrow, a) \cup \{n\} \cup (b, \rightarrow) \cup G$$

where G is an open set of A_n.

In $\mathcal{N}$, consider the product space

$$T=\prod_{n<\omega} A_n^*$$

with the weak topology (this is possible as the sequence $\langle A_n^* | n<\omega\rangle$ is in $\mathcal{N}$). Then we have the

THEOREM 2. *In $\mathcal{N}$, T is a compact Hausdorff space which is not a Baire space.*

PROOF. In [3], we constructed a model $\mathfrak{N}$ of **ZFA**, in which the Boolean prime ideal theorem holds. The model $\mathcal{N}$ constructed above is a special case of $\mathfrak{N}$. So, in $\mathcal{N}$, the Boolean prime ideal theorem holds. The Boolean prime ideal theorem is equivalent to the Tychonoff's Theorem for compact Hausdorff spaces ([2, pp. 27-30]). Since each A_n^* is compact Hausdorff in $\mathcal{N}$, T, also, is a compact Hausdorff space in $\mathcal{N}$.

In the sequel we prove that T is not a Baire space. For each $n<\omega$ put

$$\tilde{A}_n=A_0^*\times A_1^*\times \cdots \times A_n^*\times \{n+1\}\times\{n+2\}\times \cdots .$$

Then each $\tilde{A}_n$ and the sequence $\langle \tilde{A}_n | n<\omega\rangle$ are in $\mathcal{N}$, since every $\pi\in\mathcal{G}$ preserves A_n. $\tilde{A}_n$ is closed and does not contain any nonempty open set in $\mathcal{N}$; thus $\tilde{A}_n$ is nowhere dense in $\mathcal{N}$. In order to prove that T is not Baire, it suffices to show that

(****) $$T=\bigcup_{n<\omega}\tilde{A}_n .$$

in $\mathcal{N}$. Let $f\in T$ and let E be a support of f. As E is finite there is a k such that

$$E\subset A_0\cup A_1\cup \cdots \cup A_k .$$

Assume that $n>k$ and $f(n)\neq n$. Then there is a $\pi\in \mathrm{fix}(E)$ such that $\pi(f(n))\neq f(n)$. Since E is a support of f, $\pi(f(n))=(\pi f)(\pi n)=f(n)$, this is a contradiction. So if $n>k$, then $f(n)=n$; i. e. $f\in\tilde{A}_k$. As the sequence $\langle \tilde{A}_n | n<\omega\rangle$ is in $\mathcal{N}$, (****) holds in $\mathcal{N}$. □

The truth of the assertion:

T is a compact Hausdorff space which is not Baire

is determined in the set $\mathcal{P}^{\omega_1}(A)$. So we can apply the First Embedding Theorem in [2, THEOREM 6.1, p. 85] to the above permutation model to get a **ZF** version (i. e. a symmetric model in which there is a compact Hausdorff space which is not Baire). Thus we have the

THEOREM 3. *In* **ZF** (**ZFA**), *the Baire category theorem for compact Hausdorff spaces is unprovable* (*if* **ZF** *is consistent*).

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