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# ON THE NEUMANN PROBLEM FOR SOME LINEAR HYPERBOLIC SYSTEMS OF 2ND ORDER WITH COEFFICIENTS IN SOBOLEV SPACES

By

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## Introduction.

Let $\Omega$ be a domain in an $n$-dimensional Euclidean space $\boldsymbol{R}^n$, its boundary $\Gamma$ being a $C^\infty$ and compact hypersurface. Throughout the present paper, we assume that $n \geqq 2$(1). Let $x=(x_1, \cdots, x_n)$ denote points of $\boldsymbol{R}^n$ and $t$ a time variable. For differentiations we use the symbols: $\partial_t=\partial_0=\partial/\partial t$ and $\partial_j=\partial/\partial x_j$ $(j=1, \cdots, n)$. In this paper, we consider the following mixed problem:

$$\text{(N)} \quad \begin{cases} P(t)[\vec{u}(t)]=\partial_t^2\vec{u}(t)-\partial_i(A^{i0}(t)\partial_t\vec{u}(t)+A^{ij}(t)\partial_j\vec{u}(t))=\vec{f}_\Omega(t) & \text{in } (0, T)\times\Omega, \\ Q(t)[\vec{u}(t)]=\nu_i A^{ij}(t)\partial_j\vec{u}(t)+B^j(t)\partial_j\vec{u}(t)+B^0(t)\partial_t\vec{u}(t)=\vec{f}_\Gamma(t) & \text{on } (0, T)\times\Gamma, \\ \vec{u}(0)=\vec{u}_0, \qquad \partial_t\vec{u}(0)=\vec{u}_1 & \text{in } \Omega, \end{cases}$$

where $T$ is a positive constant and $\vec{u}={}^t(u_1, \cdots, u_m)$ (=the row vector of length $m$ and ${}^tM$ means the transposed vector (resp. matrix) of the vector (resp. matrix) $M$). Here and hereafter, the summation convention is understood such as the sub and superscripts $i, i', j, j'$ (resp. $p, q$) take all values 1 to $n$ (resp. 1 to $n-1$). For any vector valued function $\vec{u}={}^t(u_1, \cdots, u_m)$, we put $\partial_t^j\partial_x^\alpha\vec{u}={}^t(\partial_t^j\partial_x^\alpha u_i$ $\cdots, \partial_t^j\partial_x^\alpha u_m)$. The $\nu_i=\nu_i(x)$ are real-valued functions in $C_0^\infty(\boldsymbol{R}^n)$ such that the vector $\nu(x)=(\nu_1(x), \cdots, \nu_n(x))$ represents the unit outer normal to $\Gamma$ at $x\in\Gamma$. In the present paper, functions are assumed to be real-valued, unless ortherwise specified. Below, $I$ will always refer to the closed interval containing $[0, T]$ strictly, say, $I=[-\tau, T+\tau]$ $(\tau>0)$. And also, $K$ will always refer to the fixed integer $\geqq[n/2]+2$, which represents the order of regularitiy of solutions and coefficients of the operators $P(t)$ and $Q(t)$. The $A^{il}(t)=A^{il}(t, x)$ and $B^l(t)=B^l(t, x)$ $(l=0, 1, \cdots, n; i=1, \cdots, n)$ are $m\times m$ matrices of functions satisfying the

(1) When $n=1$, excepting the notations, we can treat the same problem without essential change. However, for the notational simplicity, we shall only treat the case where $n\geqq 2$, below.

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following five assumptions,

(A.1)$_I$ The $A^{il}$ are decomposed as follows: $A^{il}=A^{il}_\infty+A^{il}_S$ where $A^{il}_\infty\in\mathscr{B}^K(I\times\bar\Omega)$ and $A^{il}_S\in Y^{K-1,1}(I,\Omega)$; the $B^l\in Y^{K-1,1/2}(I,\Gamma)$.

Here, we should explain the notations for some function spaces used in the present paper. Let $\mathscr{B}^K(G)$ be the set of all $v\in C^K(G)$ such that $v$ and all derivatives of $v$ up to $K$ are everywhere bounded in $G$. For any time interval $J$ and Hilbert space $X$, $C^l(J,X)$, $L^\infty(J,X)$ and $\mathrm{Lip}(J,X)$ denote the sets of all $X$-valued functions which are $l$-times continuously differentiable in $J$, measurable and bounded *everywhere* in $J$ and Lipschitz continuous in $J$, in the sense of the strong topology of $X$, respectively. Since $X$ is a Hilbert space, if $u(t)\in\mathrm{Lip}(J,X)$, then the strong derivative of $u(t)$ exists almost everywhere. Usually, $L^\infty$-functions mean the measurable and almost everywhere bounded ones. However, to make may proofs as short as possible the functions are assumed to be bounded *everywhere* in the definition of $L^\infty$-functions. Let $H^r(G)$ denote the usual Sobolev space over $G$ of order $r\in\boldsymbol{R}$ defined exactly in the section of Notations below. Put

$$X^{l,r}(J,G)=\bigcap_{k=0}^{l}C^k(J,H^{l+r-k}(G))\,;\ Y^{0,r}(J,G)=L^\infty(J,H^r(G))\,;$$

$$Y^{l+1,r}(J,G)=\{u\in X^{l,r}(G)\,|\,\partial_t^j u(t)\in L^\infty(J,H^{l+1+r-j}(G))\cap\mathrm{Lip}\,(J,H^{l+r-j}(G))\quad\text{for } 0\leqq j\leqq l\}.$$

For any function space $S$, we denote a product space $S\times\cdots\times S$ by also $S$.

The second and third assumptions are the following.

(A.2)$_I$ $\quad {}^tA^{i0}=A^{i0}$ and ${}^tA^{ij}=A^{ji}$ on $I\times\bar\Omega$;

$\quad {}^tB^0=B^0$ and ${}^tB^i+B^i=0$ on $I\times\Gamma$ $(i,j=1,\cdots,n)$.

(A.3)$_{I,\delta}$ There exist positive constants $\delta_1$ and $\delta_2$ such that

$$(A^{ij}(t)\partial_j\vec v,\partial_j\vec v)+\langle B^j(t)\partial_j\vec v,\vec v\rangle\geqq\delta_1\|\vec v\|_1^2-\delta_2\|\vec v\|_0^2$$
$$\text{for any } t\in I \text{ and } \vec v\in H^2(\Omega).$$

Here and hereafter, we use the following notations:

$$(u,v)=\int_\Omega\vec u(x)\cdot\vec v(x)dx\,;\ \langle\vec u,\vec v\rangle=\int_\Gamma\vec u(x)\cdot\vec v(x)d\Gamma\,;\ \|\vec u\|_L^2=\sum_{|\alpha|\leq L}(\partial_x^\alpha\vec u,\partial_x^\alpha\vec u)$$

where "$\cdot$" denotes the usual innerproduct of $\boldsymbol{R}^m$ and $d\Gamma$ is the surface element of $\Gamma$.

The fourth and final assumptions are the following.

$(\mathrm{A}.4)_I \qquad \nu_i(x)B^i(t, x)=0 \qquad \text{for all } (t, x)\in I\times\Gamma,$

$(\mathrm{A}.5)_I \qquad (-\nu_i(x)A^{i0}(t, x)+2B^0(t, x))\eta\cdot\eta\leqq 0 \quad \text{for all } (t, x)\in I\times\Gamma \text{ and } \eta\in\boldsymbol{R}^m.$

It is essential that all the assumptions are valid on whole $I$ containing $[0, T]$ strictly. Because, in proving our main results, we use the results obtained by Shibata [9]. In that proof, it was used essentially that the coefficients are defined on some closed interval $I$ containing $[0, T]$ strictly and the assumptions (A.2)-(A.5) are valid on whole $I$ with respect to $t$. Below, if no subscripts occur on the numbers of assumptions, (A.3) and (A.N) are understood to be $(\mathrm{A}.3)_{I,\delta}$ and $(\mathrm{A.N})_I$ ($N=1, 2, 4$ and $5$), respectively. In fact, excepting Theorems 2.1, 2.2 and 5.3 and Lemma 2.3, we always state that (A.1)-(A.5) are valid.

The reason why we must consider (N) under the assumptions (A.1)-(A.5), especially (A.1), is the following: When we solve the Neumann problem for the nonlinear hyperbolic system of 2nd order, as the linearized problem, we meet the present problem. And, the key of solving the nonlinear problem lies in proving the unique existence theorem of solutions to (N) and sharp energy inequalities stated in Theorems 1.2 and 1.3 of § 1 below. Of course, such linear systems have their own interests. And also, in proving main results, we need some new technique which can be applied to treating many other problems, for example, Schrödinger equations, heat equations and so on.

T. Kato [4] treated the same linear problem in his abstract frame work and applied his linear theory to solving the Neumann problem for nonlinear hyperbolic systems of 2nd order, which was first done by Shibata [8] and Shibata and Nakamura [10]. Especially, the result due to Kato [4] attained some improvements of that due to [8] and [10] regarding the minimum order of the Sobolev space in the solutions to the nonlinear problem exist. But, Kato [4, § 14] treated only the case where nonlinear functions do not contain $t$ and $\partial_t\vec{u}$. But using the results on the linear theory in the present paper, Shibata and Kikuchi [11] got the same improvements as in Kato [4] in the case where nonlinear functions do contain $t$ and $\partial_t\vec{u}$. Our proof is elementary and completely different from Kato's one. The advantage of our approach is that the assumptions: $\partial_t^{L-1}\vec{f}_\Omega(t)\in\mathrm{Lip}([0, T), H^{-1}(\Omega))$ and $\partial_t^{L-1}\vec{f}_\Gamma(t)\in\mathrm{Lip}([0, T), H^{-1/2}(\Gamma))$ are not needed, while it seems that these assumptions are essential in the Kato's approach (cf. Theorem 1.2 below and [4, Theorem 12.4]); that some hyperbolic-parabolic coupled systems of 2nd order containing the thermoelastodynamic as an important physical example can be treated in the same manner as in the present paper.

In solving the nonlinear problem, if we know the unique existence theorem to (N) under the assumption (A.1), especially, the coefficients are Lipschits continuous (not in $C^1$) in $t$, it is very easy to show the regularity of solutions to the nonlinear problem. One can find this approach since Kato's Cortona Lecture [3].

Our idea of proving the existence of a solution $\vec{u}\in X^{2,0}$ to (N) is as follows. First, approximating the coefficients of the operators $P(t)$ and $Q(t)$ by smooth functions and using the existence theorem in the case of the operators with smooth coefficients, which was obtained by Shibata [9], we can prove the existence of a solution $\vec{u}$ in $Y^{2,0}$. Our main task is to prove that $\vec{u}\in X^{2,0}$, i.e., the continuity of second derivatives of $\vec{u}$ in $t$. To prove this, we use the idea due to Ikawa [2] (originally goes back to Mizohata's work on the Dirichlet problem in 1966). Namely, we mollify $\vec{u}$ with respect to $t$ by Friedrichs' method and prove that the sequence of mollified functions converges to $\vec{u}$ uniformly in $t$. The key of proving the convergence lies in obtaining the right continuity of the second derivatives of $\vec{u}$ at $t=0$. By employing the arguments due to Majda [5, pp. 44], we can get this right continuity.

Our idea of proving the further regularities of solutions in $X^{2,0}$ to (N) is the following. Differentiate (N) $l$ times $(0\leqq l\leqq K-2)$ in $t$ formally and consider the resulting equations as the $K-1$ systems: $\partial_t^l\{P(t)[\vec{u}(t)\}=\partial_t^l\vec{f}_\Omega(t)$ with boundary conditions: $\partial_t^l\{Q(t)[\vec{u}(t)]\}=\partial_t^l\vec{f}_\Gamma(t)$ $(l=0, 1, \cdots, K-2)$ for unknowns $\vec{u}, \partial_t\vec{u}, \cdots, \partial_t^{K-2}\vec{u}$. The system: $\partial_t^{K-2}\{P(t)[\vec{u}(t)]\}=\partial_t^{K-2}\vec{f}_\Omega(t)$ with boundary condition: $\partial_t^{K-2}\{Q(t)[\vec{u}(t)]\}=\partial_t^{K-2}\vec{f}_\Gamma(t)$ can be regarded as a hyperbolic system for unknown $\partial_t^{K-2}\vec{u}(t)$, and other equations can be regarded as an elliptic system for unknowns $\vec{u}, \cdots, \partial_t^{K-3}\vec{u}$. These systems forms a "hyperbolic-elliptic" system. With the help of the existence theorems obtained in §§ 2.3 and 5, we can solve this system by the method of successive approximations. And then, we can prove that $\vec{u}\in X^{K,0}$. It is first for Shibata [8] and Shibata-Nakamura [10] to treat such a "hyperbolic-elliptic" system. Kato also treated this system in his abstract frame work.

**Notations.**

Now, we shall explain our basic notations. To denote differentiations of higher order, we use the symbols:

$$\bar{D}^L\bar{D}_x^M\vec{u}=(\partial_t^j\partial_x^\alpha\vec{u}\ ;\ j+|\alpha|\leqq L+M,\ j\leqq L)\ ;\ \bar{D}^L\bar{D}_x^0\vec{u}=\bar{D}^L\vec{u}\ ;\ \bar{D}^0\bar{D}_x^M\vec{u}=\bar{D}_x^M\vec{u}\ .$$

For any $r\in\boldsymbol{R}$, we put

$$\|v\|_{R^n,r}^2=\int_{R^n}|\hat{v}(\xi)|^2(1+|\xi|^2)^r d\xi\,;\; H^r(\boldsymbol{R}^n)=\{v\in\mathcal{S}'\,|\,\|v\|_{R^n,r}<\infty\}$$

where $\hat{v}$ is the Fourier transform of $v$. For any domain $G\subset\boldsymbol{R}^n$, we put

$$H^r(G)=\{u\,|\,u(x)=U(x) \text{ in } G \text{ for some } U\in H^r(\boldsymbol{R}^n)\}\,;$$
$$\|u\|_{G,r}=\inf\,\{\|U\|_{R^n,r}\,|\,u=U \text{ on } G\}.$$

As is well-known, if $r$ is a non-negative integer and $G=\boldsymbol{R}^n$, $\boldsymbol{R}_+^n$ or $\Omega$, then $\|v\|_{G,r}$ is equivalent to the usual norm:

$$\sum_{|\alpha|\leq r}\int_G|\partial_x^\alpha v(x)|^2dx\,,$$

where $\boldsymbol{R}_+^n=\{x=(x_1,\cdots,x_n)\,|\,x_n>0\}$. For the notational simplicity, we use the abbreviation: $\|v\|_r=\|v\|_{\Omega,r}$. For any integer $l\geqq 0$ and $\sigma\in(0,1)$, put $\mathscr{B}^{l+\sigma}(\bar{G})=\{v\in\mathscr{B}^l(\bar{G})\,|\,|v|_{\infty,l+\sigma,G}<\infty\}$, where

$$|v|_{\infty,l,G}=\sum_{|\alpha|\leq l}\sup_{x\in G}|\partial_x^\alpha v(x)|\,;$$
$$|v|_{\infty,l+\sigma,G}=|v|_{\infty,l,G}+\sum_{|\alpha|=l}\sup\{|\partial_x^\alpha v(x)-\partial_x^\alpha v(y)|\,|x-y|^{-\sigma}\,|\,x,\,y\in G,\,x\neq y\}.$$

Especially, we write $\|\cdot\|_{\infty,l+\sigma}=|\cdot|_{\infty,l+\sigma,\Omega}$ and $|\cdot|_{\infty,l+\sigma,I}=|\cdot|_{\infty,l+\sigma,I\times\Omega}$ $(0\leqq\sigma<1)$.

Since $\Gamma$ is a $C^\infty$ and compact hypersurface, we may assume that there exist finite number of open sets $\mathcal{O}_l$ in $\boldsymbol{R}^n$, $\omega_l$ in $\boldsymbol{R}^{n-1}$, $\rho_l\in C^\infty(\omega_l)$ and integers $d(l)\in[1,n]$ $(l=1,\cdots,N_0)$ such that $\mathcal{O}_l\cap\Gamma=\{x_{d(l)}=\rho_l(x') \text{ for } x'\in\omega_l\}$ and $\mathcal{O}_l\cap\Omega=\{x_{d(l)}>\rho_l(x') \text{ for } x'\in\omega_l\}$ where $x'=(x_1,\cdots,x_{d(l)-1},x_{d(l)+1},\cdots,x_n)$. Let us define $\Phi_{lk}(x)$, $k=1,\cdots,n$, $l=1,\cdots,N_0$ as follows: $\Phi_{lk}(x)=x_k$ for $1\leqq k\leqq d(l)-1$; $\Phi_{lk}(x)=x_{k+1}$ for $d(l)\leqq k\leqq n-1$; $\Phi_{ln}(x)=x_{d(l)}-\rho_l(x')$. Then, we may assume that $\Phi_l(x)=(\Phi_{l1}(x),\cdots,\Phi_{ln}(x))$ are $C^\infty$-diffeomorphisms of $\mathcal{O}_l$ onto $Q(\sigma_l)=\{y=(y_1,\cdots,y_n)\in\boldsymbol{R}^n\,|\,|y'|=|(y_1,\cdots,y_{n-1})|<\sigma_l,\,|y_n|<\sigma_l\}$ such that $\Phi_l(\mathcal{O}_l\cap\Omega)=Q_+(\sigma_l)=\{y\in Q(\sigma_l)\,|\,y_n>0\}$ and $\Phi_l(\mathcal{O}_l\cap\Gamma)=\{y\in Q(\sigma_l)\,|\,y_n=0\}$. There will be no confusion as to whether $Q(\cdot)$ denotes the boundary operator or the set defined just now, because this will always be clear from the context. Note that the Jacobian of the change of variables: $y=\Phi_l(x)$ is equal to 1, i.e., $dx=dy$. Let $\Psi_l$ be the inverse map of the $\Phi_l$. Let $\Phi_k$ and $\phi_k'$ $(k=0,1,\cdots,N_0)$ be functions in $C_0^\infty(\boldsymbol{R}^n)$ having the following properties:

(No. 1) $\operatorname{supp}\phi_0\subset\operatorname{supp}\phi_0'\subset\Omega$; $\operatorname{supp}\phi_l\subset\operatorname{supp}\phi_l'\subset\mathcal{O}_l$ for $l=1,\cdots,N_0$;

$$\sum_{k=0}^{N_0}\phi_k(x)^2=1 \quad\text{and}\quad \sum_{k=l}^{N_0}\phi_k'(x)=1 \qquad \text{on } \bar{\Omega}\,.$$

Put

$$\langle\!\langle v\rangle\!\rangle_r^2=\sum_{k=1}^{N_0}\|v_k\|_{R^{n-1},r}^2 \qquad \text{where } v_k=v(\Psi_k(y',0))\phi_k(\Psi_k(y',0)).$$

Note that there exists a constant $C>0$ such that

$$C^{-1}\langle\!\langle v\rangle\!\rangle_0^2\leqq\int_\Gamma|v(x)|^2 d\Gamma\leqq C\langle\!\langle v\rangle\!\rangle_0^2,$$

and that each $H^r(\Gamma)$ is a Hilbert space equipped with norm: $\langle\!\langle\cdot\rangle\!\rangle_r$. For any functions $v(x)$ and $w(t, x)$ defined on $\Gamma$ and $I\times\Gamma$, we put

$$\langle\!\langle v\rangle\!\rangle_{\infty,l+\sigma}=\sum_{k=1}^{N_0}|v_k|_{\infty,l+\sigma,Q_+(\sigma_k)}\,;\ \langle w\rangle_{\infty,l+\sigma,I}=\sum_{k=1}^{N_0}|w_k|_{\infty,l+\sigma,I\times Q_+(\sigma_k)},$$

where $v_k=v(\Psi_k(y',0))\phi_k'(\Psi_k(y',0))$ and $w_k=w(t,\Psi_k(y',0))\phi_k'(\Psi_k(y',0))$.

Now, let us define the norms of $X^{L,r}(J,G)$ and $Y^{L,r}(J,G)$. Put

$$|v|_{0,r,J,G}=\sup_{t\in J}\|v(t)\|_{G,r}\,;$$

$$|v|_{L,r,J,G}=|v|_{0,L+r,J,G}+\sum_{k=0}^{L-1}\sup_{\substack{t,s\in J\\ t\neq s}}\frac{\|\partial_t^k v(t)-\partial_t^k v(s)\|_{G,L+r-1-k}}{|t-s|}\quad\text{for } L\geqq 1.$$

Let us use $|\cdot|_{L,r,J,G}$ as the norms of both $X^{L,r}(J,G)$ and $Y^{L,r}(J,G)$. If $v\in Y^{L,r}(J,G)$, from the definition of the derivatives, we have

(No. 2.a) $\|\partial_t^k v(t)\|_{G,L+r-k}\leqq|v|_{L,r,J,G}$ for almost all $t\in J$ and $1\leqq k\leqq L$.

If $v\in X^{L,r}(J,G)$, obviously we have

(No. 2.b) $|v|_{L,r,J,G}=\sum_{k=0}^{L}\sup_{t\in J}\|\partial_t^k v(t)\|_{G,L+r-k}$.

Put $|v|_{L,r,J}=|v|_{L,r,J,Q}$ and $\langle v\rangle_{L,r,J}=|v|_{L,r,J,\Gamma}$. Let us use the same notations to denote various norms of vector or matrix valued functions.

For the operators $P(t)$ and $Q(t)$ we use the following notations:

(No. 3.a) $[P(t)]_{\infty,L}=\sum_{l=0}^{n}\sum_{i=1}^{n}\sum_{k=0}^{L}\|\partial_t^k A_\infty^{il}(t)\|_{\infty,L-k}\,;$

(No. 3.b) $[P(t)|Q(t)]_{S,L,M}=\sum_{l=0}^{n}\sum_{k=0}^{L}\{\sum_{i=1}^{n}\|\partial_t^k A_S^{il}(t)\|_{L+M-k}+\langle\!\langle\partial_t^k B^l(t)\rangle\!\rangle_{L+M-k-(1/2)}\}$.

Let $M_\infty(K)$ and $M_S(K)$ be constants such that

(No. 3.c) $\sum_{i,j=1}^{n}|A_\infty^{ij}|_{\infty,K,I}\leqq M_\infty(K)\,;$

(No. 3.d) $\sum_{l=0}^{n}\{\sum_{i=1}^{n}|A_S^{il}|_{K-1,1,I}+\langle B^l\rangle_{K-1,1/2,I}\}\leqq M_S(K)$.

We use the same letter $C$ to denote different constants depending on the same set of arguments. $C=C(\cdots)$ denotes a constant depending essentially on the quantities appearing in the parentheses.

Now, let us prepare some notations to define the first energy norm of (N). Put

$$\partial'_j=\partial/\partial y_j,\ \psi_l(y)=\phi_l(\Psi_l(y)),\ Y^i_{jl}(y)=(\partial\Phi_{li}/\partial x_j)(\Psi_l(y)),$$

$$J_l(y')=\{\sum_{i=1}^{n}Y^n_{il}(y',0)^2\}^{1/2}.$$

Note that

(No. 4) $d\Gamma=J_l(y')dy'$ ; $\nu_i(x)=-Y^n_{il}(y',0)/J_l(y')$

for $x=\Psi_l(y',0)\in\mathcal{O}_l\cap\Gamma$. Since

(No. 5) $B^j(t,\Psi_l(y',0))Y^n_{jl}(y',0)=0$

as follows from (A.4) and (No. 4), we can write

$$\langle B^j(t)\partial_j\vec{u},\vec{v}\rangle$$
$$=-\sum_{l=1}^{N_0}\int_{R^n_+}\partial'_n\{\psi^2_l(y)B^j(t,\Psi_l(y',0))Y^p_{jl}(y',0)\partial'_p\vec{u}(\Psi_l(y))\cdot\vec{v}(\Psi_l(y))\}J_l(y')dy$$

If we put

(No. 6) $Q^p_l(t,y')=B^j(t,\Psi_l(y',0))Y^p_{jl}(y',0)J_l(y'),\quad p=1,\cdots,n-1;$

(No. 7) $$\mathcal{B}(t,\vec{u},\vec{v})=\sum_{l=1}^{N_0}\int_{R^n_+}\psi^2_l(y)\{Q^p_l(t,y')\partial'_n\vec{u}(\Psi_l(y))\cdot\partial'_p\vec{v}(\Psi_l(y))$$
$$Q^p_l(t,y')\partial'_p\vec{u}(\Psi_l(y))\cdot\partial'_n\vec{v}(\Psi_l(y))\}dy;$$

(No. 8) $$\mathcal{C}(t,\vec{u},\vec{v})=\sum_{l=1}^{N_0}\int[\{\partial'_p(\psi^2_l(y)Q^p_l(t,y'))\}\partial'_n\vec{u}(\Psi_l(y))\cdot\vec{v}(\Psi_l(y))$$
$$\{\partial'_n\psi^2_l(y)\}Q^p_l(t,y')\partial'_p\vec{u}(\Psi_l(y))\cdot\vec{v}(\Psi_l(y))]dy,$$

then by integration by parts with respect to $y_p$ ($p=1,\cdots,n-1$), we have

(No. 9) $\langle B^j(t)\partial_j\vec{u},\vec{v}\rangle=\mathcal{B}(t,\vec{u},\vec{v})+\mathcal{C}(t,\vec{u},\vec{v})$ for any $\vec{u}\in H^2(\Omega)$ and $\vec{v}\in H^1(\Omega)$.

By the assumption: ${}^tB^j+B^j=0$ on $I\times\Gamma$, we see that

(No. 10) $\mathcal{B}(t,\vec{u},\vec{v})=\mathcal{B}(t,\vec{v},\vec{u})$.

Furthermore, we have

(No. 11) $|\mathcal{B}(t,\vec{u},\vec{v})|\leqq CM_S(K)\|\vec{u}\|_1\|\vec{v}\|_1$;

(No. 12) $|\mathcal{C}(t,\vec{u},\vec{v})|\leqq CM_S(K)\|\vec{u}\|_1\|\vec{v}\|_0$

for all $t\in I$. In fact, since $(n-1)/2<K-(3/2)$ and the dimension of $\Gamma$ is $n-1$, we have

(No. 13.a) $\langle\!\langle A\rangle\!\rangle_{\infty,0}\leqq C\langle\!\langle A\rangle\!\rangle_{K-(3/2)}$ for any $A\in H^{K-(3/2)}(\Gamma)$.

By (No. 13.a) and (A.1) we see that $\langle\!\langle B^j(t)\rangle\!\rangle_{\infty,1}\leqq CM_S(K)$ for $j=1,\cdots,n$ and $t\in I$. Noting this and applying Schwarz's inequality to (No. 7) and (No. 8), we have (No. 11) and (No. 12). For the further references, we give the following inequality:

(No. 13.b) $\|A\|_{\infty,0}\leqq C\|A\|_{K-1}$ for any $A\in H^{K-1}(\Omega)$;

This follows from the assumption: $K\geqq[n/2]+2$ and Sobolev's imbedding Theorem, too. Put

(No. 14) $B_\lambda[t, \vec{u}, \vec{v}]=(A^{ij}(t)\partial_j\vec{u}, \partial_i\vec{v})+\mathscr{B}(t, \vec{u}, \vec{v})+\mathcal{C}(t, \vec{u}, \vec{v})+\lambda(\vec{u}, \vec{v})$.

In view of (No. 11), (No. 12) and (No. 13.b), we have

(No. 15) $|B_\lambda[t, \vec{u}, \vec{v}]|\leqq[C\{M_\infty(K)+M_S(K)\}+|\lambda|]\|\vec{u}\|_1\|\vec{v}\|_1$,

which implies that $B_\lambda$ is a continuous bilinear form on $H^1(\Omega)\times H^1(\Omega)$. Since $H^2(\Omega)$ is dense in $H^1(\Omega)$, by (No. 9) and (A.3) we have

(No. 16) $B_\lambda[t, \vec{u}, \vec{u}]\geqq\delta_1\|\vec{u}\|_1^2$ provided that $\lambda\geqq\delta_2$.

Furthermore, since $|\mathcal{C}(t, \vec{u}, \vec{u})|\leqq(\delta_1/2)\|\vec{u}\|_1^2+\{(CM_S(K))^2/2\delta_1\}\|\vec{u}\|_0^2$ as follows from (No. 12), by (No. 15) and (No. 16) we have

(No. 17) $(\delta_1/2)\|\vec{u}\|_1^2\leqq|||\vec{u}|||_{1,t}^2\leqq c_1\|\vec{u}\|_1^2$ for any $\vec{u}\in H^1(\Omega)$ and $t\in I$,

where $c_1=C(M_\infty(K), M_S(K), \delta_1)$;

(No. 18) $|||\vec{u}|||_{1,t}^2=B_{\delta_0}[t, \vec{u}, \vec{u}]-\mathcal{C}(t, \vec{u}, \vec{u})$;

(No. 19) $\delta_0=\delta_2+(CM_S(K))^2/2\delta_1$.

In view of (A.2), (No. 10) and (No. 17), $H^1(\Omega)$ is a Hilbert space equipped with norm: $|||\cdot|||_{1,t}$ and the norms $\|\cdot\|_1$ and $|||\cdot|||_{1,t}$ are equivalent for any $t\in I$. Since

$$\langle\!\langle B^j(t)-B^j(s)\rangle\!\rangle_{\infty,0}\leqq C\langle\!\langle B^j(t)-B^j(s)\rangle\!\rangle_{K-(3/2)}\leqq CM_S(K)|t-s|\,;$$

$$\|A_S^{ij}(t)-A_S^{ij}(s)\|_{\infty,0}\leqq C\|A_S^{ij}(t)-A_S^{ij}(s)\|_{K-1}\leqq CM_S(K)|t-s|$$

as follows from (No. 3) and (No. 13), and since

$$\|A_\infty^{ij}(t)-A_\infty^{ij}(s)\|_{\infty,0}\leqq CM_\infty(K)|t-s|$$

as follows from the mean value theorem, we have

(No. 20) $||||\vec{u}|||_{1,t}^2-|||\vec{u}|||_{1,s}^2|\leqq C\{M_\infty(K)+M_S(K)\}|t-s|\|\vec{u}\|_1^2$

for any $\vec{u}\in H^1(\Omega)$ and $t, s\in I$.

Now, let us define the energy norm $E(t, \vec{u}(s))$ for the operators $P(t)$ and $Q(t)$ by

(No. 21) $E(t, \vec{u}(s))=\|\partial_t\vec{u}(s)\|_0^2+|||\vec{u}(s)|||_{1,t}^2$ for any $t\in I$, $\vec{u}(s)\in X^{1,0}(J, \Omega)$.

By (No. 17) we see that there exists a $c_2=C(\delta_1, \delta_2, M_\infty(K), M_S(K))$ such that

(No. 22) $c_2^{-1}E(t, \vec{u}(s))\leqq\|\bar{D}^1\vec{u}(s)\|_0^2\leqq c_2E(t, \vec{u}(s))$

for any $\vec{u}(s)\in X^{1,0}(J, \Omega)$ and $t\in I$. In view of (No. 20), we have

(No. 23) $|E(t, \vec{u}(r))-E(s, \vec{u}(r))|\leqq C\{M_\infty(K)+M_S(K)\}|t-s|\,\|\vec{u}(r)\|_1^2$

for any $t$, $s\in I$ and $\vec{u}(r)\in X^{1,0}(J, \Omega)$.

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## §1. Compatibility condition and statements of main results.

First, we shall define the compatibility condition for (N). To do this, we define $\vec{u}_M=\vec{u}_M(x)$ $(2\leqq M\leqq L\leqq K)$ successively by the following formula:

$$\vec{u}_M=\partial_t^{M-2}\vec{f}_\Omega(0)-\sum_{l=0}^{M-2}\binom{M-2}{l}\partial_i\{\partial_t^lA^{i0}(0)\vec{u}_{M-1-l}+A^{ij}(0)\partial_j\vec{u}_{M-2-l}\}. \tag{1.1}$$

If $\vec{u}\in X^{L,0}([0, T), \Omega)$ is a solution to (N), noting (Ap. 14), we see that $\partial_t^M\vec{u}(0)=\vec{u}_M$. Here and hereafter, (Ap. N), Theorem Ap. N′ and Corollary Ap. N″ (N 1-18; N′=1, 2, 4 and 6; N″=4, 5, 7 and 8) can be found in Appendix below. We begin with

LEMMA 1.1. *Assume that* (A.1) *is valid. Let $L$ be an integer* $\in[2, K]$. *If* $\vec{u}_0\in H^L(\Omega)$, $\vec{u}_1\in H^{L-1}(\Omega)$ *and* $\bar{D}^{L-2}\vec{f}_\Omega(0)\in L^2(\Omega)$, *then* $\vec{u}_M\in H^{L-M}(\Omega)$ *for* $0\leqq M\leqq L$.

PROOF. By induction on $M$ we prove the lemma. Assume that $\vec{u}_k\in H^{L-k}(\Omega)$ for $0\leqq k\leqq M-1$. Let $0\leqq l\leqq M-2$. Applying (Ap. 1) with $\alpha=K-l-1$, $\beta=L-M+l+1$ and $\gamma=L-M+1$, we have

$$\|\partial_i(\partial_t^l A_S^{i0}(0)\vec{u}_{M-1-l})\|_{L-M}+\|\partial_i(\partial_t^l A_S^{ij}(0)\partial_j\vec{u}_{M-2-l})\|_{L-M}$$

$$\leqq C\Big\{\sum_{i=1}^n\|\partial_t^l A_S^{i0}(0)\|_{K-1-l}\|\vec{u}_{M-1-l}\|_{L-(M-l-1)}$$

$$+\sum_{i,j=1}^n\|\partial_t^l A_S^{ij}(0)\|_{K-1-l}\|\vec{u}_{M-2-l}\|_{L(M-l-2)}\Big\}.$$

Since $\partial_t^l A_S^{ij}(0)\in H^{K-1-l}(\Omega)$ for $0\leqq l\leqq K-2$ as follows from (A.1), we see easily that $\vec{u}_M\in H^{L-M}(\Omega)$, which completes the proof.

If $\vec{u}(t)\in X^{L,0}([0, T); \Omega)$ $(2\leqq L\leqq K)$ is a solution to (N), in view of (Ap. 14), we have that $\partial_t^N\{Q(t)[\vec{u}(t)]\}|_{t=0}=\partial_t^N\vec{f}_\Gamma(0)$ on $\Gamma$ for $0\leqq N\leqq L-2$. Keeping this in mind, let us define the compatibility condition of order $L-2$ to (N) as follows: We say that the data $\vec{u}_0\in H^L(\Omega)$, $\vec{u}_1\in H^{L-1}(\Omega)$, $\vec{f}_\Omega\in X^{L-2,0}([0, T), \Omega)$ and $\vec{f}_\Gamma\in X^{L-2,1/2}([0, T), \Gamma)$ satisfy the *compatibility condition of order* $L-2$ if the equalities:

$$\sum_{l=0}^N\binom{N}{l}\{\nu_i\partial_t^l A^{ij}(0)\partial_j\vec{u}_{N-l}+\partial_t^l B^0(0)\partial_j\vec{u}_{N-l}+\partial_t^l B^0(0)\vec{u}_{N+1-l}\}=\partial_t^N\vec{f}_\Gamma(0) \tag{1.2}$$

hold on $\Gamma$ for all $N\in[0, L-2]$. For the sake of simplicity, by $D^L(J)$ let us denote the set of all systems $(\vec{u}_0, \vec{u}_1, \vec{f}_\Omega, \vec{f}_\Gamma)$ of data for (N) satisfying the conditions:

(1.3.a) $\vec{u}_0\in H^L(\Omega)$; $\vec{u}_1\in H^{L-1}(\Omega)$; $\vec{f}_\Omega\in X^{L-2,0}(J, \Omega)$; $\vec{f}_\Gamma\in X^{L-2,1/2}(J, \Gamma)$;

(1.3.b) $\partial_t^{L-2}\vec{f}_\Omega\in \mathrm{Lip}(J, L^1(\Omega))$; $\partial_t^{L-2}\vec{f}_\Gamma\in \mathrm{Lip}(J, H^{1/2}(\Gamma))$;

(1.3.c) $\vec{u}_0$, $\vec{u}_1$, $\vec{f}_\Omega$ and $\vec{f}_\Gamma$ satisfy the compatibility condition of order $L-2$ to (N)

where $J$ is a time interval containing 0 and contained in $I$.

Our main purpose of this paper is to prove the following two theorems.

THEOREM 1.2. *Assume that* (A.1)-(A.5) *are valid. Let $L$ be an integer* $\in[2, K]$. *Then, for a given system* $(\vec{u}_0, \vec{u}_1, \vec{f}_\Omega, \vec{f}_\Gamma)\in D^L([0, T))$ *of data for* (N), (N) *admits a unique solution* $\vec{u}\in X^{L,0}([0, T), \Omega)$.

THEOREM 1.3. *Assume that* (A.1)-(A.5) *are valid. Let $L$ be an integer*

$\in[2, K]$ *and* $\vec{u}\in X^{L,0}([0, T), \Omega)$. *Put* $\vec{f}_{\Omega}(t)=P(t)[\vec{u}(t)]$ *and* $\vec{f}_{\Gamma}(t)=Q(t)[\vec{u}(t)]$. *Assume that*

(1.4) $\quad \partial_t^{L-2}\vec{f}_{\Omega}\in \mathrm{Lip}([0, T), L^2(\Omega))$ *and* $\partial_t^{L-2}\vec{f}_{\Gamma}\in \mathrm{Lip}([0, T), H^{1/2}(\Gamma))$.

*Then, there exists a constant* $C(T)=C(T, \delta_1, \delta_2, L, T, \Gamma, M_{\infty}(K), M_S(K))$ *such that the following two inequalities are valid for any* $t\in[0, T)$:

(a) $$\|\bar{D}^L\vec{u}(t)\|_0^2\leqq C(T)\Big\{\|\bar{D}^L\vec{u}(0)\|_0^2+|\vec{f}_{\Omega}|_{L-2,0,[0,t]}^2+\langle\vec{f}_{\Gamma}\rangle_{L-2,1/2,[0,t]}^2 + \int_0^t(\|\partial_t^{L-1}\vec{f}_{\Omega}(s)\|_0^2+\langle\!\langle\partial_t^{L-1}\vec{f}_{\Gamma}(s)\rangle\!\rangle_{1/2}^2)ds\Big\};$$

(b) $$E(t, \partial_t^{L-1}\vec{u}(t))\leqq e^{C(T)t}\Big[E(0, \partial_t^{L-1}\vec{u}(0))+C(T)\Big\{\|\bar{D}^L\vec{u}(0)\|_0^2+|\vec{f}_{\Omega}|_{L-2,0,[0,t]}^2 +\langle\vec{f}_{\Gamma}\rangle_{L-2,1/2,[0,t]}^2+\int_0^t(\|\partial_s^{L-1}\vec{f}_{\Omega}(s)\|_0^2+\langle\!\langle\partial_s^{L-1}\vec{f}_{\Gamma}(s)\rangle\!\rangle_{1/2}^2\Big\}^{1/2}\Big\{t(\|\bar{D}^L\vec{u}(0)\|_0^2 +|\vec{f}_{\Omega}|_{L-2,0,[0,t]}^2+\langle\vec{f}_{\Gamma}\rangle_{L-2,1/2,[0,t]}^2)+\int_0^t(\|\partial_s^{L-1}\vec{f}_{\Omega}(s)\|_0^2+\langle\!\langle\partial_s^{L-1}\vec{f}_{\Gamma}(s)\rangle\!\rangle_{1/2}^2)ds\Big\}^{1/2}\Big].$$

REMARK. (1) By (Ap. 14) we know that $P(t)[\vec{u}(t)]\in X^{L-2,0}([0, T); \Omega)$ and $Q(t)[\vec{u}(t)]\in X^{L-2,1/2}([0, T); \Gamma)$ provided that $\vec{u}(t)\in X^{L,0}([0, T); \Omega)$. Hence, in Theorem 1.3 we see that $\vec{f}_{\Omega}\in X^{L-2,0}([0, T), \Omega)$ and $\vec{f}_{\Gamma}\in X^{L-2,1/2}([0, T), \Gamma)$.

(2) Since $L^2(\Omega)$ and $H^{1/2}(\Gamma)$ are Hilbert spaces, (1.4) implies that $\partial_t^{L-1}\vec{f}_{\Omega}(t)$ and $\partial_t^{L-1}\vec{f}_{\Gamma}(t)$ exist in the strong derivative sense of $L^2(\Omega)$ and $H^{1/2}(\Gamma)$ for almost all $t\in[0, T)$, respectively. Furthermore, by (No. 2.a) we know that $\|\partial_t^{L-1}\vec{f}_{\Omega}(t)\|_0$ and $\langle\!\langle\partial_t^{L-1}\vec{f}_{\Gamma}(t)\rangle\!\rangle_{1/2}$ are bounded for almost all $t\in[0, T)$. Hence, $\partial_t^{L-1}\vec{f}_{\Omega}(t)$ and $\partial_t^{L-1}\vec{f}_{\Gamma}(t)$ are $L^2$ functions in $t\in(0, T)$ having their values in $L^2(\Omega)$ and $H^{1/2}(\Gamma)$, respectively.

## § 2. The first energy inequality.

The goal of this section is to prove

THEOREM 2.1. *Assume that* $(A.1)_I$, $(A.2)_I$, $(A.3)_{I,\delta}$, $(A.4)_I$ *and* $(A.5)_I$ *are valid. Let* $\vec{u}\in X^{2,0}([0, T), \Omega)$ *and put*

$$F(t, \vec{u}(t))=\int_0^t(\|P(s)[\vec{u}(s)]\|_0^2+\langle\!\langle Q(s)[\vec{u}(t)]\rangle\!\rangle_{1/2}^2)ds\,.$$

*Then, there exists a constant* $C(T)=C(T, \delta_1, \delta_2, \Gamma, M_{\infty}(K), M_S(K))$ *such that the following two estimates are valid for* $t\in[0, T)$:

(2.1) $\quad E(t, \vec{u}(t))\leqq 2e^{C(T)t}\{E(0, \vec{u}(0))+C(T)F(t, \vec{u}(t))\}$;

(2.2) $E(t, \vec{u}(t)) \leqq e^{C(T)t}[E(0, \vec{u}(0))+C(T)\{\|\bar{D}^1\vec{u}(0)\|_0^2+F(t, \vec{u}(t))\}^{1/2}F(t, \vec{u}(t))^{1/2}]$.

If the coefficients of the operators $P(t)$ and $Q(t)$ belong to $\mathscr{B}^2$, then (2.1) and (2.2) were already obtained by Shibata [9]. Namely,

THEOREM 2.2. *Let $I'=[-\tau/2, T+(\tau/2)]$. Instead of* $(A.1)_I$, *we assume that*

$(A.1)'_{I'}$ $\quad A^{il}(t, x)\in \mathscr{B}^2(I'\times\bar{\Omega})$ *and* $B^l(t, x)\in \mathscr{B}^2(I'\times\Gamma)$

*for* $l=0, 1, \cdots, n$ *and* $i=1, \cdots, n$.

*In addition,* $(A.2)_{I'}$, $(A.4)_{I'}$ *and* $(A.5)_{I'}$ *are valid. Furthermore, we assume that there exist positive constant $\delta'_1$ and $\delta'_2$ such that* $(A.3)_{I',\delta'}$ *is valid. Let $\mu$ be a small number$\in(0, [n/2]+1-(n/2))$ and $\Lambda$ be a constant such that*

$$\sum_{l=0}^{n}\left\{\sum_{i=1}^{n}|A^{il}|_{\infty,1+\mu,I'}+\langle B^l\rangle_{\infty,1+\mu,I'}\right\}\leqq\Lambda. \tag{2.3}$$

*Then, there exists a constant $C(T)=C(T, \delta'_1, \delta'_2, \Lambda, \Gamma, \mu)$ such that* (2.1) *and* (2.2) *are valid for any $\vec{u}\in X^{2,0}([0, T), \Omega)$ and $t\in[0, T)$ with this constant $C(T)$.*

REMARK. The estimate (2.1) of Theorem 2.2 was first proved by Miyatake [6] in the scalar operators case (i.e., $m=1$). But, Miyatake assumed that the coefficients of the operators are sufficiently smooth and did not show how the constant $C(T)$ in (2.1) of Theorem 2.2 depends on the coefficients of the operators. It is first for Shibata [9] to prove that the constant $C(T)$ depends essentially only on $\Lambda$, which implies that the constant $C(T)$ in Theorem 2.1 depends on $M_\infty(K)$ and $M_S(K)$. This fact is quite important to solve the corresponding nonlinear problem. The results due to [9] did not follow from [6] directly. Because, to prove that $C(T)=C(T, \Lambda, \cdots)$, to the auther it seems that one needs more ideas, in particular, sharp estimates for $L^2$-boundedness of pseudo-differential operators developed recently.

To prove Theorem 2.1 by using Theorem 2.2, we use the following lemma concerned with the approximations of the coefficients of the operators $P(t)$ and $Q(t)$.

LEMMA 2.3. *Assume that* $(A.1)_I$, $(A.2)_I$, $(A.3)_{I,\delta}$, $(A.4)_I$ *and* $(A.5)_I$ *are valid. Then, there exist a number $\Sigma_0>0$ and sequences of matrices*: $\{A^{il}_{\infty\sigma}\}\subset\mathscr{B}^\infty(I'\times\bar{\Omega})$; $\{A^{il}_{S\sigma}\}\subset C^\infty(I', H^\infty(\Omega))$; $\{B^l_\sigma\}\subset C^\infty(I', H^\infty(\Gamma))$ ($I'=[-\tau/2, T+(\tau/2)]$ *and* $\sigma\in(0, \Sigma_0)$) *having the following properties*: (a)-(f).

(a.1) $\quad \lim_{\sigma\to0}|A^{il}_{\infty\sigma}-A^{il}_\infty|_{\infty,K-1,I'}=0$; $\lim_{\sigma\to0}|A^{il}_{S\sigma}-A^{il}_S|_{K-2,1,I'}=0$;

(a.2) $$\lim_{\sigma\to 0}\langle B_\sigma^l-B^l\rangle_{K-2,1/2,I'}=0;$$

(b.1) $$\sum_{l=0}^{n}\sum_{i=1}^{n}|A_{\infty\sigma}^{il}|_{\infty,K,I'}\leqq CM_\infty(K);\ \sum_{l=0}^{n}\sum_{i=1}^{n}|A_{S\sigma}^{il}|_{K-1,1,I'}\leqq CM_S(K);$$

(b.2) $$\sum_{l=0}^{n}\langle B_\sigma^l\rangle_{K-1,1/2,I'}\leqq CM_S(K),$$

*for any* $\sigma\in(0,\Sigma_0)$.

(c) *There exists a sequence* $\{\kappa(\sigma)\}$ *of positive numbers which tends to zero as* $\sigma\to 0$ *and has the following property: If we put*

$$A_\sigma^{i0}(t,x)=A_{\infty\sigma}^{i0}(t,x)+A_{S\sigma}^{i0}(t,x)-\kappa(\sigma)\nu_i(x)I_m$$

*where* $I_m$ *is the* $m\times m$ *unit matrix, then* $A_\sigma^{i0}(t,x)$ *and* $B_\sigma^0(t,x)$ *satisfy* $(\mathrm{A}.5)_{I'}$ *for any* $\sigma\in(0,\Sigma_0)$.

(d) *If we put*

$$A_\sigma^{ij}(t,x)=A_{\infty\sigma}^{ij}(t,x)+A_{S\sigma}^{ij}(t,x)$$

*then there exist constants* $\delta_1'$ *and* $\delta_2'$ *depending only on* $\delta_1$, $\delta_2$, $M_\infty(K)$ *and* $M_S(K)$ *and independent of* $\sigma$ *such that* $A_\sigma^{ij}(t,x)$ *and* $B_\sigma^j(t,x)$ *satisfy* $(\mathrm{A}.3)_{I',\delta'}$ *for any* $\sigma\in(0,\Sigma_0)$.

(e) $\nu_i(x)B_\sigma^i(t,x)=0$ *for any* $(t,x)\in I'\times\Gamma$ *and* $\sigma\in(0,\Sigma_0)$, i.e., $(\mathrm{A}.4)_{I'}$ *is valid.*

(f) $A_\sigma^{il}$ *and* $B_\sigma^l$ *satisfy the* $(\mathrm{A}.2)_{I'}$ *for any* $\sigma\in(0,\Sigma_0)$ *and* $i=1,\cdots,n$; $l=0,1,\cdots,n$.

Deferring the proof of Lemma 2.3, we shall first give a

PROOF OF THEOREM 2.1. Let $A_{\infty\sigma}^{il}$, $A_{S\sigma}^{il}$, $B_\sigma^l$, $\Sigma_0$ and $\kappa(\sigma)$ ($i=1,\cdots,n$; $l=0,1,\cdots,n$) be the same as in Lemma 2.3. Let $\mu$ be a small positive number $\in(0,[n/2]+1-(n/2))$ and $\sigma\in(0,\Sigma_0)$. Since $1+\mu<2<K$, by (b) of Lemma 2.3 we have

(2.4.a) $$|A_{\infty\sigma}^{ij}|_{\infty,1+\mu,I'}+|A_{\infty\sigma}^{i0}-\kappa(\sigma)\nu_iI_m|_{\infty,1+\mu,I'}\leqq C\{M_\infty(K)+1\}.$$

By Corollaries Ap. 7 and Ap. 8 and (b) of Lemma 2.3, we have also that

(2.4.b) $$|A_{S\sigma}^{il}|_{\infty,1+\mu,I'}\leqq C|A_{S\sigma}^{il}|_{K-1,1,I'}\leqq CM_S(K);$$

(2.4.c) $$\langle B_\sigma^l\rangle_{\infty,1+\mu,I'}\leqq C\langle B_\sigma^l\rangle_{K-1,1/2,I'}\leqq CM_S(K).$$

From these points of view, let us put $\Lambda=C\{M_\infty(K)+M_S(K)+1\}$. Then, Lemma 2.3 implies that for each $\sigma\in(0,\Sigma_0)$, $A_\sigma^{il}$ and $B_\sigma^l$ satisfy all the assumptions of Theorem 2.2. Note that $\Lambda$ and constants $\delta_1'$ and $\delta_2'$ depend on $M_\infty(K)$, $M_S(K)$, $\delta_1$ and $\delta_2$, but independent of $\sigma$. Put

$$\text{(2.5)}\qquad \begin{aligned} &P_\sigma(t)[\vec{u}(t)]=\partial_t^2\vec{u}(t)-\partial_i(A_\sigma^{i0}(t)\partial_t\vec{u}(t)+A_\sigma^{ij}(t)\partial_j\vec{u}(t));\\ &Q_\sigma(t)[\vec{u}(t)]=\nu_iA_\sigma^{ij}(t)\partial_j\vec{u}(t)+B_\sigma^j(t)\partial_j\vec{u}(t)+B_\sigma^0(t)\partial_t\vec{u}(t).\end{aligned}$$

If we denote the energy corresponding to $P_\sigma(t)$ and $Q_\sigma(t)$ by $E_\sigma$, then by Theorem 2.2, we see that there exists a constant $C(T)=C(T, \delta_1, \delta_2, \Gamma, M_\infty(K), M_S(K))$ independent of $\sigma$ such that

(2.6) $E_\sigma(t, \vec{u}(t))\leqq 2e^{C(T)t}\{E_\sigma(0, \vec{u}(0))+C(T)F_\sigma(t, \vec{u}(t))\}$;

(2.7) $E_\sigma(t, \vec{u}(t))\leqq e^{C(T)t}\{E_\sigma(0, \vec{u}(0))+C(T)(\|\bar{D}^1\vec{u}(0)\|_0^2+F_\sigma(t, \vec{u}(t))^{1/2}F_\sigma(t, \vec{u}(t))^{1/2})\}$,

where $F_\sigma(t, \vec{u}(t))=\int_0^t(\|P_\sigma(s)[\vec{u}(s)]\|_0^2+\langle\!\langle Q_\sigma(s)[\vec{u}(s)]\rangle\!\rangle_{1/2}^2)ds$.

Now, we shall prove that

(2.8) $E_\sigma(t, \vec{u}(t))\to E(t, \vec{u}(t))$; $F_\sigma(t, \vec{u}(t))\to F(t, \vec{u}(t))$ as $\sigma\to 0$ for all $t\in[0, T)$.

Noting the definition of energy (cf. (No. 21)) and using the definitions (No. 7) and (No. 18), we have

$$\text{(2.9.a)}\quad |E_\sigma(t, \vec{u}(t))-E(t, \vec{u}(t))|\leqq C\Big\{\sum_{j=1}^n\langle\!\langle B_\sigma^j(t)-B^j(t)\rangle\!\rangle_{\infty,0}$$
$$+\sum_{i,j=1}^n(\|A_{\infty\sigma}^{ij}(t)-A_\infty^{ij}(t)\|_{\infty,0}+\|A_{S\sigma}^{ij}(t)-A_S^{ij}(t)\|_{\infty,0}\Big\}\|\vec{u}(t)\|_1^2\leqq CU_\sigma(t)\|\vec{u}(t)\|_1^2$$

where

(2.9.b) $U_\sigma(t)=[P_\sigma(t)-P(t)]_{\infty,K-1}+[P_\sigma(t)-P(t)\,|\,Q_\sigma(t)-Q(t)]_{S,K-2,1}$ (cf. (No. 3)).

Here, we have used (No. 13). Thus, by (a) of Lemma 2.3 we have the first part of (2.8). Applying (Ap. 1)-(Ap. 3) with $\alpha=K-1$ and $\beta=\gamma=1$, we have

$$|F_\sigma(t, \vec{u}(t))-F(t, \vec{u}(t))|\leqq C\int_0^t U_\sigma(s)\|\bar{D}^1\vec{u}(s)\|_1^2ds\quad (C=C(M_\infty(K), M_S(K))).$$

Since $\vec{u}\in X^{2,0}([0, T); \Omega)$, by (a) of Lemma 2.3 we have the second part of (2.8). Hence, letting $\sigma\to 0$ in (2.6) and (2.7) and using (2.8), we have Theorem 2.1.

To complete the proof of Theorem 2.1, we give a

PROOF OF LEMMA 2.3. First, we shall discuss about the approximations of $B^l$ ($l=0, 1, \cdots, n$). Let $\phi_k$ be functions satisfying (No. 1). Put

$$\tilde{B}_k^l(t, y')=\begin{cases}\phi_k^2(\Psi_k(y', 0))B^l(t, \Psi_k(y', 0)) & \text{for } |y'|\leqq\sigma_k,\\ 0 & \text{for } |y'|>\sigma_k,\end{cases}$$

for $k=1, \cdots, N_0$ and $l=0, 1, \cdots, n$. By (No. 1) we have

$$\text{(2.10)}\qquad B^l(t, x)=\sum_{k=1}^{N_0}\phi_k^2(x)B^l(t, x)=\Sigma'\phi_k^2(\Psi_k(y', 0))B^l(t, \Psi_k(y', 0)),$$

where the summation $\Sigma'$ is taken over all $k$ such that $x=\Psi_k(y', 0)\in\mathcal{O}_k\cap\Gamma$.

Since supp $\phi_k \subset \mathcal{O}_k$, without loss of generality, we may assume that supp $\phi_k(\Psi_k(y)) \subset Q(\sigma_k')$ with some $\sigma_k' \in (0, \sigma_k)$. As a result, since $B^l \in Y^{K-1,1/2}(I, \Gamma)$, $\tilde{B}_k^l \in Y^{K-1,1/2}(I, \boldsymbol{R}^{n-1})$ and supp $\tilde{B}_k^l(t) \subset \{|y'| < \sigma_k'\}$ for all $t \in I$. Furthermore, we may assume that $Y_{ik}^n(y', 0) \neq 0$ on $\{|y'| \leqq \sigma_k\}$ for some $i$, say $i=n$. By (No. 5) we have

$$\tilde{B}_k^n(t, y') = -Y_{nk}^n(y', 0)^{-1}\left\{\sum_{p=1}^{n-1} \tilde{B}_k^p(t, y') Y_{pk}^n(y', 0)\right\}. \tag{2.11}$$

Let $\rho(t, y')$ be a function in $C_0^\infty(\boldsymbol{R} \times \boldsymbol{R}^{n-1})$ such that supp $\rho \subset \{|t|^2 + |y'|^2 < 1\}$, $\rho \geqq 0$ and $\iint \rho(t, y')dtdy' = 1$. Put $\rho_\sigma(t, y') = \sigma^{-n}\rho(t\sigma^{-1}, y'\sigma^{-1})$ and

$$[\tilde{B}_k^l]_\sigma(t, y') = \iint \rho_\sigma(t-s, y'-z')\tilde{B}_k^l(s, z')dsdz' \quad \text{for } l=0, 1, \cdots, n-1$$

(i.e., we mollify each component of $\tilde{B}_k^l$ by means of the usual Friedrichs' method). In view of (2.11), we put

$$[\tilde{B}_k^n]_\sigma(t, y') = -Y_{nk}^n(y', 0)^{-1}\left\{\sum_{p=1}^{n-1} [\tilde{B}_k^l]_\sigma(t, y') Y_{lk}^n(y', 0)\right\}. \tag{2.12}$$

Since $0 < \sigma_k' < \sigma_k$, there exists a $\Sigma_1 > 0$ such that $[\tilde{B}_k^l]_\sigma(t, y')$ are well-defined for $(t, y') \in I' \times \boldsymbol{R}^{n-1}$ and supp $[\tilde{B}_k^l]_\sigma(t, y') \subset \{|y'| < \sigma_k\}$ for any $t \in I'$ and $\sigma \in (0, \Sigma_1)$. Furthermore, $[\tilde{B}_k^l]_\sigma \in C^\infty(I', H^\infty(\boldsymbol{R}^{n-1}))$. From the second part of (A.2) it follows that

$${}^t[\tilde{B}_k^0]_\sigma(t, y') = [\tilde{B}_k^0]_\sigma(t, y'); \; {}^t[\tilde{B}_k^i]_\sigma(t, y') + [\tilde{B}_k^i]_\sigma(t, y') = 0 \tag{2.13}$$

for all $(t, y') \in I' \times \boldsymbol{R}^{n-1}$ and $\sigma \in (0, \Sigma_1)$. Put

$$B_{k\sigma}^l(t, x) = [\tilde{B}_k^l]_\sigma(t, \Phi_k(x)) \quad \text{for } x \in \mathcal{O}_k \cap \Gamma \text{ and } =0 \text{ otherwise.}$$

Since $[\tilde{B}_k^l]_\sigma(t, y') = 0$ for $|y'| \geqq \sigma_k$ and $t \in I'$, $B_{k\sigma}^l(t, x)$ are not only well-defined but also in $C^\infty(I', H^\infty(\Gamma))$. Put

$$B_\sigma^l(t, x) = \sum_{k=1}^{N_0} B_{k\sigma}^l(t, x).$$

Then, by (2.12) and (2.13) we see easily that ${}^tB_\sigma^0(t, x) = B_\sigma^0(t, x)$ and ${}^tB_\sigma^i(t, x) + B_\sigma^i(t, x) = 0$ $(i=1, \cdots, n)$; $\nu_i B_\sigma^i(t, x) = 0$ for any $(t, x) \in I' \times \Gamma$. Namely, the third and fourth parts of $(A.2)_{I'}$ and $(A.4)_{I'}$ are valid for any $\sigma \in (0, \Sigma_1)$. Obviously, we have that $\langle B_{k\sigma}^l \rangle_{K-1,1/2,I'} \leqq C\langle \phi_k^2 B^l \rangle_{K-1,1/2,I'} \leqq CM_S(K)$ and $\langle B_{k\sigma}^l - \phi_k^2 B^l \rangle_{K-2,1/2,I'} \to 0$ as $\sigma \to 0$ for $l=0, 1, \cdots, n-1$. With the help of (2.11) and (2.12), we see also that $\langle B_{k\sigma}^n \rangle_{K-1,1/2,I'} \leqq CM_S(K)$ and $\langle B_{k\sigma}^n - \phi_k^2 B^n \rangle_{K-1,1/2,I'} \to 0$ as $\sigma \to 0$. Noting (2.10), by these results we see easily that (a.2) and (b.2) of Lemma 2.3 are valid.

Now, we consider the approximations of $A^{il}$. In view of (A,1) and (A.2), without loss of generality we may assume that

(2.14) $\quad {}^tA_U^{i0}=A_U^{i0}$ and ${}^tA_U^{ij}=A_U^{ji}$ for $U=\infty$ and $S$ and $i, j=1, \cdots, n$.

By well-known Lions' method of extending functions defined on $\Omega$ to whole $\boldsymbol{R}^n$, we have that there exist $[A_\infty^{il}]\in\mathscr{B}^K(I\times\boldsymbol{R}^n)$ and $[A_S^{il}]\in Y^{K-1,1}(I, \boldsymbol{R}^n)$ such that $A_\infty^{il}=[A_\infty^{il}]$ and $A_S^{il}=[A_S^{il}]$ on $I\times\Omega$, and

(2.15)
$$|[A_\infty^{il}]|_{\infty,K,I\times R^n}\leqq C|A_\infty^{il}|_{\infty,K,I};$$
$$|[A_S^{il}]|_{K-1,1,I,R^n}\leqq C|A_S^{il}|_{K-1,1,I} \quad (C=C(K, \Gamma)).$$

Furthermore, in view of (2.14), we may assume that

(2.16) ${}^t[A_U^{i0}]=[A_U^{i0}]$ and ${}^t[A_U^{ij}]=[A_U^{ij}]$ for $U=\infty$ and $S$ and $i, j=1, \cdots, n$.

By using Friedrichs' method mentioned previously we mollify $[A_\infty^{ij}]$ and $[A_S^{il}]$ with respect to $(t, x)$. Then, noting (2.15), we see that there exist a small constant $\Sigma_2$ and sequences $\{A_\infty^{il}\}\subset\mathscr{B}^\infty(I'\times\bar{\Omega})$; $\{A_S^{il}\}\subset C^\infty(I', H^\infty(\Omega))$ $(\sigma\in(0, \Sigma_2))$ such that

(2.17) $\quad |A_{\infty\sigma}^{il}-A_\infty^{il}|_{\infty,K-1,I'}\to 0$ and $|A_{S\sigma}^{il}-A_S^{il}|_{K-2,1,I'}\to 0$ as $\sigma\to 0$;

(2.18) $\quad \sum_{l=0}^n\sum_{i=1}^n|A_{\infty\sigma}^{il}|_{\infty,K,I'}\leqq CM_\infty(K)$ and $\sum_{l=0}^n\sum_{i=1}^n|A_{S\sigma}^{il}|_{K-1,1,I'}\leqq CM_S(K)$ for $\sigma\in(0, \Sigma_2)$.

In view of (2.16), obviously we have

(2.19) $\quad {}^tA_{U\sigma}^{i0}=A_{U\sigma}^{i0}$ and ${}^tA_{U\sigma}^{ij}=A_{U\sigma}^{ji}$ for $U=\infty$ and $S$ and $i, j=1, \cdots, n$.

In particular, (2.7) and (2.18) mean that (a.1) and (b.1) of Lemma 2.3 are valid. Hence, we have proved (a), (b), (e) and (f) of Lemma 2.3 if we choose $\Sigma_0$ so that $\Sigma_0\leqq\min(\Sigma_1, \Sigma_2)$.

Now, we shall prove (c) of Lemma 2.3. Noting (A.1) and (A.5), we have

$$\{-\nu_i(x)(A_{\infty\sigma}^{i0}(t, x)+A_{S\sigma}^{i0}(t, x))+2B_\sigma^0(t, x)\}\eta\cdot\eta$$
$$\geqq(-\nu_i(x)A^{i0}(t, x)+2B^0(t, x))\eta\cdot\eta-\kappa(\sigma)|\eta|^2\geqq-\kappa(\sigma)|\eta|^2$$

where

$$\kappa(\sigma)=\langle\!\langle\nu_i\rangle\!\rangle_{\infty,0}(|A_{\infty\sigma}^{i0}-A_\infty^{i0}|_{\infty,0,I'}+|A_{S\sigma}^{i0}-A_S^{i0}|_{\infty,0,I'})+2\langle B_\sigma^0-B^0\rangle_{\infty,0,I'}.$$

Obviously, by (No. 13) and (a) of Lemma 2.3 we know that $\kappa(\sigma)\to 0$ as $\sigma\to 0$. Since the $\nu(x)$ is the unit outer normal of $\Gamma$ at $x\in\Gamma$, $|\nu(x)|^2=1$ for $x\in\Gamma$. Hence, if we put $A_\sigma^{i0}(t, x)=A_{\infty\sigma}^{i0}(t, x)+A_{S\sigma}^{i0}(t, x)-\kappa(\sigma)\nu_i(x)I_m$, then we see that (c) is valid for any $\sigma\in(0, \Sigma_2)$. Furthermore, putting $A_\sigma^{ij}(t, x)=A_{\infty\sigma}^{ij}(t, x)+A_{S\sigma}^{ij}(t, x)$, from (2.19) it follows that the first and second parts of $(\mathrm{A.2})_{I'}$ are valid for $\sigma\in(0, \Sigma_2)$.

Our final task is to prove (d) of Lemma 2.3. Let $\mathscr{B}_\sigma(t, \vec{u}, \vec{v})$ and $\mathcal{C}_\sigma(t, \vec{u}, \vec{v})$ be bilinear forms defined by replacing $B^j$ by $B_\sigma^j$ in (No. 7) and (No. 8), respectively. In the same way as in (2.9), we have

(2.20) $$|\mathcal{B}_\sigma(t, \vec{u}, \vec{v})-\mathcal{B}(t, \vec{u}, \vec{v})| \leqq C U_\sigma(t)\|\vec{u}\|_1\|\vec{v}\|_1$$

for any $t\in I'$, $\sigma\in(0, \Sigma_1)$ and $\vec{u}$, $\vec{v}\in H^1(\Omega)$. Noting (No. 8), we have

(2.21) $$|\mathcal{C}_\sigma(t, \vec{u}, \vec{v}| \leqq C \sum_{j=1}^{n}\langle B_\sigma^j\rangle_{\infty,1,I'}\|\vec{u}\|_1\|\vec{v}\|_0$$

for any $t\in I'$, $\sigma\in(0, \Sigma_1)$ and $\vec{u}$, $\vec{v}\in H^1(\Omega)$. Furthermore, we have

(2.22) $$\langle B_\sigma^j(t)\partial_j\vec{u}, \vec{v}\rangle=\mathcal{B}_\sigma(t, \vec{u}, \vec{v})+\mathcal{C}_\sigma(t, \vec{u}, \vec{v})$$

for $\vec{u}\in H^2(\Omega)$ and $\vec{v}\in H^1(\Omega)$ (cf. (No. 9)).

By $(A.3)_I$, (No. 9) and (2.22), we have

(2.23) $(A_\sigma^{ij}(t)\partial_j\vec{u}, \partial_i\vec{u})+\langle B_\sigma^j(t)\partial_j\vec{u}, \vec{u}\rangle\geqq\delta_1\|\vec{u}\|_1^2-\delta_2\|\vec{u}\|_0^2-I_1-I_2$ for $\vec{u}\in H^2(\Omega)$,

where $I_1=|((A_\sigma^{ij}(t)-A^{ij}(t))\partial_j\vec{u}\ {}'\partial_i\vec{u})|+|\mathcal{B}_\sigma(t, \vec{u}, \vec{u})-\mathcal{B}(t, \vec{u}, \vec{u})|$ and $I_2=|\mathcal{C}_\sigma(t, \vec{u}, \vec{v})-\mathcal{C}(t, \vec{u}, \vec{v})|$. Noting (2.20), (2.21) and (No. 12) and using (No. 13) and (b.2) of Lemma 2.3, we have that $I_1\leqq CU_\sigma(t)\|\vec{u}\|_1^2$ and $I_2\leqq CM_S(K)\|\vec{u}\|_1\|\vec{u}\|_0$. In view of (a) of Lemma 2.3, there exists a $\Sigma_3$ such that $I_1\leqq(\delta_1/4)\|\vec{u}\|_1^2$ for $\sigma\in(0, \Sigma_3)$. Since $I_2\leqq(\delta_1/4)\|\vec{u}\|_1^2+\{(CM_S(K))^2/\delta_1\}\|\vec{u}\|_0^2$, combining these facts and (2.23) implies that $(A.3)_{I',\delta'}$ is valid for any $\sigma\in(0, \Sigma_3)$, where $\delta_1'=\delta_1/2$ and $\delta_2'=\delta_2+\{(CM_S(K))^2/\delta_1\}$. Note that $\delta_1'$ and $\delta_2'$ are independent of $\sigma$. If we take $\Sigma_0=\min(\Sigma_1, \Sigma_2, \Sigma_3)$, the we have completed the proof of Lemma 2.3.

## §3. On some fundamental results on elliptic boundary value problems.

In this section, we shall prove some results on elliptic systems, which will be used in later sections. In the paragraphes 3.1 and 3.2, we shall discuss the fundamental principles from which the differentiability of weak solutions in the interior of $\Omega$ and near the boundary follows readily. These two paragraphes are independent of other sections, but to prove results stated in the rest of §3, the theorems in §§3.1 and 3.2 play an essential role. In the paragraph 3.3, we shall investigate a unique existence theorem of solutions to some elliptic boundary value problems in $\Omega$. In the final paragraph, we shall prove the unique existence theorem and time-dependence of solutions to some elliptic boundary value problem with parameter $t$ as a time mentioned in the final part of Introduction.

3.1 Differentiability in the interior of $\Omega$. Let $a^{ij}(x)$ be $m\times m$ matrices of functions satisfying the following properties:

(a.1.1) Each of $a^{ij}(x)$ is decomposed as follows: $a^{ij}(x)=a_\infty^{ij}(x)+a_S^{ij}(x)$, where

$$a_\infty^{ij} \in \mathcal{B}^{K-1}(\bar{\Omega}), \quad a_S^{ij} \in H^{K(n)}(\Omega).$$

Here and hereafter, $K(n)=\max([n/2]+2, K-1)$.

(a.1.2) There exist constants $d_1$ and $d_2>0$ such that

$$(a^{ij}\partial_j \vec{v}, \partial_i \vec{v}) \geqq d_1 \|\vec{v}\|_1^2 - d_2 \|\vec{v}\|_0^2 \quad \text{for any } \vec{v} \in H^1_{(0)}(\Omega).$$

Here and hereafter, we put

$$H^L_{(0)}(\Omega) = \{\vec{v} \in H^L(\boldsymbol{R}^n) \mid \text{dist}(\text{supp}\, v, \Gamma) \geqq \varepsilon \text{ for some } \varepsilon>0\}.$$

First, we consider the differentiability of $\vec{u} \in H^1(\Omega)$ satisfying the variational equation:

$$(3.1) \qquad (a^{ij}\partial_j \vec{u}, \partial_i \vec{v}) = (\vec{f}, \vec{v}) \quad \text{for all } \vec{v} \in H^1(\boldsymbol{R}^n).$$

In this and next paragraphes, we use the notations: $[\vec{v}]_h^k = (\vec{v}(x+he_k) - \vec{v}(x))h^{-1}$; $\vec{v}|_h^k = \vec{v}(x+he_k)$ where $e_k=(0, \cdots, 0, 1, \cdots, 0)$ are the $k$-th coordinate vectors of $\boldsymbol{R}^n$.

THEOREM 1.3. *Assume that* (a.1.1) *and* (a.1.2) *are valid. Let $L$ be an integer* $\in [2, K]$. *Let* $\vec{u} \in H^{L-1}(\Omega)$ *satisfy* (3.1) *and*

$$(3.2) \qquad \text{dist}(\text{supp}\, \vec{u}, \Gamma) \geqq \varepsilon \quad \textit{for some } \varepsilon>0.$$

*If* $\vec{f} \in H^{L-2}(\Omega)$, *then* $\vec{u} \in H^L(\Omega)$ *and*

$$(3.3) \qquad \|\vec{u}\|_L \leqq C(d_1, d_2, \gamma_{\infty, K-1}, \gamma_{S, K(n)}, L)\{\|\vec{f}\|_{L-2} + \|\vec{u}\|_{L-1}\}.$$

*Here,* $\gamma_{\infty, K-1}$ *and* $\gamma_{S, K(n)}$ *are constants such that*

$$(3.4) \qquad \sum_{i,j=1}^n \|a_\infty^{ij}\|_{\infty, K-1} \leqq \gamma_{\infty, K-1}; \quad \sum_{i,j=1}^n \|a_S^{ij}\|_{K(n)} \leqq \gamma_{S, K(n)}.$$

PROOF. Let $\delta=(\delta_1, \cdots, \delta_n)$ be any multi-index such that $|\delta|=L-2$. Then, $\partial_x^\delta \vec{u}$ satisfies the variational equation:

$$(3.5) \qquad (a^{ij}\partial_j(\partial_x^\delta \vec{u}), \partial_i \vec{v}) = (\vec{F}_\delta, \vec{v}) \quad \text{for any } \vec{v} \in H^1(\boldsymbol{R}^n)$$

where

$$\vec{F}_\delta = \partial_x^\delta \vec{f} - \sum_{\omega<\delta} \binom{\delta}{\omega} \partial_x^{\delta-\omega} a^{ij} \partial_j \partial_x^\omega \vec{u}$$

($\omega=(\omega_1, \cdots, \omega_n)$ are also multi-index and $\omega<\delta$ means that $\omega_i \leqq \delta_i$ for all $i=1, \cdots, n$ and $|\omega|<|\delta|$). In fact, if $\vec{v} \in H^{L-1}(\boldsymbol{R}^n)$, from (3.1) we have that $(a^{ij}\partial_j \vec{u}, \partial_i(-\partial_x)^\delta \vec{v}) = (\vec{f}, (-\partial_x)^\delta \vec{v})$. Noting (3.2), by integration by parts we see (3.5) immediately. As will be seen soon, $\vec{F}_\delta \in L^2(\Omega)$. Hence, since $H^{L-1}(\Omega)$ is dense in $H^1(\Omega)$, (3.5) follows immediately.

Now, we shall prove that $\vec{F}_\delta \in L^2(\Omega)$ and

$$\|\vec{F}_\delta\|_0 \leqq \|\vec{f}\|_{L-2} + C(L)\{\gamma_{\infty, K-1} + \gamma_{S, K(n)}\}\|\vec{u}\|_{L-1}. \tag{3.6}$$

Recall that $|\delta| = L-2$. Let $\omega < \delta$. Applying (Ap. 1) with $\alpha = K(n) - |\delta - \omega|$, $\beta = L-2-|\omega|$ and $\gamma = 1$, we have that $\|\partial_i(\partial_x^{\delta-\omega} a_S^{ij} \partial_j \partial_x^\omega \vec{u})\|_0 \leqq C\|a_S^{ij}\|_{K(n)}\|\vec{u}\|_{L-1}$. From this, (3.6) follows immediately.

Now, we shall prove that $\partial_x^\delta \vec{u} \in H^2(\Omega)$ and

$$\|\partial_x^\delta \vec{u}\|_2 \leqq C\|\vec{f}\|_{L-2} + \|\vec{u}\|_{L-1}\} \quad \text{where } C = C(d_1, d_2, L, \gamma_{\infty, K-1}, \gamma_{S, K(n)}). \tag{3.7}$$

Since $\delta$ is any multi-index such that $|\delta| = L-2$, the theorem follows from (3.7) immediately. For the notational simplicity, we write $\vec{w} = \partial_x^\delta \vec{u}$. Let $h$ be any number satisfying the condition: $0 < |h| < \delta/2$. Since $(a^{ij}\partial_j \vec{w}, \partial_i[\vec{v}]_{-h}^k) = (\vec{F}_\delta, [\vec{v}]_{-h}^k)$ as follows from (3.5), by the change of variables: $x + he_k \to x$, we have that $(a^{ij}\partial_j[\vec{w}]_h^k, \partial_i \vec{v}) = -([a^{ij}]_h^k \partial_j \vec{w}, \partial_i \vec{v}\,|_{-h}^k) - (\vec{F}_\delta, [\vec{v}]_{-h}^k)$. Note that $\|\partial_j \vec{w}\|_0 \leqq \|\vec{u}\|_{L-1}$; $\|\partial_i \vec{v}\,|_{-h}^k\|_{R^n, 0} \leqq \|\vec{v}\|_{R^n, 1}$; $\|[\vec{v}]_{-h}^k\|_{R^n, 0} \leqq \|\vec{v}\|_{R^n, 1}$. Since $|[a^{ij}]_h^k| \leqq \|a^{ij}\|_{\infty, 1} \leqq C\{\gamma_{\infty, K-1} + \gamma_{S, K(n)}\}$ as follows from (No. 13.b), by Schwarz's inequality and (3.6) we have

$$|(a^{ij}\partial_j[\vec{w}]_h^k, \partial_i \vec{v})| \leqq C\{\|\vec{f}\|_{L-2} + \|\vec{u}\|_{L-1}\}\|\vec{v}\|_{R^n, 1} \tag{3.8}$$

where $C = C(\gamma_{\infty, K-1}, \gamma_{S, K(n)}, L)$. Since dist $(\operatorname{supp}[\vec{w}]_h^k, \Gamma) > \varepsilon/2$ provided that $0 < |h| < \varepsilon/2$, by (a.1.2) we have

$$\|[\vec{w}]_h^k\|_1 \leqq (d_1)^{-1}\{|(a^{ij}\partial_j[\vec{w}]_h^k, \partial_i[\vec{w}]_h^k)| + d_2\|[\vec{w}]_h^k\|_0^2\}. \tag{3.9}$$

Since $\|[\vec{w}]_h^k\|_0 \leqq \|\vec{u}\|_{L-1}$, combining (3.8) with $\vec{v} = [\vec{w}]_h^k$ and (3.9), we have that $\|[\vec{w}]_h^k\|_1 \leqq C\{\|\vec{f}\|_{L-2} + \|\vec{u}\|_{L-1}\}$ where $C = C(d_1, d_2, L, \gamma_{\infty, K-1}, \gamma_{S, K(n)})$. From this it follows that $\vec{w} = \partial_x^\delta \vec{u} \in H^2(\Omega)$ and (3.7) is valid, which completes the proof.

As an easy corollary of Theorem 3.1, we shall give a theorem on further differentiability of $\vec{u}$ satisfying the equation:

$$-\partial_i(a^{ij}(x)\partial_j \vec{u}(x)) = \vec{f}(x) \quad \text{in } \Omega. \tag{3.10}$$

COROLLARY 3.2. *Assume that* (a.1.1) *and* (a.1.2) *are valid. Let $L$ be an integer* $\in [3, K]$, $\vec{u} \in H^{L-1}(\Omega)$ *and* $\vec{f} \in H^{L-2}(\Omega)$. *If* $\vec{u}$ *satisfies* (3.2) *and* (3.10), *then* $\vec{u} \in H^L(\Omega)$ *and* (3.3) *is valid.*

PROOF. Multiplying (3.10) by $\vec{v}$, integrating over $\boldsymbol{R}^n$ and noting (3.2), by integration by parts we have that $\vec{u}$ satisfies (3.1). Hence, Theorem 3.1 implies immediately Corollary 3.2.

3.2 Differentiability in $\boldsymbol{R}_{+}^{n}$. In this paragraph, first we consider the differentiability of a solution to the variational equation:

(3.11) $\quad B[\vec{u}, \vec{v}]=(\vec{f}_1, \vec{v})'+\langle\vec{f}_2, \vec{v}(\cdot, 0)\rangle'+(\vec{f}_3^i, \partial_i\vec{v})'$ for any $\vec{v}\in H^1(\boldsymbol{R}_{+}^{n})$.

Here, $\vec{f}_1$, $\vec{f}_2$ and $\vec{f}_3^i$ are given functions; $B$ is a bilinear form of the form:

$$B[\vec{u}, \vec{v}]=(a^{ij}\partial_j\vec{u}, \partial_i\vec{v})'+(b^p\partial_n\vec{u}, \partial_p\vec{v})'-(b^p\partial_p\vec{u}, \partial_n\vec{v})'.$$

Here and hereafter, for the notational simplicity, we use the following abbreviations:

$$(\vec{u}, \vec{v})'=\int_{\boldsymbol{R}_{+}^{n}}\vec{u}(x)\cdot\vec{v}(x)dx\,;\ \langle\vec{u}, \vec{v}\rangle'=\int_{\boldsymbol{R}^{n-1}}\vec{u}(x')\cdot\vec{v}(x')dx' \quad (x'=(x_1, \cdots, x_{n-1}))\,;$$

$$\|\cdot\|_r'=\|\cdot\|_{\boldsymbol{R}_{+}^{n}, r}\,;\ \langle\!\langle\cdot\rangle\!\rangle_r'=\|\cdot\|_{\boldsymbol{R}^{n-1}, r}'.$$

Let $a^{ij}$ and $b^p$ $(i, j=1, \cdots, n\,; p=1, \cdots, n-1)$ be $m\times m$ matrices of functions satisfying the following assumptions:

(a.2.1) The $a^{ij}$ are decomposed as follows: $a^{ij}=a_\infty^{ij}+a_S^{ij}$ where $a_\infty^{ij}\in\mathscr{B}^{K-1}(\overline{\boldsymbol{R}_{+}^{n}})$ and $a_S^{ij}\in H^{K(n)}(\boldsymbol{R}_{+}^{n})$.

(a.2.2) $b^p\in H^{K(n)}(\boldsymbol{R}_{+}^{n})$.

(a.2.3) ${}^t a^{ij}=a^{ji}$.

(a.2.4) Let $\sigma$ be a positive constant. There exist positive constants $d_3$ and $d_4$ which may depend on $\sigma$ such that

$$B[\vec{v}, \vec{v}]\geqq d_3(\|\vec{v}\|_1')^2-d_4(\|\vec{v}\|_0')^2 \quad \text{for any } \vec{v}\in H_\sigma^1(\boldsymbol{R}_{+}^{n}).$$

Here and hereafter, we set

$$H_\varepsilon^L(\boldsymbol{R}_{+}^{n})=\{\vec{v}\in H^L(\boldsymbol{R}_{+}^{n}) \mid \operatorname{supp} v\subset Q(\varepsilon)\} \ (Q(\varepsilon)=\{x\in\boldsymbol{R}^n \mid |x'|<\varepsilon, \ |x_n|<\varepsilon\}).$$

As a corresponding theorem to Theorem 3.1, we shall prove

THEOREM 3.3. *Assume that* (a.2.1)-(a.2.4) *are valid. Let* $L$ *be an integer* $\in[2, K]$. *Assume that* $\vec{f}_1\in H^{L-2}(\boldsymbol{R}_{+}^{n})$, $\vec{f}_2\in H^{L-(3/2)}(\boldsymbol{R}^{n-1})$ *and* $\vec{f}_3^i\in H^{L-1}(\boldsymbol{R}_{+}^{n})$ $(i=1, \cdots, n)$. *If* $\vec{u}\in H_\varepsilon^{L-1}(\boldsymbol{R}_{+}^{n})$ *for some* $\varepsilon\in(0, \sigma)$ *and satisfies* (3.11), *then* $\vec{u}\in H^L(\boldsymbol{R}_{+}^{n})$ *and*

(3.12)
$$\|\vec{u}\|_L'\leqq C\{\|\vec{f}_1\|_{L-2}'+\langle\!\langle\vec{f}_2\rangle\!\rangle_{L-(3/2)}'+\sum_{i=1}^{n}\|\vec{f}_3^i\|_{L-1}'+\|\vec{u}\|_{L-1}'\}$$

*where* $C=C(d_3, d_4, L, \gamma_{\infty, K-1}', \gamma_{S, K(n)}')$. *Here,* $\gamma_{\infty, K-1}'$ *and* $\gamma_{S, K(n)}'$ *are constants such that*

$$\sum_{i,j=1}^{n}|a_\infty^{ij}|_{\infty, K-1, \boldsymbol{R}^n}\leqq\gamma_{\infty, K-1}'\,;\ \sum_{i,j=1}^{n}\|a_S^{ij}\|_{K(n)}'+\sum_{p=1}^{n-1}\|b^p\|_{K(n)}'\leqq\gamma_{S, K(n)}'.$$

PROOF. Let $k$ be any integer $\in[1, n-1]$ and $\alpha'=(\alpha_1, \cdots, \alpha_{n-1})$ be any multi-index such that $|\alpha'|=L-2$. First, we shall prove that $\partial_k\partial_{x'}^{\alpha'}\vec{u}=\partial_k\partial_1^{\alpha_1}\cdots\partial_{n-1}^{\alpha_{n-1}}\vec{u}\in H^1(\boldsymbol{R}_+^n)$ that

$$\|\partial_k\partial_{x'}^{\alpha'}\vec{u}\|_1'\leqq C\Delta. \tag{3.13}$$

In the present proof, for the notational simplicity we use the same letter $C$ to denote various constants depending at most on $L$, $d_3$, $d_4$, $\gamma'_{\infty,K-1}$ and $\gamma'_{S,K(n)}$ and put $\Delta=\|\vec{f}_1\|_{L-2}'+\langle\!\langle\vec{f}_2\rangle\!\rangle_{L-(3/2)}'+\sum_{i=1}^{n}\|\vec{f}_3^i\|_{L-1}'+\|\vec{u}\|_{L-1}'$. To prove (3.13), we shall use the fact that $\partial_{x'}^{\alpha'}\vec{u}$ satisfies the variational equations:

$$B[\vec{w}, \vec{v}]=(\vec{F}_1, \vec{v})'+\langle\vec{F}_2, \vec{v}(\cdot, 0)\rangle'+(\vec{F}_3^i, \partial_i\vec{v})' \quad \text{for any } \vec{v}\in H^1(\boldsymbol{R}_+^n) \tag{3.14}$$

where $F_1=\partial_{x'}^{\alpha'}\vec{f}_1$; $F_2=\partial_{x'}^{\alpha'}\vec{f}_2$; $w=\partial_{x'}^{\alpha'}\vec{u}$;

$$\vec{F}_3^p=\partial_{x'}^{\alpha'}\vec{f}_3^p-\sum_{\alpha'<\beta'}\binom{\alpha'}{\beta'}\{\partial_{x'}^{\alpha'-\beta'}a^{pj}\partial_j\partial_{x'}^{\beta'}\vec{u}+\partial_{x'}^{\alpha'-\beta'}b^p\partial_n\partial_{x'}^{\beta'}\vec{u}\} \quad (p=1, \cdots, n-1);$$

$$F_3^n=\partial_{x'}^{\alpha'}\vec{f}_3^n-\sum_{\beta'<\alpha'}\binom{\alpha'}{\beta'}\{\partial_{x'}^{\alpha'-\beta'}a^{nj}\partial_j\partial_{x'}^{\beta'}\vec{u}-\partial_{x'}^{\alpha'-\beta'}b^p\partial_p\partial_{x'}^{\beta'}\vec{u}\}.$$

In fact, if $\vec{v}\in H^{L-1}(\boldsymbol{R}_+^n)$, replacing $\vec{v}$ by $(-\partial_{x'})^{\alpha'}\vec{v}$ in (3.11) and applying integration by parts, we have (3.14). As will be seen below, $F_3^i\in H^1(\boldsymbol{R}_+^n)$. Furthermore, $F_2\in H^{1/2}(\boldsymbol{R}^{n-1})$ and $F_1\in L^2(\boldsymbol{R}_+^n)$. Since $H^{L-1}(\boldsymbol{R}_+^n)$ is dense in $H^1(\boldsymbol{R}_+^n)$, (3.41) is also valid for any $\vec{v}\in H^1(\boldsymbol{R}_+^n)$.

Applying (Ap. 1) with $\sigma=K(n)-|\alpha'-\beta'|$, $\beta=L-2-|\beta'|$ and $\gamma=1$ for $\beta'<\alpha'$, we have

$$\|\partial_{x'}^{\alpha'-\beta'}a_S^{ij}\partial_j\partial_{x'}^{\beta'}\vec{u}\|_1'\leqq C\gamma'_{S,K(n)}\|\vec{u}\|_{L-1}; \quad \|\partial_{x'}^{\alpha'-\beta'}b^p\partial_i\partial_{x'}^{\beta'}\vec{u}\|_1'\leqq C\gamma'_{S,K(n)}\|\vec{u}\|_{L-1}$$

for $i=1, \cdots, n$ and $p=1, \cdots, n-1$. From this it follows immediately that

$$\|\vec{F}_1\|_0'+\langle\!\langle\vec{F}_2\rangle\!\rangle_{1/2}'+\sum_{i=1}^{n}\|\vec{F}_3^i\|_1'\leqq C\Delta. \tag{3.15}$$

Now, we prove (3.13). Let $h$ satisfy the condition: $0<|h|<\sigma-\varepsilon$. By the change of variables: $x+he_k\to x$, from (3.14) we have

$$\begin{aligned} B[[\vec{w}]_h^k, \vec{v}]=&-([a^{ij}]_{-h}^k\partial_j\vec{w}, \partial_i\vec{v}|_{-h}^k)'-([b^p]_{-h}^k\partial_n\vec{w}, \partial_p\vec{v}|_{-h}^k)' \\ &+([b^p]_{-h}^k\partial_p\vec{w}, \partial_n\vec{v}|_{-h}^k)'-(F_1, [\vec{v}]_{-h}^k)' \\ &-\langle F_2, [\vec{v}(\cdot, 0)]_{-h}^k\rangle'+([F_3^i]_h^k, \partial_i\vec{v})'. \end{aligned} \tag{3.16}$$

By Schwarz's inequality and Theorem Ap. 2-(1), we have

$$\begin{aligned} |\langle F_2, [\vec{v}(\cdot, 0)]_{-h}^k\rangle'|&\leqq\langle\!\langle F_2\rangle\!\rangle_{1/2}'\langle\!\langle[\vec{v}(\cdot, 0)]_{-h}^k\rangle\!\rangle_{-1/2}' \\ &\leqq\langle\!\langle F_2\rangle\!\rangle_{1/2}'\langle\!\langle\vec{v}(\cdot, 0)\rangle\!\rangle_{1/2}'\leqq C\langle\!\langle\vec{F}_2\rangle\!\rangle_{1/2}'\|\vec{v}\|_1'. \end{aligned} \tag{3.17}$$

Hence, applying Schwarz's inequality to other terms of the right-hand side of (3.16) and using (3.15) and (3.17), we have

$$|B[[\vec{w}]_h^k, \vec{v}]| \leqq C\Delta \|\vec{v}\|_1'. \tag{3.18}$$

Here, we have also used the facts that $\|\partial_j \vec{w}\|_0 \leqq \|\vec{u}\|_{L-1}$;

$$|[a^{ij}]_{-h}^k| \leqq |a^{ij}|_{\infty,1,R_+^n} \leqq |a_\infty^{ij}|_{\infty,1,R_+^n} + C\|a_S^{ij}\|_{K(n)}'; \; |[b^p]_{-h}^k| \leqq C\|b^p\|_{K(n)}'$$

(cf. (No. 13.b)). Since $[\vec{w}]_h^k$ vanishes for $|x| \geqq \sigma$ as follows from the assumptions: $\vec{u} \in H_\varepsilon^{L-1}(\boldsymbol{R}_+^n)$, by (a.2.4) we have

$$(\|[\vec{w}]_h^k\|_1')^2 \leqq (d_3)^{-1}\{B[[\vec{w}]_h^k, [\vec{w}]_h^k] + d_4(\|[\vec{w}]_h^k\|_0')^2\}. \tag{3.19}$$

Substituting the inequality: $(\|[\vec{w}]_h^k\|_0')^2 \leqq \|\vec{u}\|_{L-1}\|[\vec{w}]_h^k\|_0'$ into (3.19), putting $\vec{v} = [\vec{w}]_h^k$ in (3.18) and combining the two resulting inequalities, we have that $\|[\vec{w}]_h^k\|_1' \leqq C\Delta$. From this it follows immediately that $\partial_k \vec{w} = \partial_k \partial_{x'}^{\alpha'} \vec{u} \in H^1(\boldsymbol{R}_+^n)$ and (3.13) is valid.

Now, by induction on $N$ we shall prove that $\partial_{x'}^{\alpha'} \partial_n^N \vec{u} \in L^2(\Omega)$ and

$$\|\partial_{x'}^{\alpha'} \partial_n^N \vec{u}\|_0' \leqq C\Delta \tag{3.20}$$

for any integer $N \in [0, L]$ and multi-index $\alpha' = (\alpha_1, \cdots, \alpha_{n-1})$ such that $|\alpha'| = L - N$. As was already proved, (3.20) is valid for $N=0$ and 1. Thus, assume that $2 \leqq N \leqq L$ and that the assertion is valid for smaller values of $N$. First, we prove that $a^{nn}(x)$ is a nonsingular matrix for all $x \in \overline{Q_+(\sigma)}$ ($Q_+(\sigma) = Q(\sigma) \cap \boldsymbol{R}_+^n$) and

$$|a^{nn}(x)^{-1}| \leqq C \quad \text{for all } x \in \overline{Q_+(\sigma)}. \tag{3.21}$$

To prove this, we need

LEMMA 3.4. *Let $G$ be a domain in $\boldsymbol{R}^n$ and $P^{ij}(x)$ be $m \times m$ maritces of functions in $C^0(G)$. Assume that ${}^tP^{ij} = P^{ji}$ and that there exist positive constants $c_3$ and $c_4$ such that* $\mathrm{Re}\int_G P^{ij}(x)\partial_j \vec{v}(x) \cdot \partial_i \vec{v}(x)dx \geqq c_3\|\vec{v}\|_{G,1}^2 - c_4\|\vec{v}\|_{G,0}^2$ *for any $\vec{v} \in C_0^\infty(G)$ which may be complex-valued. Then, $P^{ij}(x)\xi_i\xi_j \geqq c_3|\xi|^2 I_m$ for any $x \in G$ and $\xi = (\xi_1, \cdots, \xi_n) \in \boldsymbol{R}^n$.*

This lemma is well-known and for its proof, see Shibata [9]. Since

$$\begin{aligned} \langle b^p \partial_p \vec{u}, \vec{v} \rangle' = -\int_{R_+^n} \partial_n (b^p \partial_p \vec{u} \cdot \vec{v}) dx = -(b^p \partial_p \vec{u}, \partial_n \vec{v}) + (b^p \partial_n \vec{u}, \partial_p \vec{v})' \\ + ((\partial_p b^p)\partial_n \vec{u}, \vec{v})' - ((\partial_n b^p)\partial_p \vec{u}, \vec{v})' \end{aligned} \tag{3.22}$$

as follows from the integration by parts with respect to $x_p$ ($p=1, \cdots, n-1$), we see that for any $\vec{v}\in C_0^\infty(Q_+(\sigma))$

$$B[\vec{v},\ \vec{v}]=(a^{ij}\partial_j\vec{v},\ \partial_i\vec{v})'-((\partial_p b^p)\partial_n\vec{v},\ \vec{v})'+((\partial_n b^p)\partial_p\vec{v},\ \vec{v})'$$
$$\leqq(a^{ij}\partial_j\vec{v},\ \partial_i\vec{v})'+C\gamma'_{S,K(n)}\|\vec{v}\|_1'\|\vec{v}\|_0'.$$

Combining this and (a.2.4) implies that

$$(a^{ij}\partial_j\vec{w},\ \partial_i\vec{w})'\geqq(d_3/2)(\|\vec{w}\|_1')^2-d_4'(\|\vec{w}\|_0')^2$$

for any $\vec{w}\in C_0^\infty(Q_+(\sigma))$ which may be complex-valued where $d_4'=d_4+(C\gamma'_{S,K(n)})^2/2d_3$. Applying Lemma 2.4 and noting that $a^{ij}$ is continuous on $\overline{Q_+(\sigma)}$ (cf. Sobolev's imbedding theorem), we have that $a^{ij}(x)\xi_i\xi_j\geqq(d_3/2)|\xi|^2I_m$ for any $x\in\overline{Q_+(\sigma)}$ and $\xi=(\xi_1,\cdots,\xi_n)\in \boldsymbol{R}^n$. In particular, if we put $\xi=(0,\cdots,0,1)$, we have that $a^{nn}(x)\geqq(d_3/2)I_m$ for any $x\in\overline{Q_+(\sigma)}$, which means that $a^{nn}(x)$ is non-singular for all $x\in\overline{Q_+(\sigma)}$ and that all eigenvalues of $a^{nn}$ are bounded by $d_3/2$ from below. Since

$$a^{nn}(x)^{-1}=\{\det(a^{nn}(x))\}^{-1}\text{ cofactor matrix of }a^{nn}(x),$$

we have (3.21).

Let $\vec{v}\in C_0^\infty(Q_+(\sigma))$ and replace $\vec{v}$ by $(-\partial_x)^\delta\vec{v}$ in (3.11) where $\delta=(\alpha',\ N-2)$. Then, by integration by parts we have

$$(a^{nn}\partial_n^2\partial_x^\delta\vec{u},\ \vec{v})'=(\vec{G},\ \vec{v})' \tag{3.23}$$

where

$$\vec{G}=-(\partial_n a^{nn})\partial_n\partial_x^\delta\vec{u}-\partial_n(a^{nn}\partial_p\partial_x^\delta\vec{u})-\partial_p(a^{pj}\partial_j\partial_x^\delta\vec{u})-\partial_p(b^p\partial_n\partial_x^\delta\vec{u})$$
$$-\partial_n(b^p\partial_p\partial_x^\delta\vec{u})-\sum_{\omega<\delta}\binom{\delta}{\omega}\{\partial_i(\partial_x^{\delta-\omega}a^{ij}\partial_j\partial_x^\omega\vec{u})\partial_p(\partial_x^{\delta-\omega}b^p\partial_p^\omega\vec{u})\}$$
$$-\partial_n(\partial_x^{\delta-\omega}b^p\partial_p^\omega\vec{u})\}+\partial_x^\delta\vec{f}_1+\sum_{i=1}^n\partial_i(\partial_x^\delta\vec{f}_2^i).$$

Note that $|\delta|=L-2$, because $|\alpha'|=L-N$. Applying (Ap. 1) with $\alpha=K(n)-|\delta-\omega|$, $\beta=L-2-|\omega|$ and $\gamma=1$ for $\omega<\delta$, we have

$$\|\partial_i(\partial_x^{\delta-\omega}a_S^{ij}\partial_j\partial_x^\omega\vec{u})\|_0'+\|\partial_p(\partial_x^{\delta-\omega}b^p\partial_n^\omega\vec{u})\|_0'+\|\partial_n(\partial_x^{\delta-\omega}b^p\partial_p^\omega\vec{u})\|_0'\leqq C\gamma'_{S,K(n)}\|\vec{u}\|_{L-1}'.$$

Hence, by the inductive assumption we have that $\vec{G}\in L^2(\boldsymbol{R}_+^n)$ and $\|\vec{G}\|_0'\leqq C\Delta$. Accordingly, since (3.23) is valid for any $\vec{v}\in C_0^\infty(Q_+(\sigma))$ and $\operatorname{supp}\vec{u}\subset Q(\sigma)$, $\|a^{nn}\partial_n^2\partial_x^\delta\vec{u}\|_0'\leqq C\Delta$. Thus, (3.21) implies that $\partial_n^2\partial_x^\delta\vec{u}\in L^2(\boldsymbol{R}_+^n)$ and that $\|\partial_n^2\partial_x^\delta\vec{u}\|_0'\leqq C\Delta$. Since $\partial_n^2\partial_x^\delta\vec{u}=\partial_{x'}^{\alpha'}\partial_n^N\vec{u}$, we have proved (3.20), which completes the proof of the theorem.

As an application of Theorem 3.3, we shall prove further differentiability of a solution $\vec{u}$ to the boundary value problem:

(3.24.a) $$-\partial_i(a^{ij}(x)\partial_j\vec{u}(x))=\vec{f}(x) \qquad \text{in } \boldsymbol{R}^n_+,$$

(3.24.b) $$-a^{nj}(x', 0)\partial_j\vec{u}(x', 0)+c^p(x')\partial_p\vec{u}(x', 0)=\vec{g}(x') \quad \text{on } \boldsymbol{R}^{n-1}.$$

Here, $a^{ij}(x)$ and $c^p(x')$ are $m\times m$ matrices of functions satisfying the following assumptions:

(a.2.5) (a.2.1) and (a.2.3) are valid;

(a.2.6) $c^p\in H^{K(n)-(1/2)}(\boldsymbol{R}^{n-1})$;

(a.2.7) there exist positive constants $d_5$ and $d_6$ such that

$$(a^{ij}\partial_j\vec{v}, \partial_i\vec{v})'+\langle c^p\partial_p\vec{v}(\cdot, 0), \vec{v}(\cdot, 0)\rangle'\geqq d_5(\|\vec{v}\|_1')^2-d_6(\|\vec{v}\|_0')^2$$

for any $\vec{v}\in H^2_\sigma(\boldsymbol{R}^n_+)$.

Note that the unit outer normal of the boundary of $\boldsymbol{R}^n_+$ is $(0, \cdots, 0, -1)$. The following theorem can be deduced form Theorem 3.3, which is corresponding to Corollary 3.2. It is independent of the text, but for the further references we state and prove it

THEOREM 3.4. *Assume that* (a.2.5)-(a.2.7) *are valid. Let $L$ be an integer* $[3, K]$, $\vec{u}\in H^{L-1}_\varepsilon(\boldsymbol{R}^n_+)$ *for some* $\varepsilon\in(0, \sigma)$, $\vec{f}\in H^{L-2}(\boldsymbol{R}^n_+)$ *and* $\vec{g}\in H^{L-(3/2)}(\boldsymbol{R}^{n-1})$. *If $\vec{u}$ satisfies* (3.24), *then* $\vec{u}\in H^L(\boldsymbol{R}^n_+)$ *and*

(3.25) $$\|\vec{u}\|_L'\leqq C\{\|\vec{f}\|_{L-2}'+\langle\!\langle\vec{g}\rangle\!\rangle_{L-(3/2)}'+\|\vec{u}\|_{L-1}'\},$$

*where* $C=C(d_5, d_6, L, \gamma'_{\infty, K-1}, \gamma''_{S, K(n)})$. *Here,* $\gamma'_{\infty, K-1}$ *is the same as in Theorem* 3.3 *and* $\gamma''_{S, K(n)}$ *is a constant such that*

$$\sum_{i,j=1}^n\|a^{ij}_S\|_{K(n)}'+\sum_{p=1}^{n-1}\langle\!\langle c^n\rangle\!\rangle_{K(n)-(1/2)}'\leqq\gamma''_{S, K(n)}.$$

PROOF. We shall reduce (3.24) to (3.11). Let $b^p(x)$ be $m\times m$ matrices of functions such that $b^p(x', 0)=c^p(x')$ for almost all $x'\in\boldsymbol{R}^{n-1}$ and

(2.26) $$\|b^p\|_{K(n)}'\leqq C(r)\langle\!\langle c^p\rangle\!\rangle_{K(n)-(1/2)}'\leqq C\gamma''_{S, K(n)}.$$

The existence of such $b^p$ is assured by Theorem Ap. 3. Since $\langle c^p\partial_p\vec{u}(\cdot, 0), \vec{v}(\cdot, 0)\rangle'=\langle b^p\partial_p\vec{u}(\cdot, 0), \vec{v}(\cdot, 0)\rangle'$, by employing the same argument as in (3.22) we have

(3.27) $$\langle c^p\partial_p\vec{u}(\cdot, 0), \vec{v}(\cdot, 0)\rangle'=(b^p\partial_n\vec{u}, \partial_p\vec{v})'-(b^p\partial_p\vec{u}, \partial_n, \vec{v})'$$
$$+((\partial_pb^p)\partial_n\vec{u}, \vec{v})'-((\partial_nb^p)\partial_p\vec{u}, \vec{v})'.$$

Multiplying (3.24.a) by $\vec{v}$ and integrating over $\boldsymbol{R}^n_+$, by integration by part and (3.27) we have

(3.28) $B[\vec{u}, \vec{v}]=(\vec{f}, \vec{v})'+\langle\vec{f}_2, v(\cdot, 0)\rangle'+(\vec{f}_3^i, \partial_i\vec{v})'$ for any $\vec{v}\in H^1(\boldsymbol{R}_+^n)$

where

$$B[\vec{u}, \vec{v}]=(a^{ij}\partial_j\vec{u}, \partial_i\vec{v})+(b^p\partial_n\vec{u}, \partial_p\vec{v})'-(b^p\partial_p\vec{u}, \partial_n\vec{v})';$$

$$\vec{f}_1=\vec{f}+(\partial_n b^p)\partial_p\vec{u}-(\partial_p b^p)\partial_n\vec{u}; \ \vec{f}_2=\vec{g}; \ \vec{f}_3^i=0.$$

To apply Theorem 3.3, we must prove that $\vec{f}_1\in H^{L-2}(\boldsymbol{R}_+^n)$ and (a.2.4) is valid. Applying (Ap. 1) with $\alpha=K(n)-1$, $\beta=L-2$ and $\gamma=L-2$ and using (3.26), we have

$$\|(\partial_i b^p)\partial_j\vec{u}\|_{L-2}\leqq C\|b^p\|_{K(n)}\|\vec{u}\|_{L-1}\leqq C\gamma''_{S,K(n)}\|\vec{u}\|_{L-1}$$

for $i, j=1, \cdots, n$, from which we have immediately that $\vec{f}_1\in H^{L-2}(\boldsymbol{R}_+^n)$ and $\|\vec{f}_1\|'_{L-2}\leqq C\{\|\vec{f}\|'_{L-2}+\gamma''_{S,K(n)}\|\vec{u}\|'_{L-1}\}$. Since $H_\sigma^2(\boldsymbol{R}_+^n)$ is dense in $H_\sigma^1(\boldsymbol{R}_+^n)$, to prove that (a.2.4) is valid in the present case, it is sufficient to show that there exists a $d'_6>0$ depending only on $d_5$, $d_6$ and $\gamma''_{S,K(n)}$ such that

(3.29) $B[v, v]\geqq(d_5/2)(\|\vec{v}\|'_1)^2-d'_6(\|\vec{v}\|'_0)^2$ for any $\vec{v}\in H_\sigma^2(\boldsymbol{R}_+^n)$.

In the same way as in (3.22)′ (or (3.27)), we have that for any $\vec{v}\in H_\sigma^2(\boldsymbol{R}_+^n)$, $B[\vec{v}, \vec{v}]=(a^{ij}\partial_j\vec{v}, \partial_i\vec{v})'+\langle c^p\partial_p\vec{v}(\cdot, 0), \vec{v}(\cdot, 0)\rangle'-((\partial_p b^p)\partial_n\vec{v}, \vec{v})'+((\partial_n b^p)\partial_p\vec{v}, \vec{v}')$. Since $|-((\partial_p b^p)\partial_n\vec{v}, \vec{v})'+((\partial_n b^p)\partial_p\vec{v}, \vec{v})'|\leqq C\gamma''_{S,K(n)}\|\vec{v}\|'_1\|\vec{v}\|'_0$ as follows from Schwarz's inequality, (No. 13.b) and (3.26), by (a.2.7) we have (3.29) with $d'_6=d_6+(C\gamma''_{S,K(n)})^2/2d_5$. Thus, the present bilinear form $B$ satisfies all the assumptions of Theorem 3.3, and $\vec{f}_1$, $\vec{f}_2$, $\vec{f}_3^i$ and $\vec{u}$ do, too. Theorem 3.3 implies Theorem 3.4 immediately.

3.3 Unique existence theorem of solutions to some elliptic boundary value problem.

In this paragraph, we consider the following boundary value problem of elliptic system of 2nd order in $\Omega$:

(3.30.a) $-\partial_i(P^{ij}(x)\partial_j\vec{u}(x))+P_\Omega^j(x)\partial_j\vec{u}(x)+P_\Omega^{n+1}(x)\vec{u}(x)+\lambda\vec{u}(x)=\vec{g}_\Omega(x)$ in $\Omega$,

(3.30.b) $\nu_i(x)P^{ij}(x)\partial_j\vec{u}(x)+P_\Gamma^j(x)\partial_j\vec{u}(x)+P_\Gamma^{n+1}(x)\vec{u}(x)=\vec{g}_\Gamma(x)$ on $\Gamma$.

Here, $P^{ij}(x)$, $P_\Omega^l(x)$ and $P_\Gamma^l(x)$ $(i, j=1, \cdots, n; l=1, \cdots, n+1)$ are $m\times m$ matrices of functions having the following properties (a.3.1)-(a.3.5):

(a.3.1) $P^{ij}$ and $P_\Omega^l$ are decomposed as follows: $P^{ij}=P_\infty^{ij}+P_S^{ij}$ and $P_\Omega^l=P_{\Omega,\infty}^l+P_{\Omega,S}^l$ where $P_\infty^{ij}\in\mathscr{B}^{K-1}(\bar{\Omega})$; $P_{\Omega,\infty}^l\in\mathscr{B}^{K-2}(\bar{\Omega})$; $P_S^{ij}\in H^{K(n)}(\Omega)$; $P_{\Omega,S}^l\in H^{K'(n)}(\Omega)$ $(K(n)=\max([n/2]+2, K-1)$ and $K'(n)=\max([n/2]+1, K-2))$.

(a.3.2) $P_\Gamma^j\in H^{K(n)-(1/1)}(\Gamma)$ $(j=1, \cdots, n)$; $P_\Gamma^{n+1}\in H^{K(2/2)}(\Gamma)$.

(a.3.3) ${}^tP^{ij}=P^{ji}$.

(a.3.4) There exist positive constants $d_7$ and $d_8$ such that

$$(P^{ij}\partial_j\vec{u},\ \partial_i\vec{u})+\langle P_\Gamma^j\partial_j\vec{u},\ \vec{u}\rangle\geqq d_7\|\vec{u}\|_1^2-d_8\|\vec{u}\|_0^2 \quad \text{for any } \vec{u}\in H^2(\Omega).$$

(a.3.5) $\nu_i(x)P^i(x)=0$ for $x\in\Gamma$.

Since the operators $P(t)$ and $Q(t)$ of the original problem (N) are homogeneous, it suffices to consider the case where$P_\Omega^l=P_\Gamma^{n+1}=0$. But, to the auther it seems that there are no litratures of treating with (3.30) exactly even in the case where $P_S^{ij}=P_{\Omega,S}^l=0$ and $P_\Gamma^l\in\mathcal{B}^{K-1}(\Omega)$ (namely, the smooth coefficients case). Thus, we dare to treat with the general operators for the further references. Let $\gamma_{\infty,K-1}(\Omega)$ and $\gamma_{S,K}(\Omega)$ be constants such that

$$\sum_{i,j=1}^{n}\|P_\infty^{ij}\|_{\infty,K-1}+\sum_{l=1}^{n-1}\|P_{\Omega,\infty}^l\|_{\infty,K-2}\leqq\gamma_{\infty,K-1}(\Omega);$$

$$\sum_{i,j=1}^{n}\|P_S^{ij}\|_{K(n)}+\sum_{l=1}^{n+1}\|P_{\Omega,S}^l\|_{K'(n)}+\sum_{j=1}^{n}\langle\!\langle P_\Gamma^j\rangle\!\rangle_{K(n)-(1/2)}+\langle\!\langle P_\Gamma^{n+1}\rangle\!\rangle_{K-(3/2)}\leqq\gamma_{S,K}(\Omega).$$

The purpose of this paragraph is to prove

THEOREM 3.6. *Let $L$ be an integer $\in[2, K]$. Assume that* (a.3.1)-(a.3.5) *are valid. Then, there exists a $\lambda_0>0$ depending only on $d_7$, $d_8$, $\gamma_{\infty,K-1}(\Omega)$ and $\gamma_{S,K}(\Omega)$ such that for any $\lambda>\lambda_0$, $\vec{g}_\Omega\in H^{L-2}(\Omega)$ and $\vec{g}_\Gamma\in H^{L-(3/2)}(\Gamma)$,* (3.41) *admits a unique solution $\vec{u}\in H^L(\Omega)$ having the estimate*:

$$\|\vec{u}\|_L\leqq C\{\vec{g}_\Omega\|_{L-2}+\langle\!\langle\vec{g}_\Gamma\rangle\!\rangle_{L-(3/2)}\} \tag{3.31}$$

*where $C=C(d_7, d_8, \Gamma, L, \gamma_{\infty,K-1}(\Omega), \gamma_{S,K}(\Omega))$.*

The following is an easy corollary of Theorem 3.6 and will be used to derive the "a priori estimate" of derivatives with respect to $x$ in the original problem (N).

COROLLARY 3.7. *Assume that* (A.1)-(A.4) *are valid. Let $L$ be an integer $\in[2, K]$ and $\vec{u}(t)\in L^\infty(J; H^L(\Omega))$ where $J$ is a time interval $\subset I$. Then,*

$$\|\vec{u}(t)\|_L\leqq C\{\|\partial_i(A^{ij}(t)\partial_j\vec{u}(t))\|_{L-2} + \langle\!\langle\nu_iA^{ij}(t)\partial_j\vec{u}(t)+B^j(t)\partial_j\vec{u}(t)\rangle\!\rangle_{L-(3/2)}+\|\vec{u}(t)\|_{L-1}\} \tag{3.32}$$

for any $t\in J$, where $C=C(\delta_1, \delta_2, M_\infty(K), M_S(K))$.

PROOF OF COROLLARY 3.7. If we put $P^{ij}=A^{ij}(t)$; $P_\Gamma^j=B^j(t)$; $P_\Omega^l=0$; $P_\Gamma^{n+1}=0$ $(i, j=1, \cdots, n;\ l=, \cdots, n+1)$, the assumptions: (A.1)-(A.4) implies that (a.3.1)-(a.3.5) are valid for each $t\in J$. Furthermore, put $\vec{g}_\Omega=-\partial_i(A^{ij}(t)\partial_j\vec{u}(t))+\lambda\vec{u}(t)$; $\vec{g}_\Gamma=\nu_iA^{ij}(t)\partial_j\vec{u}(t)+B^j(t)\partial_j\vec{u}(t)$; $\gamma_{\infty,K-1}(\Omega)=M_\infty(K)$; $\gamma_{S,K-1}(\Omega)=M_S(K)$. Note that the present constants $\gamma_{\infty,K-1}(\Omega)$ and $\gamma_{S,K-1}(\Omega)$ are independent of $t$. Hence, Corollary 3.7 follows from Theorem 3.6 immediately.

PROOF OF THEOREM 3.6. For the notational simplicity, we use the same letter $C$ to denote various constants depending on $d_7$, $d_8$, $\Gamma$, $L$, $\gamma_{\infty,K-1}(\Omega)$ and $\gamma_{S,K}(\Omega)$. First, we shall prove the existence of a unique weak solution in $H^1(\Omega)$. To do this, let us define the bilinear form corresponding to (3.30). Using the notations defined in the section of Notations, noting (a.3.5) and employing the same arguments as in (No. 5)-(No. 9), we have

$$\langle P_\Gamma^j \partial_j \vec{u},\ \vec{v}\rangle = \mathcal{P}(\vec{u},\ \vec{v}) + Q(\vec{u},\ \vec{v}) \tag{3.33}$$

for $\vec{u}\in H^2(\Omega)$ and $\vec{v}\in H^1(\Omega)$, where

$$R_k^p(y) = P_\Gamma^j(\Psi_k(y',\ 0))Y_{jk}^p(y',\ 0)J_k(y'),\quad p=1,\ \cdots,\ n-1;$$

$$\begin{aligned}\mathcal{P}(\vec{u},\ \vec{v}) = \sum_{k=1}^{N_0}\int_{R_+^n} & \phi_k^2(y)\{R_k^p(y')\partial_n'\vec{u}(\Psi_k(y))\cdot\partial_p'\vec{v}(\Psi_k(y)) \\ & -R_k^p(y')\partial_p'\vec{u}(\Psi_k(y))\cdot\partial_n'\vec{v}(\Psi_k((y))\}dy;\end{aligned}$$

$$\begin{aligned}Q(\vec{u},\ \vec{v}) = \sum_{k=1}^{N_0}\int_{R_+^n} & [\{\partial_p'(\phi_k^2(y)R_k^p(y'))\}\partial_n'\vec{u}(\Psi_k(y))\cdot\vec{v}(\Psi_k(y)) \\ & -\{\partial_n'\phi_k^2(y)\}R_k^p(y')\partial_p'\vec{u}(\Psi_k(y))\cdot\vec{v}(\Psi_k(y))]dy.\end{aligned}$$

By Schwarz's inequality and (No. 13.a) we see that

$$|\mathcal{P}(\vec{u},\ \vec{v})| \leqq C\|\vec{u}\|_1\|\vec{v}\|_1; \tag{3.34}$$

$$|Q(\vec{u},\ \vec{v})| \leqq C\|\vec{u}\|_1\|\vec{v}\|_0. \tag{3.35}$$

In particular, $\mathcal{P}$ and $Q$ are continuous bilinear forms on $H^1(\Omega)\times H^1(\Omega)$ and $H^1(\Omega)\times L^2(\Omega)$, respectively. Keeping (3.33) in mind, let us define the bilinear form $\boldsymbol{P}_\lambda$ corresponding to (3.30) as follows:

$$\begin{aligned}\boldsymbol{P}_\lambda[\vec{u},\ \vec{v}] = & (P^{ij}\partial_j\vec{u},\ \partial_i\vec{v}) + (P_\Omega^j\partial_j\vec{u} + P_\Omega^{n+1}\vec{u},\ \vec{v}) + \lambda(\vec{u},\ \vec{v}) \\ & + \mathcal{P}(\vec{u},\ \vec{v}) + Q(\vec{u},\ \vec{v}) + \langle P_\Gamma^{n+1}\vec{u},\ \vec{v}\rangle.\end{aligned}$$

Obviously, by Schwarz's inequality, (3.34), (3.35), (No. 13) and Corollary Ap. 4-(1).

$$|\boldsymbol{P}_\lambda[\vec{u},\ \vec{v}]| \leqq \{C+|\lambda|\}\|\vec{u}\|_1\|\vec{v}\|_1 \quad \text{for } \vec{u},\ \vec{v}\in H^1(\Omega), \tag{3.36}$$

from which it follows that $\boldsymbol{P}_\lambda$ is a continuous bilinear form on $H^1(\Omega)\times H^1(\Omega)$. Let us prove the coercivity of the $\boldsymbol{P}_\lambda$. Namely,

$$\boldsymbol{P}_\lambda[\vec{u},\ \vec{u}] \geqq (d_7/2)\|\vec{u}\|_1^2 \quad \text{for any } \vec{u}\in H^1(\Omega) \text{ provided that } \lambda\geqq\lambda_0 \tag{3.37}$$

with some $\lambda_0$ which is a constant depending only on $d_7$, $d_8$, $\gamma_{\infty,K-1}(\Omega)$ and

$\gamma_{S,K}(\Omega)$. Since $H^2(\Omega)$ is dense in $H^1(\Omega)$, it suffices to show that (3.37) is valid for any $\vec{v}\in H^2(\Omega)$. Since

$$|\langle P_{\Gamma}^{n+1}\vec{u},\ \vec{u}\rangle|\leqq C\langle\!\langle \vec{u}\rangle\!\rangle_0^2\leqq\varepsilon\|\vec{u}\|_1^2+C(n,\ \varepsilon)\|\vec{u}\|_0^2\quad\text{for any } \varepsilon>0$$

as follows from (No. 13.a) and Corollary Ap. 4-(2), noting (3.33) and (3.35), we have

$$\begin{aligned}\boldsymbol{P}_{\lambda}[\vec{u},\ \vec{u}]\geqq&(P^{ij}\partial_j\vec{u},\ \partial_i\vec{u})+\langle P_{\Gamma}^j\partial_j\vec{u},\ \vec{u}\rangle+\lambda\|\vec{u}\|_0^2\\&-C\|\vec{u}\|_1\|\vec{u}\|_0-\varepsilon\|\vec{u}\|_1^2-C(n,\ \varepsilon)\|\vec{u}\|_0^2\end{aligned}$$

for any $\vec{u}\in H^2(\Omega)$. Since $C\|\vec{u}\|_1\|\vec{u}\|_0\leqq(d_7/4)\|\vec{u}\|_1^2+(C^2/d_7)\|\vec{u}\|_0^2$, taking $\varepsilon=d_7/4$, from (a.3.4) we have (3.37). In view of (3.36) and (3.37), the $\boldsymbol{P}_{\lambda}$ is a coercive bilinear form on $H^1(\Omega)\times H^1(\Omega)$. By well-known Lax and Milgram theorem we know that there exists a unique solution $\vec{u}\in H^1(\Omega)$ of the variational equation:

(3.38) $$\boldsymbol{P}_{\lambda}[\vec{u},\ \vec{v}]=(\vec{g}_{\Omega},\ \vec{v})+\langle\vec{g}_{\Gamma},\ \vec{v}\rangle\quad\text{for any } \vec{v}\in H^1(\Omega)\,.$$

Especially, putting $\vec{v}=\vec{u}$ in (3.38) and using (3.37), we have

(3.39) $$\|\vec{u}\|_1\leqq C\{\|\vec{g}_{\Omega}\|_0+\langle\!\langle\vec{g}_{\Gamma}\rangle\!\rangle_{1/2}\}\leqq C\Delta$$

where $\Delta=\|\vec{g}_{\Omega}\|_{L-2}+\langle\!\langle\vec{g}_{\Gamma}\rangle\!\rangle_{L-(3/2)}$.

Now, by induction on $N\in[1,\ L]$ we shall prove that $\vec{u}\in H^N(\Omega)$ and

(3.40) $$\|\vec{u}\|_N\leqq C\Delta\,.$$

As has been seen, when $N=1$, the assertion is valid. Assume that $2\leqq N\leqq L$, $\vec{u}\in H^{N-1}(\Omega)$ and

(3.41) $$\|\vec{u}\|_{N-1}\leqq C\Delta\,.$$

We shall use Theorems 3.1 and 3.3. Let $\phi_k$, $k=0,\ 1,\ \cdots,\ N_0$, be the functions satisfying (No. 1). First, we shall prove that $\vec{u}_0=\phi_0\vec{u}\in H^N(\Omega)$ and

(3.42) $$\|\vec{u}_0\|_N\leqq C\Delta\,,$$

by using Theorem 3.1. To do this, we shall prove that

(3.43) $$(P^{ij}\partial_j\vec{u}_0,\ \partial_i\vec{w})=(\vec{f}_0,\ \vec{w})\quad\text{for any } \vec{w}\in H^1(\boldsymbol{R}^n);$$

(3.44) $$\|\vec{f}_0\|_{N-2}\leqq C\Delta\,,$$

where $\vec{f}_0=-\partial_i(P^{ij}(\partial_j\phi_0)\vec{u})-P^{ij}\partial_j\vec{u}\partial_i\phi_0+\phi_0\vec{g}_{\Omega}-\phi_0(P_{\Omega}^j\partial_j\vec{u}+P_{\Omega}^{n+1}\vec{u}+\lambda\vec{u})$. First, we note that

(3.45) $$\mathcal{P}(\vec{v},\ \vec{w})+Q(\vec{v},\ \vec{w})=0\quad\text{for any } \vec{v}\in H^1(\Omega) \text{ and } \vec{w}\in H^1_{(0)}(\Omega)$$

where $H^L_{(0)}(\Omega)$ is the same as in § 3.1. In fact, since $\langle P^j\partial_j\vec{v},\ \vec{w}\rangle=0$ for any $\vec{v}\in H^2(\Omega)$ and $\vec{w}\in H^1_{(0)}(\Omega)$, (3.45) follows from (3.33) when $\vec{v}\in H^2(\Omega)$. Since $H^2(\Omega)$

is dense in $H^1(\Omega)$ and $\mathscr{P}$ and $Q$ are continuous, (3.45) is also valid for any $\vec{v}\in H^1(\Omega)$ and $\vec{w}\in H^1_{(0)}(\Omega)$. Let $\vec{w}\in H^1(\boldsymbol{R}^n)$. Since

$$(3.46)\qquad (P^{ij}\partial_j\vec{u}_0,\ \partial_i\vec{w})=(P^{ij}\partial_j\vec{u},\ \partial_i(\phi_0\vec{w}))-(\partial_i(P^{ij}(\partial_j\phi_0)\vec{u}),\ \vec{w})-(P^{ij}\partial_j\vec{u}\partial_i\phi_0,\ \vec{w}),$$

noting that $\phi_0\vec{w}\in H^1_{(0)}(\Omega)$, $(\vec{g}_\Omega,\ \phi_0\vec{w})+\langle\vec{g}_\Gamma,\ \phi_0\vec{w}\rangle=(\phi_0\vec{g},\ \vec{w})$ and $\langle P_\Gamma^{n+1}\vec{u},\ \phi_0\vec{w}\rangle=0$, by (3.38), (3.45) and (3.46), we have (3.43).

Applying (Ap. 1) with $\alpha=K(n)$ and $\beta=\gamma=N-1$, we have

$$(3.47.\mathrm{a})\qquad \|\partial_i(P_S^{ij}(\partial_j\phi_0)\vec{u})\|_{N-2}\leqq C\|P_S^{ij}\|_{K(n)}\|\vec{u}\|_{N-1}\,.$$

Applying (Ap. 1) with $\alpha=K(n)$ and $\beta=\gamma=N-2$, we have

$$(3.47.\mathrm{b})\qquad \|P_S^{ij}\partial_j\vec{u}\partial_i\phi_0\|_{N-2}\leqq C\|P_S^{ij}\|_{K(n)}\|\vec{u}\|_{N-1}\,.$$

Applying (Ap. 1) with $\alpha=K'(n)$ and $\beta=\gamma=N-2$, we have

$$(3.47.\mathrm{c})\qquad \|P_{\Omega,S}^j\partial_j\vec{u}+P_{\Omega,S}^{n+1}\vec{u}\|_{N-2}\leqq C\sum_{l=1}^{n+1}\|P_{\Omega,S}^l\|_{K'(n)}\|\vec{u}\|_{N-2}\,.$$

(3.44) follows immediately from (3.47) and (3.41).

To use Theorem 3.1, we must check the conditions (a.1.1) and (a.1.2). However, in this case, (a.1.1) follows from (a.3.1) obviously. If $\vec{v}\in H^2_{(0)}(\Omega)$, then from (a.3.4) it follows that

$$(3.48)\qquad (P^{ij}\partial_j\vec{v},\ \partial_i\vec{v})\geqq d_7\|\vec{v}\|_1^2-d_8\|\vec{v}\|_0^2\,,$$

because $\langle P^j\partial_j\vec{v},\ \vec{v}\rangle=0$. Since $H^2_{(0)}(\Omega)$ is dense in $H^1_{(0)}(\Omega)$, (3.48) is valid for any $\vec{v}\in H^1_{(0)}(\Omega)$. Hence, in the present case, (a.1.2) is also valid. Applying Theorem 3.1 to (3.43) and using (3.44) and (3.41), we see easily that $\vec{u}_0\in H^N(\Omega)$ and (3.42) is valid.

Now, we consider $\vec{u}_k(y)=\phi_k(\Psi_k(y))\vec{u}(\Psi_k(y))$ $(k=1,\cdots,N_0)$. By Theorem 3.3, we shall prove that $\vec{u}_k\in H^N(\boldsymbol{R}_+^n)$ and that

$$(3.49)\qquad \|\vec{u}_k\|'_N\leqq C\Delta\,.$$

Here and hereafter, for the notational simplicity, we use the same abbreviation: $\|\cdot\|'_r$ as in §3.2. Likewise for $\langle\!\langle\cdot\rangle\!\rangle'_r$, $(\cdot,\ \cdot)'$, $\langle\cdot,\ \cdot\rangle'$. For given $\vec{v}(x)$ and $W(y)$, we write $\vec{V}(y)=\vec{v}(\Psi_k(y))$ and $\vec{w}(x)=W(\Phi_k(x))$. For the notational simplicity, put $H_\sigma^L(\boldsymbol{R}_+^n)=\{\vec{V}\in H^L(\boldsymbol{R}_+^n)\,|\,\mathrm{supp}\,\vec{V}\subset Q(\sigma)\}$ and $H_\sigma^L(\Omega)=\{\vec{v}\in H^L(\Omega)\,|\,\mathrm{supp}\,\vec{V}\subset Q(\sigma)\}$. Since $\mathrm{supp}\,\vec{u}_k\subset Q(\sigma_k)$, we may assume that $\mathrm{supp}\,\vec{u}_k\subset Q(\sigma'_k)$ for some $\sigma'_k\in(0,\ \sigma_k)$. Let $\rho_k(y)\in C_0^\infty(Q(\sigma_k))$ such that $\rho_k(y)=1$ on $Q(\sigma''_k)$ for some $\sigma''_k\in(\sigma'_k,\ \sigma_k)$. Recall that the Jacobian of the transformation: $y=\Phi_k(x)$ is equal to 1, i.e., $dx=dy$. Noting (3.33) and (a.3.5), for any $\vec{v}\in H^2_{\sigma''_k}(\Omega)$ and $\vec{w}\in H^1(\Omega)$ we have

(3.49) $(P^{ij}\partial_j\vec{v},\ \partial_i\vec{w})+\mathscr{P}(\vec{v},\ \vec{w})+Q(\vec{v},\ \vec{w})=(P^{ij}\partial_j\vec{v},\ \partial_i\vec{w})+\langle P_I^j\partial_j\vec{v},\ \vec{w}\rangle$

$$=(a^{ij}\partial_j'\vec{V},\ \partial_i'W)'+\langle c^p\partial_p'\vec{V}(\cdot,0),\ W(\cdot,0)\rangle'$$

where $\partial_j'=\partial/\partial y_j$; $a^{ij}(y)=\rho_k(y)Y^i_{i'k}(y)Y^j_{j'k}(y)P^{i'j'}(\Psi_k(y))$; $y'=(y_1,\cdots,y_{n-1})$; $c^p(y')=\rho_k(y',0)P_I^j(\Psi_k(y',0))Y^p_{jk}(y',0)J_k(y',0)$. In view of Theorem Ap. 3 and (a.3.2), there exist $b^p(y)\in H^{K(n)}(\boldsymbol{R}^n_+)$ such that $b^p(y',0)=c^p(y')$ for almost all $y'\in\boldsymbol{R}^{n-1}$ and

(3.50) $$\|b^p\|'_{K(0)}\leqq C\langle\!\langle c^p\rangle\!\rangle'_{K(n)-(1/2)}\leqq C\gamma_{S,K}(\Omega).$$

And also, we have

(3.51) $$|a^{ij}_\infty|_{\infty,K-1,\boldsymbol{R}^n_+}\leqq C\gamma_{\infty,K-1}(\Omega);\ \|a^{ij}_S\|'_{K(n)}\leqq C\gamma_{S,K}(\Omega)$$

where $a^{ij}_U(y)=\rho_k(y)Y^i_{i'k}(y)Y^j_{j'k}(y)P^{i'j'}_U(\Psi_k(y))$ for $U=\infty$ and $S$. Put

$$B[\vec{V},W]=(a^{ij}\partial_j'\vec{V},\ \partial_i'W)'+(b^p\partial_n'\vec{V},\ \partial_p'W)'-(b^p\partial_p'\vec{V},\ \partial_n'W)'.$$

From (3.49) we have

(3.52) $$B[\vec{V},W]=(P^{ij}\partial_j\vec{v},\ \partial_i\vec{w})+\mathscr{P}(\vec{v},\ \vec{w})+Q(\vec{v},\ \vec{w})$$
$$-((\partial_p'b^p)\partial_n'\vec{V},\ W)'+((\partial_n'b^p)\partial_p'\vec{V},\ W)'$$

for any $\vec{V}\in H^1_{\sigma_k'}(\boldsymbol{R}^n_+)$ and $W\in H^1(\boldsymbol{R}^n_+)$. In fact, if $\vec{V}\in H^2_{\sigma_k''}(\boldsymbol{R}^n_+)$ and $W\in H^1(\boldsymbol{R}^n_+)$, employing the some arguments as in (3.27), from (3.49) we have (3.52). Since $H^2_{\sigma_k''}(\boldsymbol{R}^n_+)$ is dense in $H^1_{\sigma_k'}(\boldsymbol{R}^n_+)$ $(0<\sigma_k'<\sigma_k''<\sigma_k)$ (3.52) is also valid for any $\vec{V}\in H^1_{\sigma_k'}(\boldsymbol{R}^n_+)$ and $W\in H^1(\boldsymbol{R}^n_+)$.

Employing the same arguments as above, from (3.49) and (a.3.4), we have also

$$B[\vec{V},\vec{V}]=(P^{ij}\partial_j\vec{v},\ \partial_i\vec{v})+\langle P_I^j\partial_j\vec{v},\ \vec{v}\rangle-(\{\partial_p'b^p\}\partial_n'\vec{V},\ \vec{V})'+(\{\partial_n'b^p\}\partial_p'\vec{V},\ \vec{V})'$$
$$\geqq d_7\|\vec{v}\|_1^2-d_8\|\vec{v}\|_0^2-C\gamma_{S,K}\|\vec{V}\|_1'\|\vec{V}\|_0'\quad\text{for any } \vec{V}\in H^2_{\sigma_k''}(\boldsymbol{R}^n_+).$$

where we have used (No. 13.b) and (3.50). Since $H^1(\boldsymbol{R}^n_+\cap Q(\sigma_k))$ and $H^1((\Omega)\cap\mathcal{O}_k)$ are homeomorphic by the transformation: $y=\Phi_k(x)$, i.e., there exists a constant $c_5>0$ such that

$$c_5^{-1}\|\vec{V}\|_{\boldsymbol{R}^n_+\cap Q(\sigma_k),1}\leqq\|\vec{v}\|_{\Omega\cap\mathcal{O}_k,1}\leqq c_5\|\vec{V}\|_{\boldsymbol{R}^n_+\cap Q(\sigma_k),1},$$

and since $H^2_{\sigma_k''}(\boldsymbol{R}^n_+)$ is dense in $H^1_{\sigma_k'}(\boldsymbol{R}^n_+)$ (cf. $\sigma_k'<\sigma_k''$), we have

(3.53) $$B[\vec{V},\vec{V}]\geqq d_9(\|\vec{V}\|_1')^2-d_{10}(\|\vec{V}\|_0')^2\quad\text{for any } \vec{V}\in H^1_{\sigma_k'}(\boldsymbol{R}^n_+)$$

with some positive constants $d_9$ and $d_{10}$ which depend only on $d_7$, $d_8$, $\gamma_{\infty,K-1}(\Omega)$, $\gamma_{S,K}(\Omega)$ and $\Gamma$. Combining (3.50), (3.51), (3.53) and (a.3.3), we see that the present bilinear form $B$ satisfies (a.2.1)-(a.2.4) of § 3.2.

Now, we shall prove that

(3.54) $\quad B[\vec{u}_k, \vec{V}]=(\vec{f}_1, \vec{V})'+\langle\vec{f}_1, \vec{V}(\cdot, 0)\rangle'+(\vec{f}_3^i, \partial_i'\vec{V})'$ for any $\vec{V}\in H^1(\boldsymbol{R}_+^n)$;

(3.55) $\quad \|\vec{f}_1\|_{N-2}'\leqq C\Delta$;

(3.56) $\quad \langle\!\langle\vec{f}_2\rangle\!\rangle_{N-(3/2)}'\leqq C\Delta$;

(3.57) $\quad \|\vec{f}_3^i\|_{N-1}'\leqq C\Delta$,

where

$$\vec{f}_1(y)=(\phi_k\vec{g}_\Omega)(\Psi_k(y))-(\phi_k(P_\Omega^j\partial_j\vec{u}+P_\Omega^{n+1}\vec{u}+\lambda\vec{u}))(\Psi_k(y))$$
$$-(P^{ij}(\partial_j\phi_k)\partial_j\vec{u})(\Psi_k(y))-\{\partial_p'b^p(y)\}\partial_n'\vec{u}_k(y)+\{\partial_n'b^p(y)\}\partial_p'\vec{u}_k(y);$$
$$\vec{f}_2(y')=(\phi_k\vec{g}_\Gamma)(\Psi_k(y', 0))+(P_\Gamma^j(\partial_j\phi_k)\vec{u})(\Psi_k(y', 0))-(\phi_kP_\Gamma^{n+1}\vec{u})(\Psi_k(y', 0));$$
$$\vec{f}_3^i(y)=(P^{i'j}(\partial_j\phi_k)\vec{u})(\Psi_k(y))Y_{i'k}^i(y).$$

To prove (3.54), we use the formula:

(3.58)
$$(P^{ij}\partial_j(\phi_k\vec{v}), \partial_i\vec{w})+\mathscr{P}(\phi_k\vec{v}, \vec{w})+Q(\phi_k\vec{v}, \vec{w})$$
$$=\boldsymbol{P}_\lambda[\vec{v}, \phi_k\vec{w}]+(P^{ij}(\partial_j\phi_k)\vec{v}, \partial_i\vec{w})-(P^{ij}(\partial_i\phi_k)\partial_j\vec{v}, \vec{w})$$
$$+\langle P_\Gamma^j(\partial_j\phi_k)\vec{v}, \vec{w}\rangle-(\phi_k\{P_\Omega^j\partial_j\vec{v}+P_\Omega^{n+1}\vec{v}+\lambda\vec{v}\}, \vec{w})-\langle\phi_kP_\Gamma^{n+1}\vec{v}, \vec{w}\rangle$$

for any $\vec{v}$ and $\vec{w}\in H^1(\Omega)$. Noting the definition of $\boldsymbol{P}_\lambda$ and (3.33), for $\vec{v}\in H^2(\Omega)$ and $\vec{w}\in H^1(\Omega)$ we can check (3.58) easily. Since $H^2(\Omega)$ is dense and since the both hand sides of (3.58) are continuous bilinear forms on $H^1(\Omega)\times H^1(\Omega)$ (note Corollary Ap. 4-(2) and (No. 13.a)), (3.58) is also valid for any $\vec{v}$ and $\vec{w}\in H^1(\Omega)$. Since $\vec{u}_k(y)=\phi_k(\Psi_k(y))\vec{u}(\Psi_k(y))\in H_{\sigma'}^1(\boldsymbol{R}_+^n)$, combining (3.52), (3.58) and (3.38) and making the change of variables: $x=\Psi_k(y)$, we have (3.54).

Now, we check (3.55)-(3.57). Employing the same arguments as in (3.47) and using (3.50) and (3.41), we have (3.55) easily. Applying (Ap. 2) with $\sigma=K-1$ and $\beta=\gamma=N-1$, we have

$$\langle\!\langle P_\Gamma^l\vec{u}\rangle\!\rangle_{N-(3/2)}\leqq C\langle\!\langle P_\Gamma^l\rangle\!\rangle_{K-(3/2)}\|\vec{u}\|_{N-1} \quad \text{for } l=1, \cdots, n+1.$$

(3.56) follows immediately from this fact and (3.41). Applying (Ap. 1) with $\alpha=K(n)$ and $\beta=\gamma=N-1$ and using (3.41), we have (3.57).

Hence, applying Theorem 3.3 to (3.54) and using (3.55)-(3.57), we can conclude that $\vec{u}_k\in H^N(\boldsymbol{R}_+^n)$ and that (3.49) is valid. Noting (No. 1) and combining (3.42) and (3.49), we see easily that $\vec{u}\in H^N(\Omega)$ and (3.40) is valid.

Finally, we shall prove that $\vec{u}$ satisfies (3.30). First, noting that $\vec{u}\in H^2(\Omega)$ and taking $\vec{v}\in C_0^\infty(\Omega)$ in (3.38), by the divergence theorem we see that $\vec{u}$ satisfies (3.30.a). Then, applying the divergence theorem to (3.38) again and substituting

(3.30.a) into the resulting equations, we have

$$\langle \nu_i P^{ij}\partial_j \vec{u} + P^j_{\Gamma}\partial_j \vec{u} + P^{n+1}_{\Gamma}\vec{u} - \vec{g}_1, \vec{v}\rangle = 0 \quad \text{for any } \vec{v}\in H^1(\Omega).$$

In view of Corollary Ap. 5, for any $\vec{w}\in H^{1/2}(\Gamma)$, there exists a $\vec{v}\in H^1(\Omega)$ such that $\vec{v}(x)=\vec{w}(x)$ for almost all $x\in\Gamma$. Combining these two facts implies that $\vec{u}$ satisfies (3.30.b). This completes the proof of Theorem 3.6.

3.4 The time dependence of solutions to some elliptic boundary value problem. In this paragraph, we consider the following problem:

$$(3.59.\mathrm{a})_M \qquad \vec{v}_{M+2}(t)=P_N(t)[\vec{v}_0(t),\cdots,\vec{v}_{N+1}(t)]+\lambda_M\vec{v}_M(t)=\vec{f}_M(t) \quad \text{in } J\times\Omega,$$

$$(3.59.\mathrm{b})_M \qquad Q_M(t)[\vec{v}_0(t),\cdots,\vec{v}_{M+1}(t)]=\vec{g}_M(t) \quad \text{on } J\times\Gamma.$$

for $0\leqq M\leqq N_1$, where $J\subset I$; $N_1$ is an integer $\in[0, K-3]$;

$$P_M(t)[\vec{w}_0,\cdots,\vec{w}_{M+1}]=\sum_{k=1}^{M}\binom{M}{k}\partial_i(\partial_t^k A^{i0}(t)\vec{w}_{M+1-k}+\partial_t^k A^{ij}(t)\partial_j\vec{w}_{M-k});$$

$$Q_M(t)[\vec{w}_0,\cdots,\vec{w}_{M+1}]=\sum_{k=0}^{M}\binom{M}{k}\{\nu_i\partial_t^k A^{ij}(t)\partial_j\vec{w}_{M-k}+\partial_t^k B^j(t)\partial_j\vec{w}_{M-k}+\partial_t^k B^j(t)\vec{w}_{M+1-k}\};$$

$\vec{v}_{N_1+1}(t)$, $\vec{v}_{N_1+2}(t)$, $\vec{f}_M(t)$ and $\vec{g}_M(t)$ $(0\leqq M\leqq N_1)$ are given functions; $\vec{v}_0(t),\cdots,\vec{v}_{N_1}(t)$ are unknown functions. The following theorem will be used in proving the further regularities of solutions to (N) with respect to $x$.

THEOREM 3.8. *Assume that* (A.1)-(A.4) *are valid. Let* $N_1$ *and* $N_2$ *be integers such that* $0\leqq N_1\leqq K-3$ *and* $N_1+2\leqq N_2\leqq K$. *Then, there exist constants* $\lambda_M$ $(0\leqq M\leqq N_1)$ *having the following properties: Let* $t$ *be any fixed time in* $J$. *If* $\vec{f}_M\in H^{N_2-M-2}(\Omega)$, $\vec{g}_M\in H^{N_2-M-(3/2)}(\Gamma)$ $(0\leqq M\leqq N_1)$, $\vec{v}_{N_1+l}\in H^{N_2-M-l}(\Omega)$ $(l=1, 2)$, *then* (3.59) *admits a unique system* $(\vec{v}_0,\cdots,\vec{v}_{N_1})\in H^{N_2}(\Omega)\times\cdots\times H^{N_2-N_1}(\Omega)$ *of solutions having the estimate*:

$$(3.60) \qquad \sum_{M=0}^{N_1}\|\vec{v}_M\|_{N_2-M}\leqq C\Big\{\sum_{l=1}^{2}\|\vec{v}_{N_1+l}\|_{N_2-N_1-l}+\sum_{M=0}^{N_1}(\|\vec{f}_M\|_{N_2-M-2}+\langle\!\langle\vec{g}_M\rangle\!\rangle_{N_2-M-(3/2)})\Big\}$$

*where* $C=C(\lambda_0,\cdots,\lambda_{N_1},\delta_1,\delta_2,M_\infty(K),M_S(K))$.

*Furthermore, in addition to what we have assumed, assume that* $N_1+3\leqq N_2\leqq K$. *If* $\vec{f}_M(t)\in X^{1,N_2-M-3}(J,\Omega)$, $\vec{g}_M(t)\in X^{1,N_2-M-(5/2)}(J,\Gamma)$ $(0\leqq M\leqq N_1)$, $\vec{v}_{N_1+l}(t)\in X^{1,N_2-N_1-l-1}(J,\Omega)$ $(l=1, 2)$, *then* (3.59) *admits a unique system* $(\vec{v}_0(t),\cdots,\vec{v}_{N_1}(t))\in X^{1,N_2-1}(J,\Omega)\times\cdots\times X^{1,N_2-N_1-1}(J,\Omega)$ *of solutions satisfying the estimates*:

$$(3.61) \qquad \sum_{M=1}^{N_1}\|\partial_t^k\vec{v}_M(t)\|_{N_2-M-k}\leqq C\sum_{h=0}^{k}\Big\{\sum_{l=1}^{2}\|\partial_t^h\vec{v}_{N_1+l}(t)\|_{N_2-N_1-l-h}$$
$$+\sum_{M=0}^{N_1}(\|\partial_t^h\vec{f}_M(t)\|_{N_2-M-h-2}+\langle\!\langle\partial_t^h\vec{g}_M(t)\rangle\!\rangle_{N_2-M-h-(3/2)})$$

*for any* $t\in J$ *and* $k=0, 1$, *where* $C=C(\lambda_0, \cdots, \lambda_{N_1}, \delta_1, \delta_2, M_\infty(K), M_S(K))$.

PROOF. By induction on $N_1$ we shall prove the first assertion. When $N_1=0$, (3.59) can be written as follows:

$$-\partial_i(A^{ij}(t)\partial_j\vec{v}_0)+\lambda_0\vec{v}_0=\vec{f}_0-\vec{v}_2+\partial_i(A^{i0}(t)\vec{v}_1) \quad \text{on } \Omega\,, \tag{3.62.a}$$

$$\nu_i A^{ij}(t)\partial_j\vec{v}_0+B^j(t)\partial_j\vec{v}_0=\vec{g}_0-B^0(t)\vec{v}_1 \quad \text{on } \Gamma\,. \tag{3.62.b}$$

Since $\|A^{ij}_\infty(t)\|_{\infty, K-1}\leqq M_\infty(K)$ and $\|A^{ij}_S(t)\|_{K(n)}+\langle\langle B^j(t)\rangle\rangle_{K(n)-(1/2)}\leqq M_S(K)$ for any $t\in J$ ($K(n)=\max[K-1, [n/2]+2]$), if the right-hand sides of (3.62.a) and (3.62.b) belong to $H^{N_2-2}(\Omega)$ and $H^{N_1-(3/2)}(\Gamma)$, respectively, then by Theorem 3.6 we see that there exists a $\lambda_0>0$ depending only on $\delta_1$, $\delta_2$, $M_\infty(K)$ and $M_S(K)$ and independent of $t\in J$ such that for any $\lambda\geqq\lambda_0$, (3.62) admits a unique solution $\vec{v}_0\in H^{N_2}(\Omega)$ and

$$\begin{aligned}\|\vec{v}_0\|_{N_2}\leqq C\{&\|\vec{f}_0\|_{N_2-2}+\langle\langle\vec{g}_0\rangle\rangle_{N_2-(3/2)}+\|\vec{v}_2\|_{N_2-2}\\ &+\|\partial_i(A^{i0}(t)\vec{v}_1)\|_{N_2-2}+\langle\langle B^0(t)\vec{v}_1\rangle\rangle_{N_2-(3/2)}\}\end{aligned} \tag{3.63}$$

where $C=C(\lambda_0, \delta_1, \delta_2, M_\infty(K), M_S(K))$. Since $\vec{v}_1\in H^{N_2-1}(\Omega)$, applying (Ap. 1) and (Ap. 3) with $\alpha=K$, $\beta=\gamma=N_2-1$, we have

$$\|\partial_i(A^{i0}_S(t)\vec{v}_1)\|_{N_2-2}\leqq C\|A^{i0}_S(t)\|_K\|\vec{v}_1\|_{N_2-1}\,;$$

$$\langle\langle B^0(t)\vec{v}_1\rangle\rangle_{N_2-(3/2)}\leqq C\langle\langle B^0(t)\rangle\rangle_{K-(1/2)}\|\vec{v}_1\|_{N_2-1}\,.$$

From this it follows immediately that the right-hand side of (3.62.a) and (3.62.b) belong to $H^{N_2-2}(\Omega)$ and $H^{N_2-(3/2)}(\Gamma)$, respectively. And then, noting (3.63), we see that the first assertion is valid for $N_1=0$.

Now, let us assume that $1\leqq N_1\leqq K-3$ and that the first assertion is valid for smaller values of $N_1$. Then, for any $N$ such that $M_1+1\leqq M\leqq K$, $\vec{f}_M\in H^{N-M-2}(\Omega)$, $\vec{g}_M\in H^{N-M-(3/2)}(\Gamma)$ $(0\leqq M\leqq N_1-1)$, $\vec{v}_{N_1}\in H^{N-N_1}(\Omega)$ and $\vec{v}_{N_1+1}\in H^{N-N_1-1}(\Omega)$, there exist constants $\lambda_0, \cdots, \lambda_{N_1-1}>0$ independent of $\vec{f}_M$, $\vec{g}_M$, $\vec{v}_{N_1}$ and $\vec{v}_{N_1+1}$ such that there exist $\vec{v}_M\in H^{M-N}(\Omega)$ $(0\leqq M\leqq N_1-1)$ satisfying the equations $(3.59.\text{a})_M$ and $(3.59.\text{b})_M$ $(0\leqq M\leqq N_1)$. Furthermore, these solutions are determined uniquely and satisfy the estimate:

$$\begin{aligned}\sum_{M=0}^{N_1-1}\|\vec{v}_M\|_{N-M}\leqq C\{&\|\vec{v}_{N_1}\|_{N-N_1}+\|\vec{v}_{N_1+1}\|_{N-N_1-1}\\ &+\sum_{M=0}^{N_1-1}(\|\vec{f}_M\|_{N-M-2}+\langle\langle\vec{g}_M\rangle\rangle_{N-M-(3/2)})\}\end{aligned} \tag{3.64}$$

where $C=C(\lambda_0, \cdots, \lambda_{N_1-1}, \delta_1, \delta_2, M_\infty(K), M_S(K))$. Let us denote solutions, obtained by putting $\vec{f}_M=\vec{g}_M=0$ $(0\leqq M\leqq N_1)$ and $\vec{v}_{N_1+1}=0$, by $R_M=R_M(\vec{v}_{N_1})$. And also, let us denote solutions, obtained by putting $\vec{v}_{N_1}=0$, by $S_M=S_M(\vec{f}_0, \cdots, \vec{f}_{N_1-1}, \vec{g}_0, \cdots, \vec{g}_{N_1-1}, \vec{v}_{N_1+1})$. Since the equations are linear, the uniqueness of solutions implies that each $R_M(\vec{v}_{N_1})$ is a linear map from $H^{N-N_1}(\Omega)$ to $H^{N-M}(\Omega)$. Furthermore, by (3.64) we have

$$\sum_{M=0}^{N_1-1}\|R_M(\vec{v}_{N_1})\|_{N-M}\leqq C\|\vec{v}_{N_1}\|_{N-N_1}; \tag{3.65}$$

$$\sum_{M=0}^{N_1-1}\|S_M(\vec{f}_0, \cdots, \vec{f}_{N_1-1}, \vec{g}_0, \cdots, \vec{g}_{N_1-1}, \vec{v}_{N_1+1})\|_{N-M} \tag{3.66}$$

$$\leqq C\{\|\vec{v}_{N_1+1}\|_{N-N_1-1}+\sum_{M=0}^{N_1-1}(\|\vec{f}_M\|_{N-M-2}+\langle\!\langle\vec{g}_M\rangle\!\rangle_{N-M-(3/2)})\}.$$

Here, $C=C(\lambda_0, \cdots, \lambda_{N_1-1}, \delta_1, \delta_2, M_\infty(K), M_S(K))$. Note that general solutions $\vec{v}_M$ can be written as follows: $\vec{v}_M=R_M+S_M$. Substituting $\vec{v}_M$ $(0\leqq M\leqq N_1-1)$ into the equations: $(3.59.a)_{N_1}$ and $(3.59.b)_{N_1}$, we have the equations for unknown $\vec{v}_{N_1}$ as follows:

$$-P_{N_1}(t)[R_0(\vec{v}_{N_1}), \cdots, R_{N_1-1}(\vec{v}_{N_1}), \vec{v}_{N_1}, 0]+\lambda_{N_1}v_{N_1}=\vec{F}_\Omega \quad \text{in } \Omega, \tag{3.67.a}$$

$$Q_{N_1}(t)[R_0(\vec{v}_{N_1}), \cdots, R_{N_1-1}(\vec{v}_{N_1}), \vec{v}_{N_1}, 0]=\vec{F}_\Gamma \quad \text{on } \Gamma, \tag{3.67.b}$$

where

$$\vec{F}_\Omega=\vec{f}_{N_1}-\vec{v}_{N_1+2}+P_{N_1}(t)[S_0, \cdots, S_{N_1-1}, 0, \vec{v}_{N_1+1}]; \tag{3.68.a}$$

$$\vec{F}_\Omega=\vec{g}_{N_1}-Q_{N_1}(t)[S_0, \cdots, S_{N_1-1}, 0, \vec{v}_{N_1+1}]. \tag{3.68.b}$$

Our task is to find a solution $\vec{v}_{N_1}\in H^{N_2-N_1}(\Omega)$. As a first step, by the variational method we prove the existence of a weak solution $\vec{v}_{N_1}\in H^1(\Omega)$. Keeping this in mind, let us consider the following variational equation:

$$V_\lambda[\vec{v}, \vec{u}]=(\vec{F}_\Omega, \vec{u})+\langle\vec{F}_\Gamma, \vec{u}\rangle \quad \text{for any } \vec{u}\in H^1(\Omega) \tag{3.69}$$

where

$$V_\lambda[\vec{v}, \vec{u}]=B_\lambda[t, \vec{v}, \vec{u}]-C_1(t, \vec{v}, \vec{u})+C_2(t, \vec{v}, \vec{u}) \tag{3.70.a}$$

$$C_1(t, \vec{v}, \vec{u})=N_1(\partial_i(\partial_t A^{i0}(t)\vec{v}), \vec{u})+(P_{N_1}(t)[R_0(\vec{v}), \cdots, R_{N_1-1}(\vec{v}), 0, 0], \vec{u}); \tag{3,70.b}$$

$$C_2(t, \vec{v}, \vec{u})=N_1\langle\partial_t B^0(t)\vec{v}, \vec{u}\rangle+\langle Q_{N_1}(t)[R_0(\vec{v}), \cdots, R_{N_1-1}(\vec{v}), 0, 0], \vec{u}\rangle; \tag{3.70.c}$$

$B_\lambda$ is the same bilinear form as in (No. 14). Here, note that $P_{N_1}(t)[R_0(\vec{v}), \cdots, R_{N_1-1}(\vec{v}), \vec{v}, 0] = \partial_i(A^{ij}(t)\partial_j\vec{v} + N_1A^{i0}(t)\vec{v}) + P_{N_1}(t)[R_0(\vec{v}), \cdots, R_{N_1-1}(\vec{v}), 0, 0]$ and $Q_{N_1}(t)[R_0(\vec{v}), \cdots, R_{N_1-1}(\vec{v}), \vec{v}, 0] = \nu_iA^{ij}(t)\partial_j\vec{v} + B^j(t)\partial_j\vec{v}+N_1\partial_t B^0(t)\vec{v} + Q_{N_1}(t)[R_0(\vec{v}), \cdots, R_{N_1-1}(\vec{v}), 0, 0]$. Let us prove that $V_\lambda$ is coercive for large $\lambda>0$. For the notational simplicity, we shall use the same letter $C'$ to denote various constants depending on $\lambda_0, \cdots, \lambda_{N_1-1}, \delta_1, \delta_2, M_\infty(K)$ and $M_S(K)$. To estimate $\mathcal{C}_1$ and $\mathcal{C}_2$, we use the following facts: Let $L$ be an integer $\in[1, N_2-N_1]$. If $\vec{v}\in H^L(\Omega)$, then

$$\|\partial_i(\partial_t A^{i0}(t)\vec{v})\|_{L-1}+\|P_{N_1}(t)[R_0(\vec{v}), \cdots, R_{N_1-1}(\vec{v}), 0, 0]\|_{L-1}\leqq C'\|\vec{v}\|_L; \tag{3.71.a}$$

$$\langle\!\langle\partial_t B^0(t)\vec{v}\rangle\!\rangle_{L-(1/2)}+\langle\!\langle Q_{N_1}(t)[R_0(\vec{v}), \cdots, R_{N_1-1}(\vec{v}), 0, 0]\rangle\!\rangle_{L-(1/2)}\leqq C'\|\vec{v}\|_L. \tag{3.71.a}$$

In fact, since $N_1+1\leqq N_1+L\leqq K$, by (3.65) with $N=N_1+L$ we know that

$$\sum_{M=0}^{N_1-1}\|R_M(\vec{v})\|_{N_1+L-M}\leqq C'\|\vec{v}\|_L. \tag{3.72}$$

Hence, letting $1\leqq k\leqq N_1$, applying (Ap. 1)-(Ap. 3) with $\alpha=K-k$, $\beta=L-1+k$ and $\gamma=L$ and using (3.72), we have

$$\|\partial_i\{\partial_t^k A_S^{i0}(t)R_{N_1+1-k}(\vec{v})\}\|_{L-1}\leqq C'\|\vec{v}\|_L; \tag{3.73.a}$$

$$\|\partial_i\{\partial_t^k A_S^{ij}(t)\partial_j R_{N_1-k}(\vec{v})\}\|_{L-1}\leqq C'\|\vec{v}\|_L; \tag{3.73.b}$$

$$\langle\!\langle\nu_i\partial_t^k A_S^{ij}(t)\partial_j R_{N_1-k}(\vec{v})\rangle\!\rangle_{L-(1/2)}\leqq C'\|\vec{v}\|_L; \tag{3.73.c}$$

$$\langle\!\langle\partial_t^k B^j(t)\partial_j R_{N_1-k}(\vec{v})\rangle\!\rangle_{L-(1/2)}\leqq C'\|\vec{v}\|_L; \tag{3.73.d}$$

$$\langle\!\langle\partial_t^k B^0(t)R_{N_1+1-k}(\vec{v})\rangle\!\rangle_{L-(1/2)}\leqq C'\|\vec{v}\|_L, \tag{3.73.e}$$

where we have put $R_{N_1}(\vec{v})=\vec{v}$ for the notational simplicity. In particular, by (3.71) with $L=1$, (No. 11), (No. 12) and (No. 12.b), we have

$$|V_\lambda[\vec{v}, \vec{u}]|\leqq C'\|\vec{v}\|_1\|\vec{u}\|_1 \quad \text{for any } \vec{v}, \vec{u}\in H^1(\Omega). \tag{3.74}$$

Recall (No. 16). Namely, we know that $B_\lambda[t, \vec{v}, \vec{v}]\geqq\delta_1\|\vec{v}\|_1^2$ for $\lambda>\delta_2$. On the other hand, by (No. 12), Schwarz's inequality and (3.73.b) and (3.73.a), we have for any $\varepsilon>0$

$$|\mathcal{C}_1(t, \vec{v}, \vec{v})|\leqq C'\|\vec{v}\|_1\|\vec{v}\|_0\leqq\varepsilon\|\vec{v}\|_1^2+\{(C')^2/4\varepsilon\}\|\vec{v}\|_0^2. \tag{3.75.a}$$

And also, noting Corollary Ap. 4-(2), by Schwarz's inequality and (3.73.b) we have

$$|\mathcal{C}_2(t, \vec{v}, \vec{v})|\leqq C'\langle\!\langle\vec{v}\rangle\!\rangle_0^2\leqq C'\{\varepsilon\|\vec{v}\|_1^2+C(\varepsilon, n)\|\vec{v}\|_0^2\}. \tag{3.75.b}$$

Combining (No. 16) and (3.75) and taking $\varepsilon>0$ so small, we see easily that there exists a $\lambda^{(1)}>0$ depending only on $\lambda_0, \cdots, \lambda_{N_1}, \delta_1, \delta_2, M_\infty(K)$ and $M_S(K)$ such that

$$V_\lambda[\vec{v}, \vec{v}]\geqq(\delta_1/2)\|\vec{v}\|_1^2 \quad \text{for any } \vec{v}\in H^1(\Omega) \text{ and } \lambda>\lambda^{(1)}. \tag{3.76}$$

Combining (3.74) and (3.76) implies that $V_\lambda$ is a coercive bilinear from on $H^1(\Omega)\times H^1(\Omega)$ for $\lambda>\lambda^{(1)}$.

Now, we shall prove that

$$\|\vec{F}_\Omega\|_{N_2-N_1-2}+\langle\!\langle\vec{F}_\Gamma\rangle\!\rangle_{N_2-N_1-(3/2)}\leqq C'\Lambda \tag{3.77}$$

where

$$\Lambda=\sum_{l=1}^{2}\|\vec{v}_{N_1+l}\|_{N_2-N_1-l}+\sum_{M=0}^{N_1}(\|\vec{f}_M\|_{N_2-M-2}+\langle\!\langle\vec{g}_M\rangle\!\rangle_{N_2-M-(3/2)}).$$

Recall the definitions of $\vec{F}_\Omega$ and $\vec{F}_\Gamma$ (cf. (3.68)). Applying (Ap. 1)-(Ap. 3) with $\alpha=K-k$, $\beta=N_2-N_1-1+k$ and $\gamma=N_2-N_1-1$, we have for $2\leqq k\leqq N_1$,

$$\|\partial_i(\partial_t^k A_S^{i0}(t)S_{N_1+1-k})\|_{N_2-N_1-2}\leqq CM_S(K)\|S_{N_1+1-k}\|_{N_2-N_1-1+k}; \tag{3.78.a}$$

$$\|\partial_i(\partial_t^k A_S^{ij}(t)\partial_j S_{N_1-k})\|_{N_2-N_1-2}\leqq CM_S(K)\|S_{N_1-k}\|_{N_2-N_1+k}; \tag{3.78.b}$$

$$\langle\!\langle\nu_i\partial_t^k A_S^{ij}(t)\partial_j S_{N_1-k}\rangle\!\rangle_{N_2-N_1-(3/2)}\leqq CM_S(K)\|S_{N_1-k}\|_{N_2-N_1+k}; \tag{3.78.c}$$

$$\langle\!\langle\partial_t^k B^j(t)\partial_j S_{N_1-k}\rangle\!\rangle_{N_2-N_1-(3/2)}\leqq CM_S(K)\|S_{N_1-k}\|_{N_2-N_1+k}; \tag{3.78.d}$$

$$\langle\!\langle\partial_t^k B^0(t)S_{N_1+1-k}\rangle\!\rangle_{N_2-N_1-(3/2)}\leqq CM_S(K)\|S_{N_1+1-k}\|_{N_2-N_1-1+k}. \tag{3.78.e}$$

And also, applying (Ap. 1) and (Ap. 3) with $\alpha=K-1$, $\beta=\gamma=N_2-N_1-1$, we have

$$\|\partial_i(\partial_t A_S^{i0}(t)\vec{v}_{N_1+1})\|_{N_2-N_1-2}\leqq CM_S(K)\|\vec{v}_{N_1+1}\|_{N_2-N_1-1}; \tag{3.79.a}$$

$$\langle\!\langle\partial_t B^0(t)\vec{v}_{N_1+1}\rangle\!\rangle_{N_2-N_1-(3/2)}\leqq CM_S(K)\|\vec{v}_{N_1+1}\|_{N_2-N_1-1}. \tag{3.79.b}$$

Combining (3.78), (3.79) and (3.66) with $N=N_2$, we have (3.77). In particular, since $N_2-N_1-1\geqq 2$, applying the well-known Lax and Milgram theorem to (3.69), we see that there exists a unique $\vec{v}$ satisfying (3.69) provided that $\lambda>\lambda^{(1)}$. Furthermore, combining (3.76), (3.77) and (3.69) with $\vec{u}=\vec{v}$, we see that $\|\vec{v}\|_1\leqq C'\Lambda$.

Now, by induction on $L\in[1, N_2-N_1]$ we shall prove that $\vec{v}\in H^1(\Omega)$ and that

$$\|\vec{v}\|_L\leqq C'\Lambda. \tag{3.80}$$

As has been seen, we know that the assertion is valid for $L=1$. Thus, we assume that $2\leqq L\leqq N_2-N_1$, $\vec{v}\in H^{L-1}(\Omega)$ and $\|\vec{v}\|_{L-1}\leqq C'\Lambda$. Let us prove that $\vec{v}\in H^L(\Omega)$ and (3.80) is valid. Keeping (3.67) and (3.69) in mind, let us consider the boundary value problem:

$$-\partial_i(A^{ij}(t)\partial_j\vec{w})+\lambda\vec{w}=\vec{G}_\Omega \quad \text{in } \Omega, \tag{3.81.a}$$

$$\nu_i A^{ij}(t)\partial_j\vec{w}+B^j(t)\partial_j\vec{w}=\vec{G}_\Gamma \quad \text{on } \Gamma, \tag{3.81.b}$$

where

$$\vec{G}_\Omega=\vec{F}_\Omega+N_1\partial_i(\partial_t A^{i0}(t)\vec{v})+P_{N_1}(t)[R_0(\vec{v}), \cdots, R_{N_1-1}(\vec{v}), 0, 0];$$

$$\vec{G}_\Omega=\vec{F}_\Omega-N_1\partial_t B^0(t)\vec{v}-Q_{N_1}(t)[R_0(\vec{v}), \cdots, R_{N_1-1}(\vec{v}), 0, 0].$$

Since $L-2\leqq N_2-N_1-2$ and $L-(3/2)\leqq N_2-N_1-(3/2)$, by (3.71) and (3.77) we know that $\vec{G}_\Omega\in H^{L-2}(\Omega)$ and $\vec{G}_\Gamma\in H^{L-(3/2)}(\Gamma)$. Furthermore, we have

$$\|\vec{G}_\Omega\|_{L-2}+\langle\!\langle\vec{G}_\Gamma\rangle\!\rangle_{L-(3/2)}\leqq C'\Lambda. \tag{3.82}$$

Hence, applying Theorem 3.6 to (3.81), we see that there exist a $\lambda^{(2)}>0$ depending only on $\delta_1$, $\delta_2$, $M_\infty(K)$ and $M_S(K)$ such that for any $\lambda>\lambda^{(2)}$, (3.81) admits a unique solution $\vec{w}$ having the estimate:

$$\|\vec{w}\|_L\leqq C(L, \lambda, \delta_1, \delta_2, M_\infty(K), M_S(K))C'\Lambda. \tag{3.83}$$

Final task is to prove that $\vec{w}=\vec{v}$ for large $\lambda>0$. Since $\vec{w}\in H^1(\Omega)\subset H^2(\Omega)$, multiplying (3.81) by $\vec{u}$, integrating the resulting formula and using (No. 9), we have

$$B_\lambda[\vec{w}, \vec{u}]=(\vec{G}_\Omega, \vec{u})+\langle\vec{G}_\Gamma, \vec{u}\rangle. \tag{3.84}$$

Since $(\vec{G}_\Omega, \vec{u})=(\vec{F}_\Omega, \vec{u})+\mathcal{C}_1(t, \vec{v}, \vec{u})$ and $\langle\vec{G}_\Gamma, \vec{u}\rangle=\langle\vec{F}_\Gamma, \vec{u}\rangle+\mathcal{C}_2(t, \vec{v}, \vec{u})$ as follows from the definitions of $\vec{G}_\Omega$ and $\vec{G}_\Gamma$ and (3.70.b and c), combining (3.69), (3.70.a) and (3.84), we have

$$B_\lambda[\vec{w}-\vec{v}, \vec{u}]=0 \quad \text{for any } \vec{u}\in H^1(\Omega). \tag{3.85}$$

Hence, putting $\vec{u}=\vec{w}-\vec{v}$ and using (No. 16), we see that $\vec{w}=\vec{v}$ provided that $\lambda>\delta_2$. Summing up, we have obtained that $\vec{v}\in H^L(\Omega)$ and (3.80) is valid provided that $\lambda\geqq\max(\delta_2, \lambda^{(1)}, \lambda^{(2)})$. Accordingly, if we take $\lambda_{N_1}=\max(\lambda^{(1)}, \lambda^{(2)}, \delta_2)$, then we have the first assertion of the theorem.

Now, we shall prove the second assertion, i.e., the dependence on $t$ of solutions. From the first assertion it follows that for each $t\in J$ (3.59) admits solutions $\vec{v}_M\in H^{N_2-M}(\Omega)$ $(0\leqq M\leqq N_1)$. From now on, we write $\vec{v}_M=\vec{v}_M(t)$. First, $\vec{v}_M(t)\in C^0(J, H^{N_2-M}(\Omega))$ $(0\leqq M\leqq N_2)$. Let $t$ and $s$ be any points in $J$ such that $t\neq s$. Putting $\vec{w}_M=\vec{v}_M(t)-\vec{v}_M(s)$ for $0\leqq M\leqq N_1+2$, by (3.59) we have

$$\begin{aligned}\vec{w}_{M+2}-P_M(t)[\vec{w}_0, \cdots, \vec{w}_{M+1}]+\lambda_M\vec{w}_M\\ =\vec{f}_M(t)-\vec{f}_M(s)+(P_M(t)-P_M(s))[\vec{v}_0(s), \cdots, \vec{v}_{M+1}(s)] \quad \text{in } \Omega\end{aligned} \tag{3.86.a}$$

$$\begin{aligned}Q_M(t)[\vec{w}_0, \cdots, \vec{w}_{M+1}]\\ =\vec{g}_M(t)-\vec{g}_M(s)-(Q_M(s)-Q_M(t))[\vec{v}_0(s), \cdots, \vec{v}_{M+1}(s)] \quad \text{on } \Gamma\end{aligned} \tag{3.86.b}$$

for $0\leqq M\leqq N_1$. Applying (3.60) to (3.86) implies that

$$(3.87)\qquad \sum_{M=0}^{N_1}\|\vec{v}_M(t)-\vec{v}_M(s)\|_{N_2-M}\leqq C\Big\{\sum_{l=1}^{2}\|\vec{v}_{N_1+l}(t)-\vec{v}_{N_1+l}(s)\|_{N_2-N_1-l}$$
$$+\sum_{M=0}^{N_1}(\|\vec{f}_M(t)-\vec{f}_M(s)\|_{N_2-M-2}+\langle\!\langle\vec{g}_M(t)-\vec{g}_M(s)\rangle\!\rangle_{N_2-M-(3/2)})+R(t,s)\Big\}.$$

where

$$R(t,s)=\sum_{M=0}^{N_1}(\|(P_M(t)-P_M(s))[\vec{v}_0(s),\cdots,\vec{v}_{M+1}(s)]\|_{N_2-2-M}$$
$$+\langle\!\langle(Q_M(t)-Q_M(s))[\vec{v}_0(s),\cdots,\vec{v}_{M+1}(s)]\rangle\!\rangle_{N_2-M-(3/2)}).$$

Recalling the notations (No. 3. a and b), let us put

$$(3.88)\quad U_\infty(t,s)=[P(t)-P(s)]_{\infty,K-1};\ U_S(t,s)=[P(t)-P(s)\,|\,Q(t)-Q(s)]_{S,K-2,1}.$$

Applying the mean value theorem to $U_\infty$ and noting the definition of Lipschits continuous functions, by (A.1) we have that

$$(3.89)\qquad U_\infty(t,s)\leqq M_\infty(K)|t-s|\,;\ U_S(t,s)\leqq M_S(K)|t-s|.$$

On the other, by (3.60) we know that

$$(3.90)\qquad \sum_{M=0}^{N_1}\|\vec{v}_M(s)\|_{N_2-M}\leqq CA',$$

where

$$A'=\sum_{l=1}^{2}|\vec{v}_{N_1+l}|_{0,N_2-N_1-l,J}+\sum_{M=0}^{N_1}(|\vec{f}_M|_{0,N_2-M-2,J}+\langle\vec{g}_M\rangle_{0,N_2-M-(3/2),J}).$$

Applying (Ap. 1)-(Ap. 3) with $\alpha=K-k-1$, $\beta=N_2-M-1+k$ and $\gamma=N_2-M-1$ $(0\leqq k\leqq M\leqq N_1)$, we have that

$$(3.91)\quad R(t,s)\leqq C\{U_\infty(t,s)+U_S(t,s)\}\sum_{M=0}^{N_1+1}\|\vec{v}_M(s)\|_{N_2-M}\leqq C\{U_\infty(t,s)+U_S(t,s)\}A'.$$

Here, we have used (3.90). Combining (3.87), (3.89) and (3.91), we see that $\vec{v}_M(t)\in C^0(J, H^{N_2-M}(\Omega))$ for $0\leqq M\leqq N_1$. Furthermore, (3.61) follows from (3.60) when $k=0$.

Finally, we shall prove that $\vec{v}_M(t)\in C^1(J, H^{N_2-N_1-M}(\Omega))$ for $0\leqq M\leqq N_1$. If $\vec{v}_M(t)\in X^{1,N_2-M-1}(J,\Omega)$, applying (A.7)-(A.9) with $M_1=K-2-k$, $M_2=N_2-M-2+k$ and $N=1$ and noting that $M_1+M_2=K+N_2-M-4\geqq K-1>n/2$ $(M+3\leqq N_1+3\leqq N_2)$, we see that $P_M(t)[\vec{v}_0(t),\cdots,\vec{v}_{M+1}(t)]\in X^{1,0}(J,\Omega)$ and $Q_M(t)[\vec{v}_0(t),\cdots,\vec{v}_{M+1}(t)]\in X^{1,1/2}(J,\Gamma)$. Thus, differentiating (3.59) once in $t$ and putting $\partial_t\vec{v}_M(t)=\vec{w}_M(t)$ $(0\leqq M\leqq N_1+2)$, we have

$$(3.92.\mathrm{a})_M\qquad \vec{w}_{M+2}(t)-P_M(t)[\vec{w}_0(t),\cdots,\vec{w}_{M+1}(t)]$$
$$=\partial_t f_M(t)+P_M'(t)[\vec{v}_0(t),\cdots,\vec{v}_{M+1}(t)]\quad\text{in } J\times\Omega,$$

(3.92.b)$_M$ $$Q_M(t)[\vec{w}_0(t), \cdots, \vec{w}_{M+1}(t)]= \partial_t\vec{g}_M(t)+Q'_M(t)[\vec{v}_0(t), \cdots, \vec{v}_{M+1}(t)] \quad \text{on } J\times\Gamma,$$

where

$$P'_M(t)[\vec{v}_0, \cdots, \vec{v}_{M+1}]=\sum_{k=0}^{M}\binom{M}{k}\partial_i\{\partial_t^{k+1}A^{i0}(t)\vec{v}_{M+1-k}+\partial_t^{k+1}A^{ij}(t)\partial_j\vec{v}_{M-k}\};$$

$$Q'_M(t)[\vec{v}_0, \cdots, \vec{v}_{M+1}]=\sum_{k=0}^{M}\binom{M}{k}\{\nu_i\partial_t^{k+1}A^{ij}(t)\partial_j\vec{v}_{M-k} +\partial_t^{k+1}B^j(t)\partial_j\vec{v}_{M-k}+\partial_t^{k+1}B^0(t)\vec{v}_{M+1-k}\}.$$

From this point of view, first we shall prove the existence of solutions $\vec{w}_M(t)\in C^0(J, H^{N_2-M-1}(\Omega))$ $(0\leq M\leq N_1)$ where $\vec{w}_{N_1+l}(t)=\partial_t\vec{v}_{N_1+l}(t)\in C^0(J, H^{N_2-N_1-l-1}(\Omega))$ are given ($l=1$ and 2); secondly, we shall prove that

(3.93) $$\lim_{\Delta t\to 0}\sum_{M=0}^{N_1}\|\vec{z}_{M,\Delta t}(t)-\vec{w}_M(t)\|_{N_2-M-1}=0$$

where $\vec{z}_{M,\Delta t}(t)=(\vec{v}_M(t+\Delta t)-\vec{v}_M(t))(\Delta t)^{-1}-\vec{w}_M(t)$.

To prove the first assertion, we use the part already proved of the second assertion of Theorem 3.8. Applying (Ap. 4)-(Ap. 6) with $\alpha=K-2-k$, $\beta=N_2-M-1+k$ and $\gamma=N_2-M-2$ and noting that $\partial_t^{k+1}A_S^{il}(t)\in C^0(J, H^{K-2-k}(\Omega))$; $\partial_t^{k+1}B^l(t)\in C^0(J, H^{K-(5/2)-k}(\Gamma))$; $\vec{v}_{M+1-k}(t)$ and $\partial_j\vec{v}_{M-k}(t)\in C^0(J, H^{N_2-M-1-k}(\Omega))$ $(i=1, \cdots, n;\ l=0, \cdots, n;\ 0\leq k\leq M\leq N_1+1)$, we see that $P'_M(t)[\vec{v}_0(t), \cdots, \vec{v}_{M+1}(t)]\in C^0(J, H^{N_2-M-3}(\Omega))$ and $Q'_M(t)[\vec{v}_0(t), \cdots, \vec{v}_{M+1}(t)]\in C^0(J, H^{N_2-M-(5/2)}(\Gamma))$ for $0\leq M\leq N_1$. Furthermore, we have

(3.94) $$\sum_{M=0}^{N_1}\{\|P'_M(t)[\vec{v}_0(t), \cdots, \vec{v}_{M+1}(t)]\|_{N_2-M-3}+\langle\!\langle Q'_M(t), \cdots, \vec{v}_{M+1}(t)]\rangle\!\rangle_{N_2-M-(5/2)}\} \leq C\{M_\infty(K)+M_S(K)\}\sum_{M=0}^{N_1+1}\|\vec{v}_M(t)\|_{N_2-M}.$$

Hence, there exists a unique system $(\vec{w}_0(t), \cdots, \vec{w}_{N_1}(t))\in C^0(J, H^{N_2-N_1-1}(\Omega))$ of solutions to (3.92) having the estimate:

(3.95) $$\sum_{M=0}^{N_1}\|\vec{w}_M(t)\|_{N_2-M-1}\leq C\sum_{k=0}^{1}\Big\{\sum_{l=1}^{2}\|\partial_t^k\vec{v}_{N_1+l}(t)\|_{N_2-N_1-l-k} +\sum_{M=0}^{N_1}(\|\partial_t^k\vec{f}_M(t)\|_{N_2-M-2-k}+\langle\!\langle\partial_t^k\vec{g}_M(t)\rangle\!\rangle_{N_2-M-(3/2)-k})\Big\}.$$

Here, we have used (3.94) and (3.61) with $k=0$.

Now, we shall prove (3.93). For the notational convenience, we put $[\vec{f}]_{\Delta t}(t)=(\vec{f}(t+\Delta t)-\vec{f}(t))(\Delta t)^{-1}-\partial_t\vec{f}(t)$ and $\vec{f}|_{\Delta t}(t)=\vec{f}(t+\Delta t)-\vec{f}(t)$. Combining (3.59) and and (3.92), we have

$$(3.96.\mathrm{a})_M \qquad \vec{z}_{M+2,\Delta t}(t)-P_M(t)[\vec{z}_{0,\Delta t}(t), \cdots, \vec{z}_{M+1,\Delta t}(t)]+\lambda_M\vec{z}_{M,\Delta t}(t) = [\vec{f}_M]_{\Delta t}(t)+H_{\Omega,M,\Delta t}(t) \quad \text{in } \Omega,$$

$$(3.96.\mathrm{b})_M \qquad Q_M(t)[\vec{z}_{0,\Delta t}(t), \cdots, \vec{z}_{M+1,\Delta t}(t)]=[\vec{g}_M]_{\Delta t}+H_{\Gamma,M,\Delta t}(t) \quad \text{on } \Gamma,$$

where $\vec{z}_{N_1+l,\Delta t}(t)=[\vec{v}_{N_1+l}]_{\Delta t}(t)$ $(l=1, 2)$;

$$H_{\Omega,M,\Delta t}(t)=\sum_{k=0}^{M}\binom{M}{k}\partial_i\{[\partial_t^k A^{i0}]_{\Delta t}(t)\vec{v}_{M+1-k}(t+\Delta t)-\partial_t^{k+1}A^{i0}(t)\vec{v}_{M+1-k}|_{\Delta t}(t)$$
$$+[\partial_t^k A^{ij}]_{\Delta t}(t)\partial_j\vec{v}_{M-k}(t+\Delta t)-\partial_t^{k+1}A^{ij}(t)\partial_j\vec{v}_{M-k}|_{\Delta t}(t)\};$$

$$H_{\Gamma,M,\Delta t}(t)=\sum_{k=0}^{M}\binom{M}{k}[\nu_i\{[\partial_t^k A^{ij}]_{\Delta t}(t)\partial_j\vec{v}_{M-k}(t+\Delta t)-\partial_t^{k+1}A^{ij}(t)\partial_j\vec{v}_{M-k}|_{\Delta t}(t)$$
$$+[\partial_t^k B^{j}]_{\Delta t}(t)\partial_j\vec{v}_{M-k}(t+\Delta t)-\partial_t^{k+1}B^{j}(t)\partial_j\vec{v}_{M-k}|_{\Delta t}(t)$$
$$+[\partial_t^k B^{0}]_{\Delta t}(t)\vec{v}_{M+1-k}(t+\Delta t)-\partial_t^{k+1}B^{0}(t)\vec{v}_{M+1-k}|_{\Delta t}(t)].$$

Then, applying (3.60) to (3.96), we have

$$(3.97) \qquad \sum_{M=0}^{N_1}\|\vec{z}_{M,\Delta t}(t)\|_{N_2-M-1}\leqq C\Big\{\sum_{l=1}^{2}\|[\vec{v}_{N_1+l}]_{\Delta t}(t)\|_{N_2-N_1-l-1}$$
$$+\sum_{M=0}^{N_1}(\|[\vec{f}_M]_{\Delta t}(t)\|_{N_2-M-3}+\langle\!\langle[\vec{g}_M]_{\Delta t}(t)\rangle\!\rangle_{N_2-M-(5/2)}$$
$$+\|H_{\Omega,M,\Delta t}(t)\|_{N_2-M-3}+\langle\!\langle H_{\Gamma,M,\Delta t}(t)\rangle\!\rangle_{N_2-M-(5/2)})\Big\}.$$

Since $\vec{v}_{N_1+l}(t)\in X^{1,N_2-N_1-l-1}(J,\Omega)$, $\vec{f}_M(t)\in X^{1,N_2-M-3}(J,\Omega)$ and $\vec{g}_M\in X^{1,N_2-M-(5/2)}(J,\Gamma)$ for $l=1, 2$ and $0\leqq M\leqq N_1$, the first, second and third terms in the brace of (3.97) tend to zero as $\Delta t\to 0$. Since $\partial_t^k A_S^{ij}(t)\in C^1(J, H^{K-2-k}(\Omega))$; $\partial_t^k B^l(t)\in C^1(J, H^{K-(5/2)-k}(\Gamma))$; $\vec{v}_{M+1-k}(t)$ and $\partial_j\vec{v}_{M-k}(t)\in C^0(J; N^{N_2-M-1+k}(\Omega))$, we see that

$$(3.98) \qquad \|[\partial_t^k A_S^{ij}]_{\Delta t}(t)\|_{K-2-k}\to 0;\ \langle\!\langle[\partial_t^k B^l]_{\Delta t}(t)\rangle\!\rangle_{K-(5/2)-k}\to 0;$$
$$\|\vec{v}_{M+1-k}|_{\Delta t}(t)\|_{N_2-M-1+k}\to 0;\ \|\partial_j\vec{v}_{M-k}|_{\Delta t}(t)\|_{N_2-M-1+k}\to 0$$

as $\Delta t\to 0$. Applying (Ap. 1)-(Ap. 3) with $\alpha=K-2-k$, $\beta=N_2-M-1+k$ and $\gamma=N_2-M-2$ and using (3.98) and (3.90), we see that

$$(3.99) \qquad \|H_{\Omega,M,\Delta t}(t)\|_{N_2-M-3}\to 0;\ \langle\!\langle H_{\Gamma,M,\Delta t}(t)\rangle\!\rangle_{N_2-M-(5/2)}\to 0$$

as $t\to 0$ for $0\leqq M\leqq N_1$. Combining (3.97) and (3.99) and letting $\Delta t\to 0$ in the resulting estimate, we have (3.93). Since (3.93) means that $\partial_t\vec{v}(t)$ exists everywhere in the strong topology of $H^{N_2-M-1}(\Omega)$ and $\partial_t\vec{v}_M(t)=\vec{w}_M(t)\in C^0(J, H^{N_2-M-1}(\Omega))$, we see that $\vec{v}_M(t)\in X^{1,N_2-M-1}(J, \Omega)$ for $0\leqq M\leqq N_1$. Furthermore, substituting $\partial_t\vec{v}_M(t)=\vec{w}_M(t)$ into (3.95), we have (3.61) with $k=1$. This completes the proof of the theorem.

## § 4. The energy inequalities of higher order

In this section, we shall prove Theorem 1.3. First, we assume that $\vec{u} \in C^\infty(J, H^L(\Omega))$ where $J=[0, T-\varepsilon]$ and $\varepsilon$ is any number $\in (0, T)$. In view (Ap. 10), we can differentiate (N) $L-1$ times in $t$. Thus, we have

$$P(t)[\partial_t^{L-1}\vec{u}(t)] = \partial_t^{L-1}\vec{f}_\Omega(t) + \vec{F}_{\Omega, L-1}(t) \quad \text{in } J\times\Omega, \tag{4.1.a}$$

$$Q(t)[\partial_t^{L-1}\vec{u}(t)] = \partial_t^{L-1}\vec{f}_\Gamma(t) + \vec{F}_{\Gamma, L-1}(t) \quad \text{on } J\times\Gamma, \tag{4.1.b}$$

where

$$\vec{F}_{\Omega, L-1}(t) = \sum_{l=0}^{L-2}\binom{L-1}{l}\partial_i(\partial_t^{L-1-l}A^{i0}(t)\partial_t^{l+1}\vec{u}(t) + \partial_t^{L-1-l}A^{ij}(t)\partial_j\partial_t^l\vec{u}(t))$$

$$\vec{F}_{\Gamma, L-1}(t) = \sum_{l=0}^{L-2}\binom{L-1}{l}\{\nu_i\partial_t^{L-1-l}A^{ij}(t)\partial_j\partial_t^l\vec{u}(t) + \partial_t^{L-1-l}B^j(t)\partial_j\partial_t^l\vec{u}(t)$$
$$+\partial_t^{L-1-l}B^0(t)\partial_t^{l+1}\vec{u}(t)\}.$$

Note that the equalities in (4.1.a) and (4.1.b) hold for almost all $t\in J$ as elements in $L^2(\Omega)$ and $H^{1/2}(\Gamma)$, respectively. Applying (Ap. 1)-(Ap. 3) with $\alpha = K-(L-1-l)$, $\beta = L-l-1$ and $\gamma=1$, we have for almost all $t \in J$

$$\sum_{k=0}^{n}\|\partial_i\{\partial_t^{L-1-l}A_S^{ik}(t)\partial_k\partial_t^l u(t)\}\|_0 \leqq C\sum_{k=0}^{n}\|\partial_t^{L-1-l}A_S^{ik}(t)\|_{K-(L-1-l)}\|\partial_k\partial_t^l\vec{u}(t)\|_{L-(l+1)};$$

$$\langle\!\langle \nu_i\partial_t^{L-1-l}A_S^{ij}(t)\partial_j\partial_t^l\vec{u}(t)\rangle\!\rangle_{1/2} \leqq C\sum_{i=1}^{n}\|\partial_t^{L-1-l}A_S^{ij}(t)\|_{K-(L-1-l)}\|\partial_j\partial_t^l\vec{u}(t)\|_{L-(l+1)};$$

$$\sum_{k=0}^{n}\langle\!\langle \partial_t^{L-1-l}B^k(t)\partial_k\partial_t^l\vec{u}(t)\rangle\!\rangle_{1/2} \leqq C\sum_{k=0}^{n}\langle\!\langle \partial_t^{L-1-l}B^k(t)\rangle\!\rangle_{L-(L-1-l)-(1/2)}\|\partial_k\partial_t^l\vec{u}(t)\|_{L-(l+1)},$$

where $\partial_t = \partial_0$. From (No. 2.a) it follows that $\|\partial_t^{L-1-l}A_S^{ik}(t)\|_{K-(L-1-l)}$, $\langle\!\langle \partial_t^{L-1-l}B^k(t)\rangle\!\rangle_{K-(L-1-l)-(1/2)} \leqq M_S(K)$ for almost all $t\in J$ and for any $i=1, \cdots, n$; $k=0, 1, \cdots, n$; $l=0, 1, \cdots, L-1$. Combining these results, we see easily that

$$\|\vec{F}_{\Omega, L-1}(t)\|_0^2 + \langle\!\langle \vec{F}_{\Gamma, L-1}(t)\rangle\!\rangle_{1/2}^2 \leqq C\{M_\infty(K)+M_S(K)\}^2\|\bar{D}^L\vec{u}(t)\|_0^2 \tag{4.2}$$

for almost all $t\in J$. Applying Theorem 2.1 to (4.1) and using (4.2), we have

$$E(t, \partial_t^{L-1}\vec{u}(t)) \leqq 2e^{C(T)t}\Big\{E(0, \partial_t^{L-1}\vec{u}(0)) + C(T)\int_0^t\|\bar{D}^L\vec{u}(s)\|_0^2 ds \tag{4.3.a}$$
$$+C(T)\int_0^t(\|\partial_t^{L-1}\vec{f}_\Omega(s)\|_0^2 + \langle\!\langle \partial_t^{L-1}\vec{f}_J(s)\rangle\!\rangle_{1/2}^2)ds\Big\};$$

(4.3.z) $$E(t, \partial_t^{L-1}\vec{u}(t)) \leqq e^{C(T)t}\Big[E(0, \partial_t^{L-1}\vec{u}(0)) + C(T)\Big\{\|\bar{D}^L\vec{u}(0)\|_0^2 + \int_0^t (\|\partial_t^{L-1}\vec{f}_\Omega(s)\|_0^2 + \langle\!\langle \partial_t^{L-1}\vec{f}_\Gamma(s)\rangle\!\rangle_{1/2}^2)ds + \int_0^t \|\bar{D}^L\vec{u}(s)\|_0^2 ds\Big\}^{1/2} \times \Big\{\int_0^t (\|\partial_t^{L-1}\vec{f}_\Omega(s)\|_0^2 + \langle\!\langle \partial_t^{L-1}\vec{f}_\Gamma(s)\rangle\!\rangle_{1/2}^2)ds + \int_0^t \|\bar{D}^L\vec{u}(s)\|_0^2 ds\Big\}^{1/2}\Big],$$

for any $t\in J$. In the present proof, $C(T)$ denotes various constants depending only on $T$, $\delta_1$, $\delta_2$, $L$, $M_\infty(K)$ and $M_S(K)$.

Now, we shall prove that

(4.3.c) $$\|\bar{D}^L\vec{u}(t)\|_0^2 \leqq C(T)\Big\{\|\bar{D}^L\vec{u}(0)\|_0^2 + |\vec{f}_\Omega|_{L-2,0,[0,t]}^2 + \langle\vec{f}_\Gamma\rangle_{L-2,1/2,[0,t]}^2 + \int_0^t (\|\partial_t^{L-1}\vec{f}_\Omega(s)\|_0^2 + \langle\!\langle \partial_t^{L-1}\vec{f}_\Gamma(s)\rangle\!\rangle_{1/2}^2)ds\Big\} \quad \text{for any } t\in J.$$

If follows from (4.3.a) and (No. 22) that

(4.4) $$\sum_{l=L-1}^{L} \|\partial_t^l\vec{u}(t)\|_{L-l}^2 \leqq C(T)\Big\{\|\bar{D}^L\vec{u}(0)\|_0^2 + \int_0^t \|\bar{D}^L\vec{u}(s)\|_0^2 ds + \int_0^t (\|\partial_t^{L-1}\vec{f}_\Omega(s)\|_0^2 + \langle\!\langle \partial_t^{L-1}\vec{f}_\Gamma(s)\rangle\!\rangle_{1/2}^2)ds\Big\} \quad \text{for any } t\in J,$$

(4.5) $$\|\partial_t^l\vec{u}(t)\|_{L-l} \leqq C\{\|\partial_t^l\vec{f}_\Omega(t)\|_{L-2-l} + \langle\!\langle \partial_t^l\vec{f}_\Gamma(t)\rangle\!\rangle_{L-(3/2)-l} + \sum_{k=1}^{2} \|\partial_t^{l+k}\vec{u}(t)\|_{L-l-k} + \|\bar{D}^{L-1}\vec{u}(t)\|_0\}$$

for $t\in J$ and $0\leqq l\leqq L-2$, where $C=C(\delta_1, \delta_2, L, M_\infty(K), M_S(K))$. Let $0\leqq l\leqq L-2$. Differentiating (N) $l$ times in $t$, we have

(4.6.a) $$-\partial_i(A^{ij}(t)\partial_j(\partial_t^l\vec{u}(t))) = \partial_t^l\vec{f}_\Omega(t) - \partial_t^{l+2}\vec{u}(t) + \partial_i(A^{i0}(t)\partial_t^{l+1}\vec{u}(t)) + \vec{G}_{\Omega,l}(t) \quad \text{in } \Omega,$$

(4.6.b) $$\nu_i A^{ij}(t)\partial_j(\partial_t^l\vec{u}(t)) + B^j(t)\partial_j(\partial_t^l\vec{u}(t) = \partial_t^l\vec{f}_\Gamma(t) - B^0(t)\partial_t^{l+1}\vec{u}(t) - \vec{G}_{\Gamma,l}(t) \quad \text{on } \Gamma,$$

where

$$\vec{G}_{\Omega,l}(t) = \begin{cases} \sum_{k=0}^{l-1}\binom{l}{k}\partial_i\{\partial_t^{l-k}A^{i0}(t)\partial_t^{k+1}\vec{u}(t) + \partial_t^{l-k}A^{ij}(t)\partial_j\partial_t^k\vec{u}(t)\} & \text{for } l\geqq 1, \\ 0 & \text{for } l=0, \end{cases}$$

$$\vec{G}_{\Gamma,l}(t) = \begin{cases} \sum_{k=0}^{l=1}\binom{l}{k}\Big\{\nu_i\partial_t^{l-k}A^{ij}(t)\partial_j\partial_t^k\vec{u}(t) + \partial_t^{l-k}B^j(t)\partial_j\partial_t^k\vec{u}(t) + \partial_t^{l-k}B^0(t)\partial_t^{k+1}\vec{u}(t)\Big\} & \text{for } l\geqq 1, \\ 0 & \text{for } l=0. \end{cases}$$

Since $\vec{u}\in C^\infty(J, H^L(\Omega)$ and $0\leqq l\leqq L-2\leqq K-2$, it follows from (Ap. 10) that the equalities of (4.6.a) and (4.6.b) hold in the sense of $L^2(\Omega)$ and $H^{1/2}(\Gamma)$ for all $t\in H$, respectively. Applying (Ap. 1)-(Ap. 3) with $\alpha=K-(l-k)$, $\beta=L-2-k$

and $\gamma=L-1-l$, we see easily that

(4.7) $\|\vec{G}_{\Omega,l}(t)\|_{L-2-l}+\langle\!\langle\vec{G}_{\Gamma,l}(t)\rangle\!\rangle_{L-(3/2)-l}\leqq C\{M_\infty(K)+M_S(K)\}\|\bar{D}^{L-1}\vec{u}(t)\|_0$

for all $t\in J$. On the other hand, we have

$$(4.8)\qquad \|\partial_i(A^{i0}(t)\partial_t^{l+1}\vec{u}(t))\|_{L-2-l},\ \langle\!\langle B^0(t)\partial_t^{l+1}\vec{u}(t)\rangle\!\rangle_{L-2-l}$$
$$\leqq C\{M_\infty(K)+M_S(K)\}\{\|\partial_t^{l+1}\vec{u}(t)\|_{L-1-l}+\|\bar{D}^{L-1}\vec{u}(t)\|_0\}\quad\text{for all } t\in J.$$

In fact, we can write symbolically

$$\|\partial_i(A_S^{i0}(t)\partial_t^{l+1}\vec{u}(t))\|_{L-2-l}\leqq\sum_{i=1}^n\|A_S^{i0}(t)\bar{D}_x^{L-1-l}\partial_t^{l+1}\vec{u}(t)\|_0$$
$$+C(L)\sum_{i=1}^n\sum_{N=0}^{L-2-l}\|\bar{D}_x^{L-1-l-N}A_S^{i0}(t)\bar{D}_x^N\partial_t^{l+1}\vec{u}(t)\|_0\,.$$

By (No. 13.b) we have

$$\|A_S^{i0}(t)\bar{D}_x^{L-1-l}\partial_t^{l+1}\vec{u}(t)\|_0\leqq\|A_S^{i0}(t)\|_{\infty,0}\|\partial_t^{l+1}\vec{u}(t)\|_{L-1-l}\leqq M_S(K)\|\partial_t^{l+1}\vec{u}(t)\|_{L-1-l}\,.$$

Let $0\leqq N\leqq L-2-l$. Applying (Ap. 1) with $\alpha=K-(L-1-l-N)$, $\beta=L-1-(N+l+1)$ and $\gamma=1$, we have

$$\|\bar{D}_x^{L-1-l-N}A_S^{i0}(t)\bar{D}_x^N\partial_t^{l+1}\vec{u}(t)\|_0\leqq CM_S(K)\|\bar{D}^{L-1}\vec{u}(t)\|_0\,.$$

Combining these facts, we see easily that the first part of (4.8) is valid. In view of Corollary Ap. 5, there exists $B_{\mathrm{ext}}^0(t)\in Y^{K-1,1}(I,\Omega)$ such that $B_{\mathrm{ext}}^0(t)=B^0(t)$ almost everywhere on $\Gamma$ and $|B_{\mathrm{ext}}^0|_{K-1,1,I}\leqq C\langle B^0\rangle_{K-1,1/2,I}\leqq CM_S(K)$. Since

$$\langle\!\langle B^0(t)\partial_t^{l+1}\vec{u}(t)\rangle\!\rangle_{L-(3/2)-l}=\langle\!\langle B_{\mathrm{ext}}^0(t)\partial_t^{l+1}\vec{u}(t)\rangle\!\rangle_{L-(3/2)-l}\leqq C\|B_{\mathrm{ext}}^0(t)\partial_t^{l+1}\vec{u}(t)\|_{L-1-l}$$

as follows from Corollary Ap. 4-(1), by employing the same arguments we see that the second part of (4.8) is valid. Hence, applying Corollary 3.7 to (4.6) and using (4.7) and (4.8), we have (4.5). Repeated use of (4.5) implies that

$$(4.9)\qquad \sum_{l=0}^L\|\partial_t^l\vec{u}(t)\|_{L-l}^2\leqq C\Big\{\sum_{l=0}^{L-2}\|\partial_t^l\vec{f}_\Omega(t)\|_{L-2-l}+\langle\!\langle\partial_t^l\vec{f}_\Gamma(t)\rangle\!\rangle_{L-(3/2)-l}^2$$
$$+\sum_{l=L-1}^L\|\partial_t^l\vec{u}(t)\|_{L-l}^2+\|\bar{D}^{L-1}\vec{u}(t)\|_0^2\Big\}$$

where $C=C(L,\delta_1,\delta_2,M_\infty(K),M_S(K))$. Since

$$\|\bar{D}^{L-1}\vec{u}(t)\|_0^2\leqq\|\bar{D}^{L-1}\vec{u}(0)\|_0^2+2\int_0^t\|\bar{D}^L\vec{u}(s)\|_0^2ds\,,$$

combining (4.4) and (4.9), we have

$$(4.10)\qquad \|\bar{D}^L\vec{u}(t)\|_0^2 \leqq \|C(T)\Big\{\|\bar{D}^L\vec{u}(0)\|_0^2 + |\vec{f}_\Omega|^2_{L-2,0,[0,t]} + \langle\vec{f}_\Gamma\rangle^2_{L-2,1/2,[0,t]}$$
$$+\int_0^t (\|\partial_t^{L-1}\vec{f}_\Omega(s)\|_0^2 + \langle\!\langle\partial_t^{L-1}\vec{f}_\Gamma(s)\rangle\!\rangle^2_{1/2})ds + \int_0^t \|\bar{D}^L\vec{u}(s)\|_0^2 ds\Big\}.$$

To get (4.3.c) from (4.10), we use the well-known

*Gronwall's inequality*: *Let $a(t)$ and $b(t)$ be non-negative functions in $L^1(a, b)$. If $b(t)$ is non-decreasing and the inequality*: $a(t) \leqq c\int_a^t a(s)ds + b(t)$ *holds for any $t \in (a, b)$ with some constant $c$ independent of $t$, then $a(t) \leqq e^{c(t-a)}b(t)$ for any $t \in (a, b)$.*

Applying Gronwall's inequality to (4.10), we see easily that (4.3.c) is valid. Furthermore, substituting (4.3.c) into (4.3.b), we have that the estimate (b) of Theorem 3.1 is valid for any $t \in J$ and $\vec{u} \in C^\infty(J, H^1(\Omega))$.

Now, we shall remove the assumption: $\vec{u} \in C^\infty(J, H^L(\Omega))$. To do this, we use the following lemma.

LEMMA 4.1. *Let $L$ be an integer $\in [2, K]$ and $\rho_\delta(t)$ be a function in $C_0^\infty([-2, -1])$ such that $\int \rho(t)dt = 1$. Put $\rho_\delta(t) = \delta^{-1}\rho(\delta^{-1}t)$, $v_\delta(t, x) = \int \rho_\delta(t-s)v(s, x)ds$ and $I_\delta(t, x) = (av)_\delta(t, x) - a(t, x)v_\delta(t, x)$. Then, the following four assertions are valid.*

1° *If $a \in \mathcal{B}^K([0, T] \times \bar{\Omega})$ and $v \in Y^{L-2,1}([0, T), \Omega)$, then $|I_\delta|_{L-2,1,[0,t]} \to 0$ and $\int_0^t \|\partial_s^{L-1} I_\delta(s)\|_1^2 ds \to 0$ as $\delta \to 0$ for any $t \in (0, T)$.*

2° *If $a \in \mathcal{B}^K([0, T] \times \Gamma)$ and $v \in Y^{L-2,1/2}([0, T), \Gamma)$, then $\langle I_\delta\rangle_{L-2,1/2,[0,t]} \to 0$ and $\int_0^t \langle\!\langle\partial_s^{L-1} I_\delta(s)\rangle\!\rangle^2_{1/2} ds \to 0$ as $\delta \to 0$ for any $t \in (0, T)$.*

3° *If $a \in Y^{K-1,1}(0, T], \Omega)$ and $v \in Y^{L-2,1}([0, T), \Omega)$, then $|I_\delta|_{L-2,1,[0,t]} \to 0$ and $\int_0^t \|\partial_s^{L-1} I_\delta(s)\|_1^2 ds \to 0$ as $\delta \to 0$ for any $t \in (0, T)$.*

4° *If $a \in Y^{K-1,1/2}([0, T], \Gamma)$ and $v \in Y^{L-2,1/2}([0, T), \Gamma)$, then $\langle I_\delta\rangle_{L-2,1/2,[0,t]} \to 0$ and $\int_0^t \langle\!\langle\partial_s^{L-1} I_\delta(s)\rangle\!\rangle^2_{1/2} ds \to 0$ as $\delta \to 0$ for any $t \in (0, T)$.*

Defering the proof of Lemma 4.1, we shall complete the proof of Theorem 1.3. Let the notation $v_\delta$ be the same as in Lemma 4.1 and put $\vec{v}_\delta = {}^t((v_1)_\delta, \cdots, (v_m)_\delta)$ for $\vec{v} = {}^t(v_1, \cdots, v_m)$. Let $\delta_0 > 0$ be a number $< (T-\varepsilon)/2$. Note that $\vec{u}_\delta \in C^\infty(J, H^L(\Omega))$ for $0 < \delta < \delta_0$ and satisfies the equations:

$$(4.11.a)\qquad P(t)[\vec{u}_\delta(t)] = (\vec{f}_\Omega)_\delta(t) - R_\delta\vec{u}(t) \quad \text{in } J \times \Omega,$$

(4.11.b) $$Q(t)[\vec{u}_\delta(t)]=(\vec{f}_\Gamma)_\delta(t)+S_\delta\vec{u}(t) \quad \text{on } J\times\Gamma\,,$$

where

(4.12.a) $$R_\delta\vec{u}(t)=\sum_{l=0}^{n}\partial_i\{A^{il}(t)\partial_l\vec{u}_\delta(t)-(A^{il}\partial_l\vec{u})_\delta(t)\}\,;$$

(4.12.b) $$S_\delta\vec{u}(t)=\nu_i(A^{ij}(t)\partial_j\vec{u}_\delta(t)-(A^{ij}\partial_j\vec{u})_\delta(t))+\sum_{l=0}^{n}B^l(t)\partial_l\vec{u}_\delta(t)-(B^l\partial_l\vec{u})_\delta(t)\,.$$

Applying (4.3.c) to (4.11) implies that

(4.13) $$\begin{aligned}\|\bar{D}^L\vec{u}_\delta(t)\|_0^2\leqq C(T)\Big\{&\|\bar{D}^L\vec{u}_\delta(0)\|_0^2+|(\vec{f}_\Omega)_\delta|^2_{L-2,0,[0,t]}+\langle(\vec{f}_\Gamma)_\delta\rangle^2_{L-2,1/2,[0,t]}\\&+\int_0^t(\|\partial_t^{L-1}(\vec{f}_\Omega)_\delta(s)\|_0^2+\langle\!\langle\partial_t^{L-1}(\vec{f}_\Gamma)_\delta(s)\rangle\!\rangle^2_{1/2})ds+|R_\delta\vec{u}|^2_{L-2,0,[0,t]}\\&+\langle S_\delta\vec{u}\rangle^2_{L-2,1/2,[0,t]}+\int_0^t(\|\partial_t^{L-1}R_\delta\vec{u}(s)\|_0^2+\langle\!\langle\partial_t^{L-1}S_\delta\vec{u}(s)\rangle\!\rangle^2_{1/2})ds\Big\}\,.\end{aligned}$$

As was stated in Remark after Theorem 1.3, we know that $\vec{f}_\Omega\in X^{L-2,0}([0,T),\Omega)$, $\vec{f}_\Gamma\in X^{L-2,1/2}([0,T),\Gamma)$, and $\partial_t^{L-1}\vec{f}_\Omega(t)$ and $\partial_t^{L-1}\vec{f}_\Gamma(t)$ are $L^2$ functions in $(0,T)$ having their values in $L^2(\Omega)$ and $H^{1/2}(\Gamma)$, respectively. Thus, we see easily that

(4.14) $$\begin{aligned}&\|\bar{D}^L(\vec{u}_\delta-\vec{u})(r)\|_0^2\to0\,;\ |(\vec{f}_\Omega)_\delta-\vec{f}_\Omega|_{L-2,0,[0,t]}+\langle(\vec{f}_\Gamma)_\delta-\vec{f}_\Gamma\rangle_{L-2,1/2,[0,t]}\to0\,;\\&\int_0^t(\|\partial_s^{L-1}((\vec{f}_\Omega)_\delta-\vec{f}_\Omega)(s)\|_0^2+\langle\!\langle\partial_s((\vec{f}_\Gamma)_\delta-\vec{f}_\Gamma)(s)\rangle\!\rangle^2_{1/2})ds\to0\end{aligned}$$

as $\delta\to0$ for any $r\in[0,T)$ and $t\in(0,T)$. And also, applying Lemma 4.1 to (4.12), we have easily that

(4.15) $$\begin{aligned}&|R_\delta\vec{u}|_{L-2,0,[0,t]}+\langle S_\delta\vec{u}\rangle_{L-2,1/2,[0,t]}\to0\,;\\&\int_0^t(\|\partial_s^{L-1}R_\delta\vec{u}(s)\|_0^2+\langle\!\langle\partial_s^{L-1}S_\delta\vec{u}(s)\rangle\!\rangle^2_{1/2})ds\to0\end{aligned}$$

as $\delta\to0$ for any $t\in(0,T)$, because $\partial_l\vec{u}\in X^{L-1,0}([0,T),\Omega)\subset Y^{L-2,1}([0,T),\Omega)$ and $\partial_l\vec{u}\in X^{L-2,1/2}([0,T),\Gamma)\subset Y^{L-2,1/2}([0,T),\Gamma)$ for $l=0,1,\cdots,n$ (the second assertion follows from Corollary Ap. 4-(1)). Letting $\delta\to0$ in (4.13), using (4.14) and (4.15) and noting that $\varepsilon$ is chosen arbitrarily, we have the estimate (a) of Theorem 1.3. In the same way, we can obtain (b) of Theorem 1.3.

PROOF OF LEMMA 4.1. The assertions 1° and 2° were essentially proved by Ikawa [2]. Noting Corollary Ap. 5, by 3° we have 4° immediately. Hence, we will prove 3° only. Noting (Ap. 15) and (No. 2.b), we have

$$|I_\delta|_{L-2,1,[0,t]}\leqq\sum_{l=0}^{L-2}\sum_{k=0}^{l}\binom{l}{k}|(\partial_t^k a\partial_t^{l-k}v)_\delta-\partial_t^k a\partial_t^{l-k}v_\delta|_{0,L-1-l,[0,t]}\,.$$

Let $0<\delta<\delta_0<(T-t)/2$. Applying (Ap. 1) with $\alpha=K-1-k$, $\beta=L-1-(l-k)$ and $\gamma=L-1-l$, we have

$$\|(\partial_t^k a\partial_t^{l-k}v)_\delta(s)-\partial_t^k a(s)\partial_t^{l-k}v_\delta(s)\|_{L-1-l}$$
$$\leqq C\sup\{\|\partial_t^k a(s-r)-\partial_t^k a(s)\|_{K-1-k}\,|\,0\leqq s\leqq t,\ r\in[-2,\ -1]\}\,|v|_{L-1,0,[0,T)}\,.$$

Since $\partial_s^k a(s)\in Y^{K-1-k,1}([0,\ T),\ \Omega)\subset X^{0,K-1-k}([0,\ T),\ \Omega)$, the uniform continuity of $\partial_s^k a(s)$ on $[0,\ t+2\delta_0]$ $(\subset[0,\ T))$ in the strong topoloty of $H^{K-1-k}(\Omega)$ implies that $|I_\delta|_{L-2,1,[0,t]}\to 0$ as $\delta\to 0$ for any $t\in[0,\ T)$.

Next, we shall prove the second assertion of 3°. Noting (Ap. 15), we have for any multi-index $\alpha$ such that $|\alpha|\leqq 1$,

$$\partial_x^\alpha\partial_s^{L-2}I_\delta(s)=\sum_{l=0}^{L-2}\sum_{\beta\leqq\alpha}\binom{L-2}{l}\binom{\alpha}{\beta}\{(\partial_x^\beta\partial_s^l a\partial_x^{\alpha-\beta}\partial_s^{L-2-l}v)_\delta(s)-\partial_x^\beta\partial_s^l a(s)(\partial_x^{\alpha-\beta}\partial_s^{L-2-l}v)_\delta(s)\}.$$

First, we consider the term where $1\leqq|\beta|+l\leqq K-1$. Applying (Ap. 7.a) with $M_1=K-1-|\beta|-l$, $M_2=l-|\alpha-\beta|$ and $N=0$, we have

$$\|\partial_s\{(\partial_x^\beta\partial_s^l a\partial_x^{\alpha-\beta}\partial_s^{L-2-l}v)_\delta(s)-\partial_x^\beta\partial_s^l(a(s)(\partial_x^{\alpha-\beta}\partial_s^{L-2-l}v)_\delta(s)\}\|_0\leqq I_\delta^1(s)+I_\delta^2(s)$$

where

$$I_\delta^1(s)=\|(\partial_x^\beta\partial_s^{l+1}a\partial_x^{\alpha-\beta}\partial_s^{L-2-l}v)_\delta(s)-\partial_x^\beta\partial_s^{l+1}a(s)(\partial_x^{\alpha-\beta}\partial_s^{L-2-l}v)_\delta(s)\|_0\,;$$
$$I_\delta^2(s)=\|(\partial_x^\beta\partial_s^l a\partial_x^{\alpha-\beta}\partial_s^{L-1-l}v)_\delta(s)-\partial_x^\beta\partial_s^l a(s)(\partial_x^{\alpha-\beta}\partial_s^{L-1-l}v)_\delta(s)\|_0\,.$$

Applying (Ap. 1) with $\alpha=K-|\beta|-l-1$, $\beta=l+1-|\alpha-\beta|$ and $\gamma=1$, we have

$$I_\delta^1(s)\leqq C\int\rho_\delta(s-r)\|\bar{D}^{K-1}\bar{D}_x^1(a(r)-a(s))\|_0\|\bar{D}^{L-1}v(r)\|_0dr\,.$$

Then, by Schwarz's inequality we have

$$\int_0^t I_\delta^1(s)^2ds\leqq C\int_0^t\Big\{\Big[\int\rho(r)\|\bar{D}^{K-1}\bar{D}_x^1(a(s-\delta r)-a(s))\|_0^2dr\Big]$$
$$\int\rho(r)\|\bar{D}^{L-1}v(s-\delta r)\|_0^2dr\Big\}ds\,.$$

Let $0<\delta<\delta_0<(T-t)/2$. Since $v\in Y^{L-1,0}([0,\ T),\ \Omega)$ and

(4.16) $\qquad s-\delta r<T$ provided that $0\leqq s\leqq t$, $-2\leqq r\leqq -1$ and $\delta<(T-t)/2$,

we have that $\int\rho(r)\|\bar{D}^{L-1}v(s-\delta r)\|_0^2dr\leqq|v|_{L-1,0,[0,T)}$ (cf. (No. 2.a)). Hence we have

$$\int_0^t I_\delta^1(s)^2ds\leqq C|v|_{L-1,0,[0,T)}\int\rho(r)dr\int_0^t\|\bar{D}^{K-1}\bar{D}_x^1(a(s-\delta r)-a(s))\|_0^2ds\,.$$

Since $a\in Y^{K-1,1}([0,\ T),\ \Omega)$, $\bar{D}^{K-1}\bar{D}_x^1 a\in L^2((0,\ T)\times\Omega)$. Noting (4.16), by the Riemann-Lebesgue theorem, we have

$$\lim_{\delta\to 0}\int_0^t \|\bar{D}^{K-1}\bar{D}_x^1(a(s-\delta r)-a(s))\|_0^2 ds\rho(r)=0 \quad \text{for all } r\in\boldsymbol{R}.$$

And also, we have

$$\int_0^t \|\bar{D}^{K-1}\bar{D}_x^1(a(s-\delta r)(-a(s))\|_0^2 ds\rho(r)\leqq 4\rho(r)\int_0^T \|\bar{D}^{K-1}\bar{D}_x^1 a(s)\|_0^2 ds\in L^1(\boldsymbol{R}).$$

Hence, by Lebesgue's dominated convergence theorem, we have

$$\lim_{\delta\to 0}\int\rho(r)dr\int_0^t \|\bar{D}^{K-1}\bar{D}_x^1(a(s-\delta r)-a(s))\|_0^2 ds=0.$$

As a result, we have obtained

$$\lim_{\delta\to 0}\int_0^t I_\delta^1(s)^2 ds=0 \quad \text{for all } t\in[0, T). \tag{4.17.a}$$

Applying (Ap. 1) with $\alpha=K-|\beta|-l$, $\beta=l-|\alpha-\beta|$ and $\gamma=1$, we have also

$$I_\delta^2(s)\leqq C\int\rho_\delta(s-r)\|\bar{D}^{K-1}\bar{D}_x^1(a(r)-a(s))\|_0\|\bar{D}^{L-1}v(r)\|_0 dr.$$

Employing the same arguments, we have

$$\lim_{\delta\to 0}\int_0^t I_\delta^2(s)^2 ds=0 \quad \text{for all } t\in[0, T). \tag{4.17.b}$$

Now, we consider the term where $|\beta|+l=0$, i.e., the term: $J_\delta=(aw)_\delta-aw_\delta$ where $w=\partial_x^\alpha\partial_s^{L-2}v$. Note that $w\in L^\infty([0, T), L^2(\Omega))$. We can write $\partial_s J_\delta=J_1+J_2$ where

$$J_1=\int\partial_r\{\rho_\delta(s-r)(a(s, x)-a(r, x))\}(w(r, x)-w(s, x))dr;$$

$$J_2=\int\rho_\delta(s-r)(\dot{a}(r, x)-\dot{a}(s, x))w(r, x)dr\,(\dot{a}(s, x)=\partial_s a(s, x)).$$

Put $I_\delta^3(s)=\|J_1\|_0$ and $I_\delta^4(s)=\|J_2\|_0$. Let $0\leqq s\leqq t<T$, $0<\delta<(T-t)/2$ and $s-r\in[-2\delta, -\delta]$. In view of Corollary Ap. 7, $a(s, x)\in\mathscr{B}^1([0, T]\times\bar{\Omega})$. Hence, by the mean value theorem we have

$$|\partial_r\{\rho_\delta(s-r)(a(s, x)-a(r, x))\}|\leqq|a|_{\infty,1,[0,T]}\{\rho_\delta(s-r)+\delta^{-1}|s-r||\rho_\delta'(s-r)|\}$$

where $\rho_\delta'(s)=\delta^{-1}\rho'(\delta^{-1}s)$. By Schwarz's inequality we have

$$I_\delta^3(s)^2\leqq C|a|_{\infty,1,[0,T]}^2\left(\int p(r)dr\right)\int p(r)\|w(s-\delta r)-w(s)\|_0^2 dr$$

where $p(r)=\rho(r)+|r||\rho'(r)|$. Since $w\in L^\infty([0, T), L^2(\Omega))\subset L^2((0, T)\times\Omega)$, by the Riemann-Lebesgue theorem, we have

$$\int_0^t \|w(s-\delta r)-w(s)\|_0^2 ds\,p(r)\to 0 \quad \text{as } \delta\to 0 \text{ for all } r\in\boldsymbol{R}.$$

As also, we have

$$\int_0^t \|w(s-\delta r)-w(s)\|_0^2 ds\, p(r) \leqq 4\int_0^T \|w(s)\|_0^2 ds\, p(r) \in L^1(\boldsymbol{R}).$$

Hence, by Lebesgue's dominated convergence theorem we have that $\int_0^t I_\delta^3(s)^2 ds \to 0$ as $\delta \to 0$. On the other hand, since $\|\dot{a}(s-\delta r)-\dot{a}(s)\|_{\infty,0} \leqq C\|\dot{a}(s-\delta r)-\dot{a}(s)\|_{K-1}$ as follows from (No. 13.b), by Schwarz's inequality we have

$$I_\delta^4(s)^2 \leqq \int \|w(s-\delta r)\|_0^2 \rho(r) dr \times \int \|\dot{a}(s-\delta r)-\dot{a}(s)\|_{K-1}^2 \rho(r) dr.$$

Noting (4.16) and the fact that $w \in L^\infty([0, T), L^2(\Omega))$, we have

$$\int_0^t I_\delta^4(s)^2 ds \leqq C|w|_{0,0,[0,T]} \int_0^t \left\{ \int \|\bar{D}_x^{K-1}\dot{a}(s-\delta r)-\bar{D}_x^{K-1}\dot{a}(s)\|_0^2 \rho(r) dr \right\} ds.$$

Since $\dot{a}(s, x) \in Y^{K-2,1}([0, T], \Omega) \subset L^\infty([0, T], H^{K-1}(\Omega))$, $\bar{D}_x^{K-1}\dot{a}(s, x) \in L^2((0, T)\times\Omega)$. Hence, employing the same arguments mentioned previously, by the Riemann-Lebesgue theorem and Lebesgue's domined convergence theorem we have

$$\lim_{\delta\to 0} \int_0^t ds \int \|\bar{D}_x^{K-1}\dot{a}(s-\delta r)-\bar{D}_x^{K-1}\dot{a}(s)\|_0^2 \rho(r) dr = 0.$$

From this it follows that $\int_0^t I_\delta^4(s)^2 ds \to 0$ as $\delta \to 0$ for all $t \in [0, T)$. Combining these results, we have

$$\lim_{\delta\to 0} \int_0^t \|\partial_s J_\delta(s)\|_0^2 ds = 0 \quad \text{for all } t \in [0, T). \tag{4.18}$$

From (4.17) and (4.18) we have Lemma 4.1.

## § 5. An existence theorem of solutions to (N) in $X^{2,0}([0, T), \Omega)$

In this section, we shall prove

THEOREM 5.1. *Assume that* (A.1)-(A.5) *are valid. Then, for a given system* $(\vec{u}_0, \vec{u}_1, \vec{f}_\Omega, \vec{f}_\Gamma) \in D^2([0, T))$ *of data,* (N) *admits a unique solution* $\vec{u} \in X^{2,0}([0, T), \Omega)$.

As a main step of our proof of Theorem 5.1, we shall prove

LEMMA 5.2. *Let* $\varepsilon$ *be any number* $\in (0, T)$ *and put* $J=[0, T-\varepsilon]$. *Assume that* (A.1)-(A.5) *are valid. Let* $(\vec{u}_0, \vec{u}_1, \vec{f}_\Omega, \vec{f}_\Gamma)$ *be data in* $D^2(J)$ *such that* $\vec{u}_1 \in H^2(\Omega)$. *Then, there exists a unique* $\vec{u}(t) \in X^{2,0}(J, \Omega)$ *satisfying the equations*:

$$\begin{aligned} &P(t)[\vec{u}(t)] = \vec{f}_\Omega(t) \quad \text{in } J\times\Omega; \qquad Q(t)[\vec{u}(t)] = \vec{f}_\Gamma(t) \quad \text{on } J\times\Gamma, \\ &\vec{u}(0) = \vec{u}_0 \quad \text{and} \quad \partial_t \vec{u}(0) = \vec{u}_1 \text{ in } \Omega. \end{aligned} \tag{5.1}$$

REMARK. In our proof of Lemma 5.2 below, we use the existence theorem of solutions to the problem for $P_\sigma(t)$ and $Q_\sigma(t)$ defined by (2.5) (cf. Theorem 5.3, below). To do this, the compatibility condition for the operators $P_\sigma(t)$ and $Q_\sigma(t)$ must be satisfied by $(\vec{u}_0, \vec{u}_1, \vec{f}_\Omega, \vec{f}_\Gamma)$. By using the assumption: $\vec{u}_1 \in H^2(\Omega)$, we shall reduce (5.1) to the problem with zero Cauchy data and $\vec{f}_\Gamma(0)=0$ on $\Gamma$, where the compatibility condition for $P_\sigma(t)$ and $Q_\sigma(t)$ is satisfied for any $\sigma$.

Deferring the proof of Lemma 5.2 and assuming that Lemma 5.2 is valid, we give a

PROOF OF THEOREM 5.1. The uniqueness of solutions follows from Theorem 2.1. To prove the existence of solutions, it is sufficient to prove that for any closed interval $[0, T-\varepsilon]$, (N) admits a unique solution $\vec{u}_\varepsilon \in X^{2,0}([0, T-\varepsilon], \Omega)$ to (5.1). For, if we put $\vec{u}(t)=\vec{u}_\varepsilon(t)$ for $0 \leqq t \leqq T-\varepsilon$, since $\vec{u}_\varepsilon(t)=\vec{u}_{\varepsilon'}(t)$ for $0 \leqq t \leqq T-\varepsilon$ provided that $0<\varepsilon'<\varepsilon<T$ as follows from the uniqueness of solutions, $\vec{u}(t)$ is well-defined, belongs $X^{2,0}([0, T), \Omega)$ and satisfies (N). Put $J=[0, T-\varepsilon]$. In view of Lemma 5.2, we shall prove that there exist sequences $\{\vec{u}_{k\sigma}\} \subset H^2(\Omega)$ ($k=0$ and 1) such that

(5.2.a) $$\|\vec{u}_{1\delta}-\vec{u}_1\|_1 \to 0 \quad \text{and} \quad \|\vec{u}_{0\delta}-\vec{u}_0\|_2 \to 0 \quad \text{as } \delta \to 0;$$

(5.2.b) $$\nu_i A^{ij}(0)\partial_j \vec{u}_{0\delta}+B^j(0)\partial_j \vec{u}_{0\delta}+B^0(0)\vec{u}_{1\delta}=\vec{f}_\Gamma(0) \quad \text{on } \Omega.$$

If we know that (5.2) is valid, since (5.2.b) means that $\vec{u}_0$, $\vec{u}_1$, $\vec{f}_\Omega$ and $\vec{f}_\Gamma$ satisfy the compatibility condition of order zero (cf. (1.2) with $N=0$), applying Lemma 5.2 implies that there exists a solution $\vec{u}_\delta(t) \in X^{2,0}(J, \Omega)$ to the equations:

(5.3.a) $$P(t)[\vec{u}_\delta(t)]=\vec{f}_\Omega(t) \text{ in } J\times\Omega; \quad Q(t)[\vec{u}_\delta(t)]=\vec{f}_\Gamma(t) \text{ on } J\times\Gamma;$$

(5.3.b) $$\vec{u}_\delta(0)=\vec{u}_0 \quad \text{and} \quad \partial_t \vec{u}_\delta(0)=\vec{u}_1 \text{ in } \Omega.$$

Applying Theorem 1.3 with $L=2$ to $\vec{u}_\delta-\vec{u}_{\delta'}$ implies that

$$|\vec{u}_\delta-\vec{u}_{\delta'}|_{2,0,J} \leqq C(T)\sum_{k=0}^{1}\|\vec{u}_{k\delta}-\vec{u}_{k\delta'}\|_{2-k}.$$

Combining that and (5.2.a), we have that $\{\vec{u}_\delta\}$ is a Cauchy sequence in $X^{2,0}(J, \Omega)$. Since $J$ is a closed interval, by the cmpleteness of $X^{2,0}(J, \Omega)$ we see that there exists a limit $\vec{u}_\varepsilon$ of $\{\vec{u}_\delta\}$ in $X^{2,0}(J, \Omega)$. Applying (Ap. 1)-(Ap. 3) with $\alpha=K-1$, $\beta=\gamma=1$, we have that $\|P(t)[\vec{u}_\delta(t)-\vec{u}_\varepsilon(t)]\|_0+\langle\!\langle Q(t)[\vec{u}_\delta(t)-\vec{u}_\varepsilon(t)]\rangle\!\rangle_{1/2} \leqq C\|\bar{D}^2[\vec{u}_\delta(t)-\vec{u}_\varepsilon(t)]\|_0$ for all $t\in J$. Hence, letting $\delta\to 0$ in (5.3), we see that $\vec{u}_\varepsilon$ satisfies (5.1).

Since $H^\infty(\Omega)$ is dense in $H^1(\Omega)$, there exists a sequence $\{\vec{u}_{1\delta}\} \subset H^\infty(\Omega)$ such that the first part of (5.2.a) is valid. Let $\vec{w}_\delta$ be solutions to the equations

(5.4.a) $$-\partial_i(A^{ij}(0)\partial_j\vec{w}_\delta)+\lambda\vec{w}_\delta=0 \quad \text{in } \Omega,$$

(5.4.b) $$\nu_i A^{ij}(0)\partial_j\vec{w}_\delta+B^i(0)_j\partial\vec{w}_\delta=\vec{g}_\delta \quad \text{on } \Gamma,$$

where $\vec{g}_\delta=\vec{f}_\Gamma(0)-\nu_i A^{ij}(0)\partial_j\vec{u}_0-B^j(0)\partial_j\vec{u}_0-B^0(0)\vec{u}_{1\delta}$. If $\lambda$ is chosen so large that we can apply Theorem 3.6 with $P^{ij}=A^{ij}(0)$, $P^j_\Gamma=B^j(0)$, $P^l_\Omega=P^{n+1}_\Gamma=0$ ($i, j=1, \cdots, n$; $l=1, \cdots, n+1$), we know that (5.4) admits a solution $\vec{w}_\delta\in H^2(\Omega)$ having the estimate: $\|\vec{w}_\delta\|_2\leqq C\langle\!\langle\vec{g}_\delta\rangle\!\rangle_{1/2}$ for each $\delta$ where $C$ is independent of $\delta$. Since $\vec{f}_\Gamma(0)=\nu_i A^{ij}(0)\partial_j\vec{u}_0+B^j(0)\partial_j\vec{u}_0+B^0(0)\vec{u}_1$ as follows from (1.2) with $N=0$, $\vec{g}_\delta=B^0(0)(\vec{u}_1-\vec{u}_{1\delta})$. Then, applying (Ap. 3) with $\alpha=K-1$, $\beta=\gamma=1$, we have that $\langle\!\langle\vec{g}_\delta\rangle\!\rangle_{1/2}\leqq C\|\vec{u}_1-\vec{u}_{1\delta}\|_1$. Since the first part of (5.2.a) is valid, we have

(5.5) $$\|\vec{w}_\delta\|_2\to 0 \quad \text{as } \delta\to 0.$$

If we put $\vec{u}_{0\delta}=\vec{u}_0+\vec{w}_\delta$, then by (5.4.b) and (5.5) we see that the second part of (5.2.a) and (5.2.b) are valid, which completes the proof of Theorem 5.1.

To prove Lemma 5.2, we shall use

THEOREM 5.3. *Let $I'=[-\tau/2, T+\tau/2]$. Assume that* $(A.1)'_{I'}$, $(A.2)_{I'}$, $(A.4)_{I'}$ *and* $(A.5)_{I'}$ *are valid, where* $(A.1)'_{I'}$ *is the same assumption as in Theorem* 2.2. *Assume that there exist positive constants $\delta'_1$ and $\delta'_2$ such that* $(A.3)_{I',\delta'}$ *is valid. Let $(\vec{u}_0, \vec{u}_1, \vec{f}_\Omega, \vec{f}_\Gamma)$ be data in $D^2([0, T])$ such that $\vec{f}_\Omega\in C^1([0, T], L^2(\Omega))$ and $\vec{f}_\Gamma\in C^1([0, T], H^{1/2}(\Gamma))$. Then, there exists a unique $\vec{u}\in X^{2,0}([0, T], \Omega)$ satisfying* (N).

Theorem 5.3 was proved by Shibata [9].

PROOF OF LEMMA 5.2. First, we shall reduce (5.1) to the problem with zero Cauchy data and $\vec{f}_\Gamma(0)=0$ on $\Gamma$. Put $\vec{U}(t)=\vec{u}_0+t\vec{u}_1$. Then, the assumption: $\vec{u}_1\in H^2(\Omega)$ implies that $\vec{U}(t)\in C^\infty(\boldsymbol{R}, H^2(\Omega))$. This assumption is used here only. In view of (Ap. 10), we have that $P(t)[\vec{U}(t)]\in Y^{1,0}(J, \Omega)$ and $Q(t)[\vec{U}(t)]\in Y^{1,1/2}(J, \Gamma)$. If we put $\vec{F}_\Omega(t)=\vec{f}_\Omega(t)-P(t)[\vec{U}(t)]$ and $\vec{F}_\Gamma(t)=\vec{f}_\Gamma(t)-Q(t)[\vec{U}(t)]$ by (1.2) with $N=0$ we see that $\vec{F}_\Gamma(0)=0$ on $\Gamma$. If $\vec{v}(t)$ is a solution to the equations: $P(t)[\vec{v}(t)]=\vec{F}_\Omega(t)$ in $J\times\Omega$; $Q(t)[\vec{v}(t)]=\vec{F}_\Gamma(t)$ on $J\times\Gamma$; $\vec{v}(0)=\partial_t\vec{v}(0)=0$ in $\Omega$, then $\vec{u}(t)=\vec{U}(t)+\vec{v}(t)$ obviously satisfies (5.1). From this point of view, it is sufficient to prove Lemma 5.2 in the case where $\vec{u}_0=\vec{u}_1=0$; $\vec{f}_\Omega(t)\in Y^{1,0}(J, \Omega)$ and $\vec{f}_\Gamma(t)\in Y^{1,1/2}(J, \Gamma)$;

(5.6) $$\vec{f}_\Gamma(0)=0 \quad \text{on } \Gamma.$$

The uniqueness of solutions follows from Theorem 2.1. Hence, we shall only prove the existence of solutions to (5.1). Let $P_\sigma(t)$ and $Q_\sigma(t)$ be operators defined by (2.5). By Lemma 2.3 we know that $P_\sigma(t)$ and $Q_\sigma(t)$ satisfy all the conditions of Theorem 5.3. To use Theorem 5.3, we must approximate $\vec{f}_\Omega$ and $\vec{f}_\Gamma$ by functions smooth in $t$. Recall that $J=[0, T-\varepsilon]$. Put

$$\vec{g}_U'(t)=\begin{cases} \vec{f}_U(T-\varepsilon), & t>T-\varepsilon\,, \\ \vec{f}_U(t), & 0\leqq t\leqq T-\varepsilon\,, \\ \vec{f}_U(0), & t<0\,, \end{cases}$$

for $U=\Omega$ and $\Gamma$. In view of (5.6), we see easily that $\vec{g}_\Omega'(t)\in Y^{1,0}(\boldsymbol{R}, \Omega)$ and $\vec{g}_\Gamma'(t)\in Y^{1,1/2}(\boldsymbol{R}, \Gamma)$. Let $\mu(t)\in C_0^\infty([-2T, 3T])$ such that $0\leqq\mu\leqq 1$ and $\mu(t)=1$ on $[-T, 2T]$. Put $\vec{g}_\Omega(t)=\mu(t)\vec{g}_\Omega'(t)$ and $\vec{g}_\Gamma(t)=\mu(t)\vec{g}_\Gamma'(t)$. Then,

(5.7.a) $\quad \vec{g}_\Omega(t)\in Y^{1,0}(\boldsymbol{R}, \Omega)$ and $\vec{g}_\Omega(t)\in Y^{1,1/2}(\boldsymbol{R}, \Gamma)$;

(5.7.b) $\quad \vec{g}_\Omega(t)=0$ for $t\notin[-2T, 3T]$ and $\vec{g}_\Gamma(t)=0$ for $t\notin[0, 3T]$;

(5.7.c) $\quad \vec{g}_\Omega(t)=\vec{f}_\Omega(t)$ and $\vec{g}_\Gamma(t)=\vec{f}_\Gamma(t)$ for $t\in J$.

Let $\kappa(t)\in C_0^\infty([1, 2])$ such that $\kappa(t)\geqq 0$ and $\int\kappa(t)dt=1$. Put

$$\vec{f}_{U\sigma}(t)=\int\kappa_\sigma(t-s)\vec{g}_U(s)ds \quad \text{for } U=\Omega \text{ and } \Gamma\,.$$

where $\kappa_\sigma(t)=\sigma^{-1}\kappa(\sigma^{-1}t)$. Since $\vec{g}_\Gamma(s)=0$ for $s<0$ and $\kappa_\sigma(-s)=0$ for $s>0$, we have

$$\vec{f}_{\Gamma\sigma}(0)=0 \quad \text{on } \Gamma \text{ for any } \sigma>0\,. \tag{5.8}$$

Obviously, we have

$$\vec{f}_{\Omega\sigma}(t)\in C_0^\infty(\boldsymbol{R}; L^2(\Omega)) \quad \text{and} \quad \vec{f}_{\Gamma\sigma}(t)\in C_0^\infty(\boldsymbol{R}, H^{1/2}(\Gamma))\,, \tag{5.9}$$

where $C_0^\infty(\boldsymbol{R}, X)$ is the set of all functions in $C_0^\infty(\boldsymbol{R})$ having its value in $X$. Furthermore, we have

$$\begin{aligned} &|\vec{f}_{\Omega\sigma}-\vec{g}_\Omega|_{0,0,\boldsymbol{R}}+\langle\vec{f}_{\Gamma\sigma}-\vec{g}_\Gamma\rangle_{0,1/2,\boldsymbol{R}} \\ &\quad+\int_{\boldsymbol{R}}(\|\partial_t(\vec{f}_{\Omega\sigma}-\vec{g}_\Omega)(t)\|_0^2+\langle\!\langle\partial_t(\vec{f}_{\Gamma\sigma}-\vec{g}_\Gamma)(t)\rangle\!\rangle_{1/2}^2)dt\to 0 \quad \text{as } \sigma\to 0\,. \end{aligned} \tag{5.10}$$

From (5.7.a) and (5.7.b) it follows immediately that $|\vec{g}_\Omega|_{0,0,\boldsymbol{R}}+\langle\vec{g}_\Gamma\rangle_{0,1/2,\boldsymbol{R}}+\int_{\boldsymbol{R}}(\|\partial_t\vec{g}_\Omega(t)\|_0^2+\langle\!\langle\partial_t\vec{g}_\Gamma(t)\rangle\!\rangle_{1/2}^2)dt<\infty$. Thus, from (5.10) we have

$$|\vec{f}_{\Omega\sigma}|_{0,0\boldsymbol{R}}+(\vec{f}_{\Gamma\sigma})_{0,1/2,\boldsymbol{R}}+\int_{\boldsymbol{R}}(\|\partial_t\vec{f}_{\Omega\sigma}(s)\|_0^2+\langle\!\langle\partial_t\vec{f}_{\Gamma\sigma}(t)\rangle\!\rangle_{1/2}^2)dt\leqq C \tag{5.11}$$

for any $\sigma\in(0, \Sigma_0)$ where $\Sigma_0$ is the same as in Lemma 2.3. In the present proof, we use the same letter $C$ to denote various constants independent of $\sigma$.

Now, let $\vec{u}_\sigma$ be solutions in $X^{2,0}([0, T], \Omega)$ to the equations:

(5.12.a) $$P_\sigma(t)[\vec{u}_\sigma(t)]=\vec{f}_{\Omega\sigma}(t) \quad \text{in } [0, T]\times\Omega,$$

(5.12.a) $$Q_\sigma(t)[\vec{u}_\sigma(t)]=\vec{f}_{\Gamma\sigma}(t) \quad \text{on } [0, T]\times\Gamma,$$

(5.12.c) $$\vec{u}_\sigma(0)=\partial_t\vec{u}_\sigma(0)=0 \qquad \text{in } \Omega.$$

Here, note that we use that $\vec{u}_\sigma$ is defined on $[0, T]$ with respect to $t$ in proving that the limit of $\vec{u}_\sigma$ belongs to $X^{2,0}(J, \Omega)$ below. In view of (5.8) and (5.9), applying Theorem 5.3 implies that (5.12) admits a unique solution $\vec{u}_\sigma\in X^{2,0}([0, T], \Omega)$ for each $\sigma\in(0, \Sigma_0)$. Furthermore, using Theorem 1.3 with $L=2$ to (5.13) and noting (b) of Lemma 2.3, we have

(5.13) $$\|\bar{D}^2\vec{u}_\sigma(t)\|_0^2\leqq C;$$

(5.14) $$E_\sigma(t, \partial_t\vec{u}_\sigma(t))\leqq e^{Ct}\{E_\sigma(0, \partial_t\vec{u}_\sigma(0))+Ct^{1/2}\}$$

for all $t\in[0, T]$, where $E_\sigma$ is the energy norm for the operators $P_\sigma(t)$ and $Q_\sigma(t)$. The main step of the present proof is summarized as follows:

LEMMA 5.4. *Put $J'=[0, T]$. Assume that* (A.1)-(A.5) *are valid. Let $\vec{u}_\sigma(t)$ be functions in $X^{2,0}(J', \Omega)$ satisfying* (5.12). *Then, there exists a $\vec{u}\in Y^{2,0}(J', \Omega)$ such that*

(5.15) $$\lim_{\sigma\to0} |\vec{u}_\sigma-\vec{u}|_{1,0,J'}=0;$$

(5.16) $$\vec{u}(0)=\partial_t\vec{u}(0)=0 \quad \text{in } \Omega;$$

(5.17.a) $$\vec{u}_\sigma(t)\to\vec{u}(t) \text{ weakly in } H^2(\Omega) \text{ as } \sigma\to0 \text{ for all } t\in J';$$

(5.17.b) $$\partial_t\vec{u}_\sigma(t)\to\partial_t\vec{u}(t) \text{ weakly in } H^1(\Omega) \text{ as } \sigma\to0 \text{ for all } t\in J';$$

(5.18) $$Q(t)[\vec{u}(t)]=\vec{g}_\Gamma(t) \text{ in the sense of } H^{1/2}(\Gamma) \text{ for all } t\in J'.$$

*Furthermore, if we put*

(5.19) $$\vec{v}(t)=\vec{g}_\Omega(t)+\partial_i(A^{i0}(t)\partial_t\vec{u}(t)+A^{ij}(t)\partial_j\vec{u}(t)),$$

*then*

(5.20) $$\partial_t^2\vec{u}_0(t)\to\vec{v}(t) \text{ weakly in } L^2(\Omega) \text{ as } \sigma\to0 \text{ for all } t\in J';$$

(5.21) $$\partial_t^2\vec{u}(t)=\vec{v}(t) \quad \text{for almost all } t\in J';$$

(5.22) $$\lim_{t\to0+} \{\|\vec{v}(t)-\vec{f}_\Omega(0)\|_0^2+\|\partial_t\vec{u}(t)\|_1^2+\|\vec{u}(t)\|_2^2\{=0.$$

Deferring the proof of Lemma 5.4, we shall prove that the $\vec{u}$ in Lemma 5.4 belongs to $X^{2,0}(J, \Omega)$ and satisfies (5.1). From (5.19) and (5.21) we see easily that

(5.23) $\quad P(t)[\vec{u}(t)]=\vec{g}_{\Omega}(t)$ in the sense of $L^2(\Omega)$ for almost all $t\in[0, T]$.

If we prove that $\vec{u}\in X^{2,0}(J, \Omega)$, by (5.18), (5.23), (Ap. 14) with $L=2$ and (5.7.c) we see that $\vec{u}$ satisfies (5.1). Hence, we shall prove that $\vec{u}\in X^{2,0}(J, \Omega)$. To do this, we use the mollifier with respect to $t$. Let $\rho(t)$ be the same function as in Lemma 4.1 and put $\vec{u}_\delta(t)=\int\rho_\delta(t-s)\vec{u}(s)ds$ where $\rho_\delta(s)=\delta^{-1}\rho(\delta^{-1}s)$. Since $\vec{u}\in Y^{2,0}(J', \Omega)\subset L^\infty(J', H^2(\Omega))$ $(J'=[0, T])$, $\vec{u}_\delta\in C^\infty(J, H^2(\Omega))$ provided that $0<\delta<(T-\varepsilon)/2$. Furthermore, noting (5.18) and (5.23) and applying Theorem 1.3 with $L=2$ to $\vec{u}_\delta-\vec{u}_{\delta'}$, we have

$$(5.24)\qquad |\vec{u}_\delta-\vec{u}_{\delta'}|^2_{2,0,J}\leqq C\{\|\bar{D}^2(\vec{u}_\delta(0)-\vec{u}_{\delta'}(0))\|_0^2+I_{\delta,\delta'}\}$$

where

$$\begin{aligned} I_{\delta'\delta'}=&|(\vec{g}_\Omega)_\delta-(\vec{g}_\Omega)_{\delta'}|_{0,0,J}+\langle(\vec{g}_\Gamma)_\delta-(\vec{g}_\Gamma)_{\delta'}\rangle_{0,1/2,J}\\ &+\int_J(\|\partial_t((\vec{g}_\Omega)_\delta(t)-(\vec{g}_\Omega)_{\delta'}(t))\|_0^2+\langle\!\langle\partial_t((\vec{g}_\Gamma)_\delta(t)-(\vec{g}_\Gamma)_{\delta'}(t))\rangle\!\rangle_{1/2}^2)dt\\ &+|R_\delta\vec{u}-R_{\delta'}\vec{u}|_{0,0,J}+\langle S_\delta\vec{u}-S_{\delta'}\vec{u}\rangle_{0,1/2,J}\\ &+\int_J(\|\partial_t(R_\delta\vec{u}(t)-R_{\delta'}\vec{u}(t))\|_0^2+\langle\!\langle\partial_t(S_\delta\vec{u}(t)-S_{\delta'}\vec{u}(t))\rangle\!\rangle_{1/2}^2)dt\,. \end{aligned}$$

Here, $R_\delta\vec{u}$ and $S_\delta\vec{u}$ are the same as in (4.12). Since $\vec{u}\in Y^{2,0}(J', \Omega)$, $\partial_l\vec{u}\in Y^{0,1}(J, \Omega)$ for $l=0, 1, \cdots, n$. Hence, we can apply Lemma 4.1 with $L=2$. As a result, noting (5.7.a) and applying Lemma 4.1 with $L=2$, we see that $I_{\delta,\delta'}\to 0$ as $\delta, \delta'\to 0$. And then, if we prove

$$(5.25)\qquad \|\bar{D}^2(\vec{u}_\delta(0)-\vec{u}_{\delta'}(0))\|_0\to 0 \quad\text{as } \delta, \delta'\to 0\,,$$

letting $\delta, \delta'\to 0$ in (5.24), we see that $\{\vec{u}_\delta\}$ is a Cauchy sequence in $X^2(J, \Omega)$, which implies that the limit $\vec{w}$ of $\{\vec{u}_\delta\}$ exists in $X^{2,0}(J, \Omega)$. However, we already knew that $\vec{u}\in Y^{2,0}(J, \Omega)\subset X^{1,0}(J, \Omega)$. This implies that $\vec{u}_\delta\to\vec{u}$ in $X^{1,0}(J, \Omega)$. Hence, we have that $\vec{u}=\vec{w}\in X^{2,0}(J, \Omega)$.

To obtain (5.25), it is sufficient to prove that

$$(5.26)\qquad \lim_{\delta\to 0}\|\partial_t^l\vec{u}_\delta(0)\|_{2-l}=0 \quad\text{for } l=0, 1 \text{ and } \lim_{\delta\to 0}\|\partial_t^2\vec{u}_\delta(0)-\vec{f}_\Omega(0)\|_0=0\,.$$

By (5.21) we know that $\partial_t^2\vec{u}_\delta(0)=\int\rho_\delta(-s)\partial_s^2\vec{u}(s)ds=\int\rho_\delta(-s)\vec{v}(s)ds$. Hence, by (5.22) we have

$$\|\partial_t^2\vec{u}_\delta(0)-\vec{f}_\Omega(0)\|_0\leqq\int\rho(-s)\|\vec{v}(\delta s)-\vec{f}_\Omega(0)\|_0ds\to 0 \quad\text{as } \delta\to 0\,,$$

where we have used the fact that $\operatorname{supp}\rho(-s)\subset[1, 2]$ (cf. Lemma 4.1). In the same way, by (5.22) we can easily prove other assertions of (5.26). Hence, if we prove Lemma 5.4, then we can complete the proof of Lemma 5.2.

PROOF OF LEMMA 5.4. First, we shall prove (5.15) and (5.16). Since

(5.27.a) $\quad P_\sigma(t)[\vec{u}_\sigma(t)-\vec{u}_{\sigma'}(t)]=(P_\sigma(t)-P_{\sigma'}(t))[\vec{u}_{\sigma'}(t))]$ in $J'\times\Omega$;

(5.27.b) $\quad Q_\sigma(t)[\vec{u}_\sigma(t)-\vec{u}_{\sigma'}(t)]=(Q_\sigma(t)-Q_{\sigma'}(t))[\vec{u}_{\sigma'}(t)]$ on $J'\times\Gamma$;

(5.27.c) $\quad \vec{u}_\sigma(0)-\vec{u}_{\sigma'}(0)=\partial_t\vec{u}_\sigma(0)-\partial_t\vec{u}_{\sigma'}(0)=0$ in $\Omega$

as follows from (5.12), applying Theorem 2.1 to (5.27) and noting (b) of Lemma 2.3, we have

$$|\vec{u}_\sigma-\vec{u}_{\sigma'}|_{1,0,J}\leqq C\int_{J'}(\|P_{\sigma'}(s)-P_\sigma(s))[\vec{u}_{\sigma'}(s)]\|_0^2 \\ +\langle\!\langle(Q_{\sigma'}(s)-Q_\sigma(s))[\vec{u}_{\sigma'}(s)]\rangle\!\rangle_{1/2}^2)ds \tag{5.28}$$

Applying (A. 1)-(A. 3) with $\alpha=K-1$ and $\beta=\gamma=1$, and using (5.13), we have

$$\|(P_{\sigma'}(s)-P_\sigma(s))[\vec{u}_{\sigma'}(s)]\|_0^2+\langle\!\langle(Q_{\sigma'}(s)-Q_\sigma(s))[\vec{u}_{\sigma'}(s)]\rangle\!\rangle_{1/2}^2\leqq CU_{\sigma,\sigma'}(s)$$

where

$$U_{\sigma,\sigma'}(s)=[P_\sigma(s)-P_{\sigma'}(s)]_{\infty,K-1}+[P_\sigma(s)-P_{\sigma'}(s)|Q_\sigma(s)-Q_{\sigma'}(s)]_{S,K-2,1}$$

(cf. (No. 3 a and b)). Substituting this into (5.28) and using (a) of Lemma 2.3, we see that $\{\vec{u}_\sigma\}$ is a Cauchy sequence in $X^{1,0}(J',\Omega)$. By the completeness of $X^{1,0}(J',\Omega)$, we can conclude that there exists a limit $\vec{u}\in X^{1,0}(J',\Omega)$ satisfying (5.15). In particular, combining (5.12.c) and (5.15) implies that (5.16) is valid.

Now, we shall prove that (5.17.a) is valid and that

(5.29.a) $\quad \|\vec{u}(t)\|_2\leqq C$ for all $t\in J'$;

(5.30.a) $\quad \vec{u}(t)$ is continuous on $J'$ in the weak topology of $H^2(\Omega)$;

(5.31.a) $\quad \|\vec{u}(t)-\vec{u}(s)\|_1\leqq C|t-s|$ for all $t,s\in J'$;

(5.32.a) $\quad \vec{u}(t)\in L^\infty(J',H^2(\Omega))\cap\mathrm{Lip}\,(J',H^1(\Omega))$.

By Pettis' theorem, we know that (5.30.a) implies that $\vec{u}(t)$ is measurable in the strong sense of $H^2(\Omega)$. Hence, (5.29.a) and (5.31.a) implies (5.32.a). (5.17.a) implies that

$$\|\vec{u}(t)\|_2\leqq\liminf_{\sigma\to0}\|\vec{u}_\sigma(t)\|_2\,.$$

Combining this and (5.13) implies (5.29.a). Since $\vec{u}_\sigma(t)\in X^{2,0}(J',\Omega)$, by the mean value theorem we have that $\|\vec{u}_\sigma(t)-\vec{u}_\sigma(s)\|_1\leqq|t-s|\int_0^1\|\partial_t\vec{u}_\sigma(s+\theta(t-s))\|_1d\theta$. Combining this and (5.13) implies that $\|\vec{u}_\sigma(t)-\vec{u}_\sigma(s)\|_1\leqq C|t-s|$. Hence, (5.31.a) follows from (5.15) immediately. Now, we shall prove (5.17.a). Let $\alpha$ be any multi-index such that $|\alpha|\leqq 2$, $\vec{w}\in L^2(\Omega)$ and $\kappa$ be any positive number. Since $C_0^\infty(\Omega)$ is dense in $L^2(\Omega)$, there exists a $\vec{z}\in C_0^\infty(\Omega)$ such that $\|\vec{w}-\vec{z}\|_0<\kappa$. Hence, we have

$$\begin{aligned}|(\partial_x^\alpha\vec{u}_\sigma(t)-\partial_x^\alpha\vec{u}_{\sigma'}(t),\ \vec{w})| &\leqq |(\partial_x^\alpha\vec{u}_\sigma(t),\ \vec{w}-\vec{z})|+|(\partial_x^\alpha\vec{u}_{\sigma'}(t),\ \vec{w}-\vec{z})|\\ &\quad+|(\vec{u}_\sigma(t)-\vec{u}_{\sigma'}(t),\ (-\partial_x)^\alpha\vec{z})|\\ &\leqq C\kappa+\|\vec{u}_\sigma(t)-\vec{u}_{\sigma'}(t)\|_0\|(-\partial_x)^\alpha\vec{z}\|_0,\end{aligned}$$

where we have used Schwarz's inequality and (5.13). Letting $\sigma$, $\sigma'\to 0$ and using (5.15) and the arbitrariness of the choice of $\kappa$, we see that $\{\partial_x^\alpha\vec{u}_\sigma\}$ is Cauchy sequence in the weak topology of $L^2(\Omega)$. Since $\sigma$ is any multi-index such that $|\alpha|\leqq 2$, we can conclude that $\{\vec{u}_\sigma\}$ is a Cauchy sequence in the topology of $H^2(\Omega)$, which implies that $\vec{u}_\sigma(t)$ converges to some $\vec{u}'(t)\in H^2(\Omega)$ weakly as $\sigma\to 0$ for all $t\in J'$. On the other hand, (5.15) implies obviously that $\vec{u}_\sigma(t)$ converges to $\vec{u}(t)$ weakly as $\sigma\to 0$ for all $t\in J'$. Thus, $\vec{u}(t)=\vec{u}'(t)\in H^2(\Omega)$ for all $t\in J'$ and (5.17.a) is valid.

Now, we prove (5.30.a). Note that (5.29.a) is now valid, because (5.17.a) has been proved. Let $\alpha$, $\kappa$, $\vec{w}$ and $\vec{z}$ be the same as above. For $t$ and $s\in J'$, we have

$$\begin{aligned}|(\partial_x^\alpha\vec{u}(t)-\partial_x^\alpha\vec{u}(s),\ \vec{w})| &\leqq |(\partial_x^\alpha\vec{u}(t),\ \vec{w}-\vec{z})|+|(\partial_x^\alpha\vec{u}(s),\ \vec{w}-\vec{z})|+|(\vec{u}(t)-\vec{u}(s),\ (-\partial_x)^\alpha\vec{z})|\\ &\leqq C\kappa+\|\vec{u}(t)-\vec{u}(s)\|_0\|(-\partial_x)^\alpha\vec{z}\|_0,\end{aligned}$$

where we have used Schwarz's inequality and (5.29.a). Since $\vec{u}(t)\in X^{1,0}(J',\ \Omega)$, letting $t\to s$ and noting that $\kappa$ is chosen arbitrarily, we have (5.30.a).

By employing the same arguments, we can prove that (5.17.b) and the following four assertions are valid:

(5.29.b) $\quad\|\partial_t\vec{u}(t)\|_1\leqq C$ for all $t\in J'$;

(5.30.b) $\quad\partial_t\vec{u}(t)$ is continuous on $J'$ in the weak topology of $H^1(\Omega)$;

(5.31.b) $\quad\|\partial_t\vec{u}(t)-\partial_t\vec{u}(s)\|_0\leqq C|t-s|$ for all $t$, $s\in J'$;

(5.32.b) $\quad\partial_t\vec{u}(t)\in L^\infty(J',\ H^1(\Omega))\cap\mathrm{Lip}\,(J',\ L^2(\Omega))$.

In particular, combining (5.32.a) and (5.32.b) implies that $\vec{u}\in Y^{2,0}(J',\ \Omega)$.

Now, we prove (5.18) and (5.20). First, note the following facts: If we define the operators $A(t)[\vec{w}_0,\ \vec{w}_1]=\partial_i(A^{i0}(t)\vec{w}_1+A^{ij}(t)\partial_j\vec{w}_0)$ and $B(t)[\vec{w}_0,\ \vec{w}_1]=\nu_iA^{ij}(t)\partial_j\vec{w}_0+B^j(t)\partial_j\vec{w}_0+B^0(0)\vec{w}_1|_\Gamma$, then $A(t)$ and $B(t)$ are bounded linear operator from $H^2(\Omega)\times H^1(\Omega)$ into $L^2(\Omega)$ and $H^{1/2}(\Gamma)$, respectively. Then facts follows immediately from (Ap. 1)-(Ap. 3) with $\alpha=K-1$ and $\beta=\gamma=1$. By (5.17) and these facts we see easily that $A(t)[\vec{u}_\sigma(t),\ \partial_t\vec{u}_\sigma(t)]\to A(t)[\vec{u}(t),\ \partial_t\vec{u}(t)]$ and $B(t)[\vec{u}_\sigma(t),$

$\partial_t\vec{u}_\sigma(t)]\to B(t)[\vec{u}(t), \partial_t\vec{u}(t)]$ weakly in $L^2(\Omega)$ and $H^{1/2}(\Gamma)$ as $\sigma\to 0$ for all $t\in J'$. On the other hand, since

$$\sum_{l=0}^{n}\|\partial_i((A_\sigma^{il}(t)-A^{il}(t))\partial_l\vec{u}_\sigma(t))\|_0+\langle\!\langle(Q_0(t)-Q(t))[\vec{u}_\sigma(t)]\rangle\!\rangle_{1/2}\leqq CU_\sigma(t) \tag{5.33}$$

as follows from (Ap. 1)-(Ap. 3) with $\alpha=K-1$ and $\beta=\gamma=1$ and (5.13), where $U_\sigma(t)$ is the same as in (2.9.b), using (a) of Lemma 2.3, we have that the left-hand side of (5.33) tends to zero as $\sigma\to 0$. Combining these two results, we have

$$\sum_{l=0}^{n}\partial_i((A_\sigma^{il}(t)\partial_l\vec{u}_\sigma(t))\to\sum_{l=0}^{n}\partial_i(A^{il}(t)\partial_l\vec{u}(t)) \quad \text{weakly in } L^2(\Omega)\,; \tag{5.34.a}$$

$$Q_\sigma(t)[\vec{u}_\sigma(t)]\to Q(t)[\vec{u}(t)] \quad \text{weakly in } H^{1/2}(\Gamma) \tag{5.34.b}$$

as $\sigma\to 0$ for all $t\in J'$. Combining (5.34.b), (5.12.b) and (5.10) implies (5.18). And also, combining (5.34.a), (5.12.a) and (5.10) and noting (5.19), we have (5.20).

Now, we shall prove that

(5.30.c) $\quad\vec{v}(t)$ is continuous on $J'$ in the weak topology of $L^2(\Omega)$.

In the same manner as above, (5.30.a and b) implies that $A(s)[\vec{u}(t), \partial_t\vec{u}(t)]\to A(s)[\vec{u}(s), \partial_t\vec{u}(s)]$ weakly in $L^2(\Omega)$ as $t\to s$. On the other hand, applying (Ap. 1) with $\alpha=K-1$ and $\beta=\gamma=1$ implies that

$$\|(A(t)-A(s))[\vec{u}(t), \partial_t\vec{u}(t)]\|_0\leqq C\{U_\infty(t, s)+U_S(t, s)\}(\|\partial_t\vec{u}(t)\|_1+\|\vec{u}(t)\|_2)$$

where $U_\infty(t, s)$ and $U_S(t, s)$ are the same as in (3.88). By (3.89) and (5.29.a and b), we see that $(A(t)-A(s))[\vec{u}(t), \partial_t\vec{u}(t)]\to 0$ strongly in $L^2(\Omega)$ as $t\to s$. Combining these two facts and noting (5.19) and the fact that $\vec{g}_\Omega(t)\in Y^{1,0}(\boldsymbol{R}, \Omega)\subset C^0(\boldsymbol{R}, L^2(\Omega))$, we have (5.30.c).

Now, we prove (5.21). Since $\vec{v}(t)\in L^\infty(J', L^2(\Omega))$, the Bochner integral $\int_0^t\vec{v}(s)ds$ exists and belong to $L^2(\Omega)$ for each $t\in J'$. Furthermore, we have

$$\left(\int_0^t\vec{v}(s)ds, \vec{w}\right)=\int_0^t(\vec{v}(s), \vec{w})ds \quad \text{for any } \vec{w}\in L^2(\Omega)\,. \tag{5.35}$$

Since $(\partial_t\vec{u}_\sigma(t), \vec{w})=\int_0^t(\partial_t^2\vec{u}_\sigma(s), \vec{w})ds$ as follows from (5.12.c) and the fact that $\vec{u}_\sigma(t)\in X^{2,0}(J', \Omega)$, letting $\sigma\to 0$ and using (5.17.b) and (5.20), we have

$$(\partial_t\vec{u}(t), \vec{w})=\int_0^t(\vec{v}(s), \vec{w})ds \quad \text{for any } \vec{w}\in L^2(\Omega)\,. \tag{5.36}$$

Combining (5.35) and (5.36) implies that $\partial_t\vec{u}(t)=\int_0^t\vec{v}(s)ds$ for all $t\in J'$, where the equality holds as functions in $t\in J'$ having their values in $L^2(\Omega)$. Since $L^2(\Omega)$

is reflexive, by Lebesgue's theorem we have (5.21).

Finally, we shall prove (5.22). To do this, we only prove that

$$\lim_{t\to 0+} \|\vec{v}(t)\|_0^2 + |||\partial_t \vec{u}(t)|||_{1,0}^2 = \|\vec{f}_\Omega(0)\|_0^2 . \tag{5.37}$$

In fact, since $L^2(\Omega)\times H^1(\Omega)$ is a Hilbert space equipped with the norm: $\|\cdot\|^2 + |||\cdot|||_{1,0}^2$ (cf. (No. 17)), (5.37) and (No. 17) implies that

$$\lim_{t\to 0+} \|\vec{v}(t)-\vec{f}_\Omega(0)\|_0^2 + \|\partial_t \vec{u}(t)\|_1^2 = 0 . \tag{5.38}$$

On the other hand, applying Corollary 3.7 with $L=2$ and noting (5.18) and (5.19), we have

$$\|\vec{u}(t)\|_2^2 \leqq C\{\|\vec{v}(t)-\vec{g}_\Omega(t)\|_0^2 + \|\partial_i(A^{i0}(t)\partial_t \vec{u}(t))\|_0^2 + \langle\!\langle \vec{g}_\Gamma(t)\rangle\!\rangle_{1/2}^2 + \|\vec{u}(t)\|_1^2\}. \tag{5.39}$$

Applying (Ap. 1) with $\alpha=K-1$ and $\beta=\gamma=1$, we have that $\|\partial_i(A^{i0}(t)\partial_t \vec{u}(t)\|_0^2 \leqq C\|\partial_t \vec{u}(t)\|_1^2$. Hence, noting that $\|\vec{v}(t)-\vec{g}_\Omega(t)\|_0 \leqq \|\vec{v}(t)-\vec{f}_\Omega(0)\|_0 + \|\vec{g}_\Omega(t)-\vec{f}_\Omega(0)\|_0$, by (5.7.a), (5.7.c), (5.6), the fact that $\vec{u}\in X^{1,0}(J, \Omega)$, (5.16), (5.38) and (5.39), we have that $\|\vec{u}(t)\|_2 \to 0$ as $t\to 0+$. Combining this and (5.38) implies (5.22).

Our idea of proving (5.37) is due to Majda [5, p. 44] essentially. First, we shall prove that

$$\|\vec{f}_\Omega(0)\|_0^2 \leqq \liminf_{t\to 0+} (\|\vec{v}(t)\|_0^2 + |||\partial_t \vec{u}(t)|||_{1,0}^2) . \tag{5.40}$$

Note that the norms of $\|\cdot\|_1$ and $|||\cdot|||_{1,0}$ are equivalent (cf. (No. 17)) and that $\vec{v}(0)=\vec{g}_\Omega(0)=\vec{f}_\Omega(0)$ and $\partial_t \vec{u}(0)=0$ (cf. (5.19), (5.16) and (5.7.c)). By (5.30.b and c), we have (5.40).

In view of (5.40), to obtain (5.37) it is sufficient to prove that

$$\limsup_{t\to 0+} (\|\vec{v}(t)\|_0^2 + |||\partial_t \vec{u}(t)|||_{1,0}^2) \leqq \|\vec{f}_\Omega(0)\|_0^2 . \tag{5.41}$$

By (5.13) and (2.9.a), we see that $|E(t, \partial_t \vec{u}_\sigma(t)) - E_\sigma(t, \partial_t \vec{u}_\sigma(t)| \leqq C U_\sigma(t)$. And also, by (No. 23) we see that $|E(t, \partial_t \vec{u}_\sigma(t)) - E(0, \partial_t \vec{u}_\sigma(t))| \leqq Ct$. Noting that $E(0, \partial_t \vec{u}_\sigma(t)) = \|\partial_t^2 \vec{u}_\sigma(t)\|_0^2 + |||\partial_t \vec{u}_\sigma(t)|||_{1,0}^2$ (cf. (No. 21)), from (5.14) we have

$$\|\partial_t^2 \vec{u}_\sigma(t)\|_0^2 + |||\partial_t \vec{u}_\sigma(t)|||_{1,0}^2 \leqq e^{Ct} E(0, \partial_t \vec{u}_\sigma(0)) + C\{U_\sigma(t) + U_\sigma(0)\} + R(t) \tag{5.42}$$

where $R(t)=e^{Ct}t^{1/2}+Ct$. By (5.12.a and c) we know that $E(0, \partial_t \vec{u}_\sigma(0)) = \|\partial_t \vec{u}_\sigma(0)\|_0^2 = \|\vec{f}_{\Omega\sigma}(0)\|_0^2$. Letting $\sigma\to 0$ in (5.42) and using (a) of Lemma 2.3, (5.7.c) and (5.10), we have

$$\limsup_{t\to 0+} (\|\partial_t^2 \vec{u}_\sigma(t)\|_0^2 + |||\partial_t \vec{u}_\sigma(t)|||_1^2) \leqq e^{Ct} \|\vec{f}_\Omega(0)\|_0^2 + R(t) . \tag{5.43}$$

With the help of (5.17.b) and (5.20), from (5.43) we have

$$\|\vec{v}(t)\|_0^2 + |||\partial_t \vec{u}(t)|||_{1,0}^2 \leqq e^{Ct} \|\vec{f}_\Omega(0)\|_0^2 + R(t) . \tag{5.44}$$

Since $e^{Ct}\to 1$ and $R(t)\to 0$ as $t\to 0+$, (5.41) follows from (5.44), which completes the proof of Lemma 5.4.

## § 6. Further regularities of solutions

Let $L$ be an integer $\in[3, K]$. In this section, we prove that for a given data $(\vec{u}_0, \vec{u}_1, \vec{f}_\Omega, \vec{f}_\Gamma)\in D^L(J)$, (N) admits a solution $\vec{u}\in X^{L,0}(J, \Omega)$, where $J=[0, T-\varepsilon]$ and $\varepsilon$ is any number $\in(0, T)$. If $\vec{u}(t)\in X^{L,0}(J, \Omega)$ satisfies (N), by (Ap. 14) we know that $P(t)[\vec{u}(t)]\in X^{L-2,0}(J, \Omega)$ and $Q(t)[\vec{u}(t)]\in X^{L-,1/2}(J, \Gamma)$. And then, differentiating (N) $L-2$ times in $t$ and putting $\partial_t^M\vec{u}(t)=\vec{v}_M(t)$ $(0\leqq M\leqq L-2)$ and $V(t)=(\vec{v}_0(t), \cdots, \vec{v}_{L-2}(t))$, we have

$$P(t)[\vec{v}_{L-2}(t)]-R_\Omega(t)[V(t)]=\partial_t^{L-2}\vec{f}_\Omega(t) \quad \text{in } J\times\Omega, \tag{6.1.a}$$

$$Q(t)[\vec{v}_{L-2}(t)]+R_\Gamma(t)[V(t)]=\partial_t^{L-2}\vec{f}_\Gamma(t) \quad \text{on } J\times\Gamma, \tag{6.1.b}$$

$$\vec{v}_{L-2}(0)=\vec{u}_{L-2}, \quad \partial_t\vec{v}_{L-2}(0)=\vec{u}_{L-1} \qquad \text{in } \Omega, \tag{6.1.c}$$

where $\vec{u}_{L-2}$ and $\vec{u}_{L-1}$ are functions defined in (1.1);

$$R_\Omega(t)[V]=\sum_{k=1}^{L-2}\binom{L-2}{k}\partial_i\{\partial_t^kA^{i0}(t)\vec{v}_{L-1-k}+\partial_t^kA^{ij}(t)\partial_j\vec{v}_{L-2-k}\},$$

$$R_\Gamma(t)[V]=\sum_{k=1}^{L-2}\binom{L-2}{k}\{\nu_i\partial_t^kA^{ij}(t)\partial_j\vec{v}_{L-2-k}+\partial_t^kB^j(t)\partial_j\vec{v}_{L-2-k}+\partial_t^kB^0(t)\vec{v}_{L-1-k}\}.$$

Furthermore, for $0\leqq M\leqq L-3$, differentiating (N) $M$-times in $t$, we have

$$\begin{aligned}\vec{v}_{M+2}(t)-P_M(t)[\vec{v}_0(t), \cdots, \vec{v}_{M+1}(t)]+\lambda_M\vec{v}_M(t)&\\ =\partial_t^M\vec{f}_\Omega(t)+\lambda_M\Big(\vec{u}_M+\int_0^t\vec{v}_{M+1}(s)ds\Big) &\quad \text{in } J\times\Omega,\end{aligned} \tag{6.2.a$_M$}$$

$$Q_M(t)[\vec{v}_0(t), \cdots, \vec{v}_{M+1}(t)]=\partial_t^M\vec{f}_\Gamma(t) \qquad \text{on } J\times\Gamma, \tag{6.2.b$_M$}$$

where $\vec{v}_{L-1}(t)=\partial_t\vec{v}_{L-2}(t)$; $P_M(t)$ and $Q_M(t)$ are the same as in (3.59); $\lambda_M$ $(0\leqq M\leqq L-3)$ are constants given in Theorem 3.8 with $N_1=L-3$ and $N_2=L$; $\vec{u}_M$ $(0\leqq M\leqq L-3)$ are functions defined in (1.1). From this point of view, we shall split our proof into two stages. First, we consider the equations (6.1) and (6.2) for unknowns $\vec{v}_M$ $(0\leqq M\leqq L-2)$. And then, we shall prove that there exist $\vec{v}_{L-2}(t)\in X^{2,0}(J, \Omega)$ and $\vec{v}_M(t)\in X^{L,L-M-1}(J, \Omega)$ $(0\leqq M\leqq L-3)$ satisfying (6.1) and (6.2). Secondly, we shall prove that $\partial_t\vec{v}_M(t)=\vec{v}_{M+1}(t)$. Then, if we put $\vec{v}_0(t)=\vec{u}(t)$, we see easily that $\vec{u}(t)\in X^{L,0}(J, \Omega)$ and satisfies (N).

1*st step*. We shall solve (6.1) and (6.2) by the method of successive approximations. Before defining the iteration scheme, we prepare the function space and some estimations. Let $Z$ be the space of all functions $V(t)=(\vec{v}_0(t), \vec{v}_{L-2}(t))$

such that

(6.3.a) $\quad \vec{v}_M(t)\in X^{1,L-M-1}(J,\Omega)\ (0\leqq M\leqq L-3);\ \vec{v}_{L-2}(t)\in X^{2,0}(J,\Omega);$

(6.3.b) $\quad \vec{v}_M(0)=\vec{u}_M\ (0\leqq M\leqq L-2)$ and $\partial_t\vec{v}_{L-2}(0)=\vec{u}_{L-1},$

By (1.2) we see easily that

(6.4) $\quad \nu_i A^{ij}(0)\partial_j\vec{u}_{L-2}+B^j(0)\partial_j\vec{u}_{L-2}+B^0(0)\vec{u}_{L-1}=\partial_t^{L-2}\vec{f}_\Gamma(0)-R_\Gamma(0)[V(0)]$ on $\Gamma$

for any $V(t)\in Z$. Furthermore, for any $V(t)$ and $V'(t)\in Z$, we have

(6.5) $\quad R_\Omega(t)[V(t)]\in Y^{1,0}(J,\Omega);\ R_\Gamma(t)[V(t)]\in Y^{1,1/2}(J,\Gamma);$

(6.6)
$$|R_\Omega(\cdot)[V(\cdot)]-R_\Omega(\cdot)[V'(\cdot)]|_{0,0,J}+\langle R_\Gamma(\cdot)[V(\cdot)]-R_\Gamma(\cdot)[V'(\cdot)]\rangle_{0,1/2,J}$$
$$+\int_J(\|\partial_t R_\Omega(t)[V(t)]-\partial_t R_\Omega(t)[V'(t)]\|_0^2$$
$$+\langle\!\langle\partial_t R_\Gamma(t)[V(t)]-\partial_t R_\Gamma(t)[V'(t)]\rangle\!\rangle_{1/2}^2)dt\leqq C\int_J(\!(V(t)-V'(t))\!)_L^2dt$$

where $C=C(M_\infty(K),M_S(K))$;

$$(\!(V(t))\!)_L^2=\sum_{M=0}^{L-3}\sum_{l=0}^{1}\|\partial_t^l\vec{v}_M(t)\|_{L-M-l}^2+\|\bar{D}^2\vec{v}_{L-2}(t)\|^2.$$

In fact, applying (Ap. 7)-(Ap. 9) with $M_1=K-k-1$, $M_2=k$ and $N=1$ for $1\leqq k\leqq L-2$ and noting that $\partial_t^k A_S^{ij}(t)\in Y^{1,K-k-1}(J,\Omega)$; $\vec{v}_t^k B^{il}(t)\in Y^{1,K-k-(3/2)}(J,\Gamma)$; $\vec{v}_{L-1-k}(t)\in X^{1,k}(J,\Omega)$; $\partial_j\vec{v}_{L-2-k}(t)\in X^{1,k}(J,\Omega)$ in the definitions of $R_\Omega$ and $R_\Gamma$, we have (6.5). Furthermore, by (Ap. 7.b)-(Ap. 9.b) we have

(6.7) $\quad \|\partial_t R_\Omega(t)[V(t)]\|_0+\langle\!\langle\partial_t R_\Gamma(t)[V(t)]\rangle\!\rangle_{1/2}\leqq C(M_\infty(K),M_S(K))(\!(V(t))\!)_L$

for almost all $t\in J$. Since $V(0)=V'(0)$ as follows form (6.3.b) we have that $R_U(0)[V(0)]=R_U(0)[V'(0)]$ for $U=\Omega$ and $\Gamma$. Noting this, we see that

$$R_U(t)[V(t)]-R_U(t)[V'(t)]=\int_0^t\partial_s\{R_U(s)[V(s)-V'(s)]\}ds\quad\text{for } U=\Omega \text{ and } \Gamma,$$

where we have used the fact that $R_U(\cdot)$ is linear in $V$. Hence, applying (6.7) implies (6.6).

Now, let us define the iteration scheme. In view of (Ap. 18), there exists a $\vec{w}(t)\in X^{2,0}(\boldsymbol{R},\Omega)$ such that $\vec{w}(0)=\vec{u}_{L-2}$ and $\partial_t\vec{w}(0)=\vec{u}_{L-1}$. Let us define $V^0(t)$ by: $V^0(t)=(\vec{u}_0,\cdots,\vec{u}_{L-3},\vec{w}(t))$. Obviously, $V^0(t)\in Z$. For $k\geqq 1$ and $V^{k-1}(t)\in Z$, let us define $\vec{v}_{L-2}(t)\in X^{2,0}(J,\Omega)$ by a solution to the equations:

(6.8.a) $\quad P(t)[\vec{v}_{L-2}^k(t)]=\partial_t^{L-2}\vec{f}_\Omega(t)+R_\Omega(t)[V^{k-1}(t)]$ in $J\times\Omega$,

(6.8.b) $\quad Q(t)[\vec{v}_{L-2}^k(t)]=\partial_t^{L-2}\vec{f}_\Gamma(t)-R_\Gamma(t)[V^{k-1}(t)]$ on $J\times\Gamma$, (6.8.c) $\quad \vec{v}^k_{L-2}(0)=\vec{u}_{L-2},\quad \partial_t\vec{v}^k_{L-2}(0)=\vec{u}_{L-1} \qquad$ in $\Omega$.

In view of (6.4) and (6.5), by Theorem 5.1 we know the existence of $\vec{v}^k_{L-2}(t)\in X^{2,0}(J,\Omega)$. Let us define $\vec{v}^k_M(t)\in X^{1,L-M-1}(J,\Omega)$ $(0\leqq M\leqq L-3)$ by solutions to the equations:

(6.9.a)
$$\vec{v}^k_{M+2}(t)-P_M(t)[\vec{v}^k_0(t),\cdots,\vec{v}^k_{M+1}(0)]+\lambda_M\vec{v}_M(t)$$
$$=\partial_t^M\vec{f}_\Omega(t)+\lambda_M\Big(\vec{u}_M+\int_0^t\vec{v}^{k-1}_{M-1}(s)ds\Big)\quad\text{in } J\times\Omega,$$

(6.9.b)
$$Q_M(t)[\vec{v}^k_0(t),\cdots,\vec{v}^k_{M+1}(t)]=\partial_t^M\vec{f}_\Gamma(t)\quad\text{on } J\times\Gamma,$$

for $0\leqq M\leqq L-3$, where $\vec{v}^k_{L-1}(t)=\partial_t\vec{v}^k_{L-2}(t)$. Since $\vec{v}^k_{L-2}\in X^{2,0}(J,\Omega)\subset X^{1,1}(J,\Omega)$; $\vec{v}^k_{L-1}\in X^{1,0}(J,\Omega)$; $\partial_t^M\vec{f}_\Omega\in X^{L-2-M,0}(J,\Omega)\subset X^{1,L-M-3}(J,\Omega)$; $\partial_t^M\vec{f}_\Gamma\in X^{L-2-M,1/2}(J,\Gamma)$ $\subset X^{1,L-M-(5/2)}(J,\Gamma)$; $\vec{u}_M+\int_0^t\vec{v}^{k-1}_{M+1}(s)ds\in X^{1,L-M-3}(J,\Omega)$ (cf. Lemma 1.1), by Theorem 3.8 with $N_1=L-3$ and $N_2=L$, we see that $\vec{v}^k_M(t)$ exist in $X^{1,L-M-1}(J,\Omega)$ for $0\leqq M\leqq L-3$. Hence, if we put $V^k(t)=(\vec{v}^k_0(t),\cdots,\vec{v}^k_{L-2}(t))$, then we see that $V^k(t)\in Z$ and we can define an iteration scheme.

Now, we shall prove that the present sequence $\{V^k(t)\}$ is a Cauchy sequence in the product space $X^{1,L-1}(J,\Omega)\times\cdots\times X^{1,2}(J,\Omega)\times X^{2,0}(J,\Omega)$. Applying Thorrem 1.3 with $L=2$ and using (6.6), we have

(6.10)
$$|v^k_{L-2}-v^{k-1}_{L-2}|_{2,0,J}\leqq C\int_J(\!(V^{k-1}(s)-V^{k-2}(s))\!)^2_L ds\,.$$

Applying (3.61), we have also

$$\sum_{k=0}^{L-3}|\vec{v}^k_M-\vec{v}^{k-1}_M|^2_{1,L-M-1,J}\leqq C\Big\{|\vec{v}^k_{L-2}-\vec{v}^{k-1}_{L-2}|^2_{2,0,J}$$
$$+\sum_{M=0}^{L-3}\Big\{\int_J\|\vec{v}^{k-1}_{M+1}(t)-\vec{v}^{k-2}_{M+1}(t)\|^2_{L-M-2}dt+|\vec{v}^{k-1}_{M+1}-\vec{v}^{k-2}_{M+1}|_{0,L-M-3,J}\Big\}\Big\}.$$

Since $\vec{v}^{k-1}_{M+1}(0)=\vec{v}^{k-2}_{M+1}(0)=\vec{u}_M$, we have

$$|\vec{v}^{k-1}_{M+1}-\vec{v}^{k-2}_{M+1}|^2_{0,L-M-3,J}\leqq 2\sum_{l=0}^{1}\int_J\|\partial_t^l\vec{v}^{k-1}_{M+1}(t)-\partial_t^l\vec{v}^{k-2}_{M+1}(t)\|^2_{L-M-3}dt\,.$$

Combining these two estimates and using (6.10), we have

(6.11)
$$(\!(\!(V^k-V^{k-1})\!)\!)^2_{L,J}\leqq C\int_J(\!(V^{k-1}(t)-V^{k-2}(t))\!)^2_L dt$$

where $(\!(\!(V)\!)\!)_{L,J}=\sup\{(\!(V(t))\!)_L\,|\,t\in J\}$. Recall that $J=[0,T-\varepsilon]$. Repeated use of (6.11) implies that

$$(\!(\!(V^k-V^{k-1})\!)\!)^2_{L,L}\leqq[\{C(T-\varepsilon)\}^{k-1}/(k-1)!](\!(\!(V^1-V^0)\!)\!)^2_{L,J}\,.$$

From this we see that $\{V^k\}$ is a Cauchy sequence in $X^{1,L-1}(J,\Omega)\times\cdots\times X^{1,2}(J,\Omega)$

$\times X^{2,0}(J, \Omega)$. As a result, there exists a limit $V(t)=(\vec{v}_0(t), \cdots, \vec{v}_{L-2}(t))$ of the sequence $\{V^k(t)\}$. In particular, $\vec{v}_M(t) \in X^{1,L-M-1}(J, \Omega)$ $(0 \leqq M \leqq L-3)$, $\vec{v}_{L-2}(t) \in X^{2,0}(J, \Omega)$, and by (6.3.b) we have

$$\vec{v}_M(0)=\vec{u}_M \ (0 \leqq M \leqq L-2)\,;\ \partial_t \vec{v}_{L-2}(0)=\vec{u}_{L-1}\,. \tag{6.12}$$

Letting $k \to \infty$ in (6.8) and (6.9) and using (Ap. 1)-(Ap. 3), we see easily that $\vec{v}_0(t), \cdots, \vec{v}_{L-3}(t)$ and $\vec{v}_{L-2}(t)$ satisfy (6.2) and (6.1).

*2nd step.* Now, we shall prove that $\partial_t \vec{v}_M(t)=\vec{v}_{M+1}(t)$ for $0 \leqq M \leqq L-3$. Applying (Ap. 7)-(Ap. 9) with $M_1=K-k-2$, $M_2=L-M-2+k$ and $N=1$ $(0 \leqq k \leqq M)$, we have that $P_M(t)[\vec{v}_0(t), \cdots, \vec{v}_{M+1}(t)] \in X^{1,0}(J, \Omega)$ and $Q_M(t)[\vec{v}_0(t), \cdots, \vec{v}_{M+1}(t)] \in X^{1,1/2}(J, \Gamma)$ for $0 \leqq M \leqq L-3$. Differentiating (6.2) once in $t$, we have

$$\begin{aligned} (6.13.\mathrm{a})_M \quad & \partial_t \vec{v}_{M+2}(t)-P_M(t)[\partial_t \vec{v}_0(t), \cdots, \partial_t \vec{v}_{M+1}(t)]+\lambda_M \partial_t \vec{v}_M(t) \\ & =\partial_t^{M+1} \vec{f}_\Omega(t)+\lambda_M \vec{v}_{M+1}(t)+P'_M(t)[\vec{v}_0(t), \cdots, \vec{v}_{M+1}(t)] \quad \text{in } J \times \Omega\,, \\ (6.13.\mathrm{b})_M \quad & Q_M(t)[\partial_t \vec{v}_0(t), \cdots, \partial_t \vec{v}_{M+1}(t)] \\ & =\partial_t^{M+1} \vec{f}_\Omega(t)-Q'_M(t)[\vec{v}_0(t), \cdots, \vec{v}_{M+1}(t)] \qquad \text{on } J \times \Gamma \end{aligned}$$

for $0 \leqq M \leqq L-3$, where $P'_M(t)$ and $Q'_M(t)$ are the same as in (3.92). When $M=L-3$, noting that $\partial_t \vec{v}_{L-1}(t)=\partial_t^2 \vec{v}_{L-2}(t)$ and using the using the identity: $\binom{L-2}{k}-\binom{L-3}{k}=\binom{L-3}{k-1}$, from (6.1) and $(6.13)_{L-3}$, we have

$$\begin{aligned} (6.14.\mathrm{a})_{L-3} \quad & -P_{L-3}(t)[\vec{w}_0(t), \cdots, \vec{w}_{L-3}(t), 0]+\lambda_{L-3} \vec{w}_{L-3}(t)=0 \quad \text{in } J \times \Omega, \\ (6.14.\mathrm{b})_{L-3} \quad & Q_{L-3}(t)[\vec{w}_0(t), \cdots, \vec{w}_{L-3}(t), 0]=0 \qquad \text{on } J \times \Gamma, \end{aligned}$$

where we have put $\vec{w}_M(t)=\partial_t \vec{v}_M(t)-\vec{v}_{M+1}(t)$ $(0 \leqq M \leqq L-3)$. When $0 \leqq M \leqq L-4$, in the same way, from $(6.13)_M$ and $(6.2)_M$ we have

$$\begin{aligned} (6.14.\mathrm{a})_M \quad & \vec{w}_{M+2}(t)-P_M(t)[\vec{w}_0(t), \cdots, \vec{w}_{M+1}(t)]+\lambda_M \vec{w}_M(t) \\ & =\lambda_{M+1} \int_0^t \vec{w}_{M+1}(s)ds \quad \text{ih } J \times \Omega, \\ (6.14.\mathrm{b})_M \quad & Q_M(t)[\vec{w}_0(t), \cdots, \vec{w}_{M+1}(t)]=0 \qquad \text{on } J \times \Gamma, \end{aligned}$$

where $\vec{w}_{L-2}(t)=\partial_t \vec{v}_{L-2}(t)-\vec{v}_{L-1}(t)=0$. Applying (3.60) with $N_1=L-3$ and $N_2=L-1$ to (6.14) and noting that $\vec{w}_{L-1}(t)=\vec{w}_{L-2}(t)=0$, we have

$$\begin{aligned} \sum_{M=0}^{L-3} |\vec{w}_M|_{0,L-M-1,[0,t]} & \leqq C \sum_{M=0}^{L-4} \int_0^t \|\vec{w}_{M+1}(s)\|_{L-M-3} ds \\ & \leqq C \int_0^t \sum_{M=0}^{L-3} |\vec{w}_M|_{0,L-M-1,[0,s]} ds \end{aligned} \tag{6.15}$$

for any $t\in J$. Applying Gronwall's inequality to (6.15) implies that

$$\sum_{M=0}^{L-3}|\vec{w}_M|_{0,L-M-1,[0,t]}=0 \quad \text{for any } t\in J.$$

From this it follows immediately that $\partial_t\vec{v}_M(t)=\vec{v}_{M+1}(t)$ for all $t\in J$ and $0\leqq M\leqq L-3$. Put $\vec{u}(t)=\vec{v}_0(t)$. Then, $\partial_t^M\vec{u}(t)=\vec{v}_M(t)\in X^{0,L-M}(J, \Omega)$ $(0\leqq M\leqq L-3)$ and $\partial_t^{L-2}\vec{u}(t)=\vec{v}_{L-2}(t)\in X^{2,0}(J, \Omega)$. Accordingly, $\vec{u}(t)\in X^{L,0}(J, \Omega)$. Substituting $\partial_t^l\vec{u}(t)=\vec{v}_l(t)$ for $l=0$, 1 and 2 into $(6.2)_0$, we see that $P(t)[\vec{u}(t)]=\vec{f}_\Omega(t)$ in $J\times\Omega$ and $Q(t)[\vec{u}(t)]=\vec{f}_\Gamma(t)$ on $J\times\Gamma$. From (6.12) it follows that $\vec{u}(0)=\vec{v}_0(0)=\vec{u}_0$ and $\partial_t\vec{u}(0)=\vec{v}_1(0)=\vec{u}_1$ in $\Omega$. Noting that $\varepsilon$ is chosen arbitrarily, we have Theorem 1.2 when $3\leqq L\leqq K$. This completes the proof of Theorem 1.2.

## Appendix. Estimates of a product of functions and trace theorem.

First of all, we state the Sobolev's imbedding theorem. To do this, we prepare some notations. For $1\leqq p\leqq\infty$, we put

$$\|u\|_{R^n,r,p}=\left\{\int_{R^n}|\mathcal{F}^{-1}[(1+|\cdot|^2)^{r/2}\mathcal{F}(u)(\cdot)](x)|^p dx\right\}^{1/p};$$

$$H_p^r(\boldsymbol{R}^n)=\{u\in\mathcal{S}'(\boldsymbol{R}^n)\,|\,\|u\|_{R^n,r,p}<\infty\},$$

where $\mathcal{F}(u)$ is the Fourier transform of $u$ and $\mathcal{F}^{-1}$ is its inversion formula. Let $G=\boldsymbol{R}_+^n$ or $\Omega$. Put

$$H_p^r(G)=\{u\mid u(x)=U(x) \text{ in } G \text{ for some } U\in H_p^r(\boldsymbol{R}^n)\};$$

$$\|u\|_{G,r,p}=\inf\{\|U\|_{R^n,r,p}\mid u(x)=U(x) \text{ in } G\}.$$

As is well-known, if $r$ is an integer$\geqq 0$ and $1<p<\infty$, then $\|u\|_{G,r,p}^p$ is equivalent to the usual norm:

$$\sum_{|\alpha|\leqq r}\int_G|\partial_x^\alpha u(x)|^p dx \quad \text{for } G=\boldsymbol{R}^n,\ \boldsymbol{R}_+^n \text{ and } \Omega.$$

In fact, if $G=\boldsymbol{R}^n$, this is well-known (cf. [1, Theorem 7]). If $G=\boldsymbol{R}_+^n$ or $\Omega$, we can extend functions defined on $G$ to whole $\boldsymbol{R}^n$ (it is well-known that under the more general assumption on the boundary of the domain we can extend functions, cf. [1]). Thus, the equivalence of two norms follows immediately (cf. [1, Theorem 12]).

*Sobelev's imbedding theorem. Let* $G=\boldsymbol{R}^n$, $\boldsymbol{R}_+^n$ *or* $\Omega$. (I) *Let* $1<p<q<\infty$ *and put* $\lambda=n(1/p-1/q)$. *Then,* $H_p^\lambda(G)$ *is continuously imbedded into* $L^q(G)$ *and* $\|u\|_{G,0,q}\leqq C(p, q, n, G)\|u\|_{G,\lambda,p}$ *for any* $u\in H_p^\lambda(G)$.

(II) *Let* $\varepsilon_1$ *and* $\varepsilon_2$ *be numbers such that* $0<\varepsilon_1<\varepsilon_2<1$. *Put* $\lambda=n/p+\varepsilon_2$. *Then,* *every function $u$ in $H_p^\lambda(G)$ coincides almost everywhere with a Hölder continuous function $v$ with exponent $\varepsilon_1$. Furthermore,*

$$|v(x+h)-v(x)| \leqq C(p, \varepsilon_1, \varepsilon_2, G)\|u\|_{G,\lambda,p}|h|^{\varepsilon_1}.$$

In the same way as in the proof of Theorem 7.1 of Mizohata [7], by using Sobolev's imbedding theorem we have

THEOREM Ap. 1. *Let $1<p\leqq\infty$. Let $r_1, \cdots, r_k$ $(k\geqq 2)$, $M$ be non-negative numbers and $L$ a non-negtive integer such that $M>n/p$ and $M\geqq r_1+\cdots+r_k+L$. Then, for $u_j\in H_p^{M-r_j}(G)$, $j=1, \cdots, k$, a product $\prod u_j\in H_p^L(G)$. Furthermore,*

$$\|\prod_{j=1}^k u_j\|_{G,L,p} \leqq C(n, G, M, p, k)\prod_{j=1}^k \|u_j\|_{G,M-r_j,p}.$$

From now on, we consider $L^2$ spaces only. For the notational simplicity, we write $\|\cdot\|_{G,r,2}=\|\cdot\|_{G,r}$ and $H_2^r(G)=H^r(G)$. Next theorems are concerned with the trace operator.

THEOREM Ap. 2 (cf. Mizohata [7, Proposition 3.6]). *Let $u\in H^1(\boldsymbol{R}_+^n)$. Then, the following are true.*

(1) $\|u(\cdot, 0)\|_{\boldsymbol{R}_+^{n-1},1/2} \leqq C\|u\|_{\boldsymbol{R}^n,1}$.

(2) *For any arbitrary $\varepsilon>0$, there exists a constant $C(n, \varepsilon)$ satisfying*

$$\|u(\cdot, 0)\|_{\boldsymbol{R}^{n-1},0} \leqq \varepsilon\|u\|_{\boldsymbol{R}_+^n,1}+C(n, \varepsilon)\|u\|_{\boldsymbol{R}_+^n,0}.$$

THEOREM Ap. 3. *Let $L$ be a non-negative integer. For any $u\in H^{L+(1/2)}(\boldsymbol{R}^{n-1})$, there exists a $U\in H^{L+1}(\boldsymbol{R}_+^n)$ such that $U(x', 0)=u(x')$ for almost all $x'=(x_1, \cdots, x_{n-1})\in\boldsymbol{R}^{n-1}$ and $\|U\|_{\boldsymbol{R}_+^n,M+1} \leqq C(L)\|u\|_{\boldsymbol{R}^{n-1},M+(1/2)}$ for any integer $M\in[0, L]$.*

PROOF. In view of Theorem Ap. 2, since $C_0^\infty(\boldsymbol{R}^{n-1})$ is dense in $H^{L+(1/2)}(\boldsymbol{R}^{n-1})$, it suffices to prove the theorem for $u\in C_0^\infty(\boldsymbol{R}^{n-1})$. Put

$$U(x)=\left(\frac{1}{2\pi}\right)^{n-1}\int_{\boldsymbol{R}^{n-1}} \exp(ix'\cdot\xi'-x_n\sqrt{1+|\xi'|^2})\mathcal{F}(u)(\xi')d\xi'$$

$(i=\sqrt{-1}$ and $\xi'=(\xi_1, \cdots, \xi_{n-1}))$. Then, we see that $U(x', 0)=u(x')$. Let $\alpha=(\alpha', \alpha_n)=(\alpha_1, \cdots, \alpha_{n-1}, \alpha_n)$ be any multi-index such that $|\alpha|\leqq L+1$. Since

$$\partial_x^\alpha U(x)=\left(\frac{1}{2\pi}\right)^{n-1}\int_{\boldsymbol{R}^{n-1}} \exp(ix'\cdot\xi'-x_n\sqrt{1+|\xi'|^2})(-\sqrt{1+|\xi'|^2})^{\alpha_n}(\xi')^{\alpha'}\mathcal{F}(u)(\xi')d\xi',$$

by Parseval's formula we have

$$\|\partial_x^\alpha U\|^2_{R^n_+,0}=\int_0^\infty\int_{R^{n-1}}\exp(-x_n\sqrt{1+|\xi'|^2})(1+|\xi'|^2)^{\alpha_n}|\xi'|^{2|\alpha'|}|\mathcal{F}(u)(\xi')|^2 d\xi' dx_n$$

$$\leqq\int_{R^{n-1}}(1+|\xi'|^2)^{|\alpha|-(1/2)}|\mathcal{F}(u)(\xi')|^2 d\xi'\leqq\|u\|_{R^{n-1},|\alpha|-(1/2)}.$$

From this we have the theorem.

Using the partition of unity near the boundary, from Theorems Ap. 2 and Ap. 3 we have the following two corollaries.

COROLLARY Ap. 4. *Let* $u\in H^1(\Omega)$. *Then, the following are true.* (1) $\langle\!\langle u\rangle\!\rangle_{1/2}\leqq C\|u\|_1$. (2) *For any arbitrary* $\varepsilon>0$, *there exists a constant* $C(n, \varepsilon, \Gamma)$ *satisfying* $\langle\!\langle u\rangle\!\rangle_0\leqq\varepsilon\|u\|_1+C(n, \varepsilon, \Gamma)\|u\|_0$,

COROLLARY Ap. 5. *Let* $L$ *be a non-negative integer and* $u\in H^{L+(1/2)}(\Gamma)$. *Then, there exists a* $U\in H^{L+1}(\Omega)$ *such that* $u(x)=U(x)$ *for almost all* $x\in\Gamma$ *and* $\|U\|_{M+1}\leqq C\langle\!\langle u\rangle\!\rangle_{M+(1/2)}$ *for any integer* $M\in[0, L]$.

Now, we shall investigate the Hölder continuity of functions in $Y^{K-1,1}$.

THEOREM Ap. 6. *Let* $J$ *be a closed interval and* $u\in Y^{1,[n/2]}(J, \boldsymbol{R}^n)$. *Then, for any* $\varepsilon\in(0, [n/2]+1-(n/2))$, $u\in\mathcal{B}^\varepsilon(J\times\boldsymbol{R}^n)$. *Furthermore,* $|u|_{\infty,\varepsilon,J\times R^n}\leqq C|u|_{1,[n/2],J,R^n}$ *where* $C=C(n, \varepsilon)$.

PROOF. For the notational simplicity, we write $\|\cdot\|_{R^n,r}=\|\cdot\|_r$. Let us denote the Fourier transform of $u(t, x)$ with respect of $x$ by $\hat{u}(t, \xi)$. Let $\gamma>0$ and $\varepsilon\in(0, 1)$ and put $\gamma'=(\gamma-\varepsilon[n/2])/(1-\varepsilon)$. By Hölder's inequality we have

$$\int|\hat{u}(t, \xi)-\hat{u}(s, \xi)|^2(1+|\xi|^2)^\gamma d\xi$$

$$=\int\{|\hat{u}(t, \xi)-\hat{u}(s, \xi)|^2(1+|\xi|^2)^{[n/2]}\}^\varepsilon\{|\hat{u}(t, \xi)-\hat{u}(s, \xi)|^2(1+|\xi|^2)^{\gamma'}\}^{1-\varepsilon}d\xi$$

$$\leqq\Big(\int|\hat{u}(t, \xi)-\hat{u}(s, \xi)|^2(1+|\xi|^2)^{[n/2]}d\xi\Big)^\varepsilon\Big(\int|\hat{u}(t, \xi)-\hat{u}(s, \xi)|^2(1+|\xi|^2)^{\gamma'}d\xi\Big)^{1-\varepsilon}.$$

By the definitions of the norms of $H^{[n/2]}(\boldsymbol{R}^n)$ and $Y^{1,[n/2]}(J, \boldsymbol{R}^n)$, we have

$$\int|\hat{u}(t, \xi)-\hat{u}(s, \xi)|^l(1+|\xi|^2)^{[n/2]}d\xi\leqq\|u(t)-u(s)\|^2_{[n/2]}\leqq|u|^2_{1,[n/2],J,R^n}|t-s|^2.$$

On the other hand,

$$\int|\hat{u}(t, \xi)-\hat{u}(s, \xi)|^2(1+|\xi|)^{\gamma'}d\xi\leqq\|u(t)-u(s)\|^2_{\gamma'}\leqq(\|u(t)\|_{\gamma'}+\|u(s)\|_{\gamma'})^2.$$

Combining these estimates, we have

$$\|u(t)-u(s)\|_{\gamma}\leqq|u|_{1,[n/2],J,R^n}^{\varepsilon}|t-s|^{\varepsilon}(\|u(t)\|_{\gamma'}+\|u(s)\|_{\gamma'})^{1-\varepsilon}.$$

Choose $\gamma$ and $\varepsilon$ so that $\gamma>n/2$, $0<\varepsilon<1$ and $\gamma'\leqq[n/2]+1$. If $\varepsilon\in(0,[n/2]+1-(n/2))$, then such a $\gamma$ exists. Thus, by Sobolev's imbedding theorem we have

$$|u(t, x)-u(s, x)|\leqq C\|u(t)-u(s)\|_{\gamma}\leqq 2^{1-\varepsilon}|u|_{1,[n/2],J,R^n}^{\varepsilon}|t-s|^{\varepsilon}.$$

On the other hand, by the Fourier inversion formula we have

$$|u(s, x)-u(s, y)|=\left|\int(e^{ix\cdot\xi}-e^{iy\cdot\xi})\hat{u}(s, \xi)d\xi\right|.$$

Note that $|e^{ix\cdot\xi}-e^{iy\cdot\xi}|\leqq 2^{1-\varepsilon}|x-y|^{\varepsilon}|\xi|^{\varepsilon}$. In fact, $|e^{ix\cdot\xi}-e^{iy\cdot\xi}|\leqq|x-y||\xi|$. Noting that $|e^{ix\cdot\xi}|=|e^{iy\cdot\xi}|=1$, we have that $|e^{ix\cdot\xi}-e^{iy\cdot\xi}|\leqq|x-y|^{\varepsilon}|\xi|^{\varepsilon}2^{1-\varepsilon}$. Hence, we have

$$\begin{aligned}|u(s, x)-u(s, y)|&\leqq 2^{1-\varepsilon}|x-y|^{\varepsilon}\int|\xi|^{\varepsilon}|\hat{u}(s, \xi)|d\xi\\&\leqq 2^{1-\varepsilon}|x-y|^{\varepsilon}\left(\int(1+|\xi|^2)^{-\gamma}d\xi\right)^{1/2}\left(\int(1+|\xi|^2)^{\gamma+\varepsilon}|\hat{u}(s, \xi)|^2d\xi\right)^{1/2}\\&=C|x-y|^{\varepsilon}\|u(s)\|_{\gamma+\varepsilon},\end{aligned}$$

where $C=C(n, \varepsilon, \gamma)$ provided that $\gamma>n/2$ and $\gamma+\varepsilon\leqq[n/2]+1$. Since $\varepsilon\in(0,[n/2]+1-(n/2))$, we can choose such a $\gamma$. Combining these two estimates, we see easily that

$$\begin{aligned}&|u(t, x)-u(s, y)|\leqq|u(t, x)-u(s, x)|+|u(s, x)-u(s, y)|\\&\leqq C(\varepsilon)|u|_{1,[n/2],J,R^n}(|t-s|^{\varepsilon}+|x-y|^{\varepsilon})\leqq C(\varepsilon)|u|_{1,[n/2],J,R^n}|(t, x)-(s, y)|^{\varepsilon},\end{aligned}$$

which implies the theorem.

COROLLARY Ap. 7. *Let* $\varepsilon\in(0,[n/2]+1-(n/2))$. *If* $v\in Y^{K-1,1}(J, \Omega)$, *then* $u\in\mathcal{B}^{1+\varepsilon}(J\times\bar{\Omega})$. *Furthermore*, $|v|_{\infty,1+\varepsilon,J}\leqq C(n, \varepsilon)|v|_{K-1,1,J}$.

PROOF. Since $K\geqq[u/2]+2$, $K\geqq 3$. Then, $Y^{K-2,1}(J, \Omega)\subset Y^{1,K-2}(J, \Omega)\subset Y^{1,[n/2]}(J, \Omega)$. Namely, $\partial_l v\in Y^{1,[n/2]}(J, \Omega)$ for $l=0, 1, \cdots, n$ ($\partial_0=\partial_t$). By using well-known Lions' method of extending functions defined on $\Omega$ to whole $R^n$, we see that there exists a $u(t, x)\in Y^{1,[n/2]}(J, R^n)$ such that $u(t, x)=\partial_l v(t, x)$ for $x\in\Omega$ and $t\in J$, and $|u|_{1,[n/2],J,R^n}\leqq C|\partial_l v|_{1,[n/2],J}$. Applying Theorem Ap. 6 implies that $\partial_l v(t, x)\in\mathcal{B}^{\varepsilon}(J\times\bar{\Omega})$. Furthermore, we have that $|\partial_l v|_{\infty,\varepsilon,J}=|u|_{\infty,\varepsilon,J}\leqq|u|_{\infty,\varepsilon,J\times R^n}\leqq C|u|_{1,[n/2],J,R^n}\leqq C|\partial_l v|_{1,[n/2],J}\leqq C|v|_{K-1,1,J}$, which completes the proof.

Combining Corollaries Ap. 5 and Ap. 7, we have

COROLLARY Ap. 8. *Let* $\varepsilon\in(0, [n/2]+1-(n/2))$. *If* $v\in Y^{K-1,1/2}(J, \Omega)$, *then* $v\in\mathscr{B}^{1+\delta}(I\times\Gamma)$. *Furthermore*, $\langle v\rangle_{\infty,1+\varepsilon,J}\leqq C(n, \varepsilon)\langle v\rangle_{K-1,1/2,J}$.

Now, we shall summarize the results on products of two functions in Sobolev spaces used in the text. Let $G=\boldsymbol{R}^n$, $\boldsymbol{R}_+^n$ or $\Omega$.

(Ap. 1) $\|A\cdot B\|_{G,\gamma}\leqq C\|A\|_{G,\alpha}\|B\|_{G,\beta}$ for any $A\in H^{\alpha}(G)$ and $B\in H^{\beta}(G)$

provided that $\alpha$, $\beta$, $\gamma$ are integers such that $\alpha$, $\beta\geqq\gamma\geqq0$ and $\alpha+\beta-\gamma>n/2$.

Let $G'=\boldsymbol{R}_+^n$ or $\Omega$. For the notational simplicity, we write

$$\|\cdot\|_{\partial G',\gamma-(1/2)}=\|\cdot\|_{\boldsymbol{R}^{n-1},\gamma-(1/2)} \quad\text{or}\quad \langle\!\langle\cdot\rangle\!\rangle_{\gamma-(1/2)}\,;\ \partial G'=\boldsymbol{R}^{n-1} \text{ or } \Gamma.$$

(Ap. 2) $\|A\cdot B\|_{\partial G',\gamma-(1/2)}\leqq C\|A\|_{G',\alpha}\|B\|_{G',\beta}$ for any $A\in H^{\alpha}(G')$ and $B\in H^{\beta}(G')$

provided that $\alpha$, $\beta$, $\gamma$ are integers such that $\alpha$, $\beta\geqq\gamma\geqq1$ and $\alpha+\beta-\gamma>n/2$.

(Ap. 3) $$\|A\cdot B\|_{\partial G',\gamma-(1/2)}\leqq C\|A\|_{\partial G',\alpha-(1/2)}\|B\|_{G',\beta}$$

for any $A\in H^{\alpha-(1/2)}(\partial G')$ and $B\in H^{\beta}(G')$ provided that $\alpha$, $\beta$, $\gamma$ are integers such that $\alpha$, $\beta\geqq\gamma\geqq1$ and $\alpha+\beta-\gamma>n/2$. In fact, (Ap. 1) follows immediately from Theorem Ap. 1 with $k=2$, $L=\gamma$, $M=\alpha+\beta+\gamma$, $r_1=\beta-\gamma$ and $r_2=\alpha-\gamma$. By Corollary Ap. 4-(1), we know that $\|A\cdot B\|_{\partial G',\gamma-(1/2)}\leqq C\|A\cdot B\|_{G',\gamma}$. Hence, (Ap. 2) follows from (Ap. 1). By Corollary Ap. 5, we know that there exists an $A'\in H^{\gamma}(G')$ such that $A'=A$ almost everywhere on $\partial G'$ and $\|A'\|_{G,\gamma}\leqq C\|A\|_{\partial G',\gamma-(1/2)}$. Since $\|A\cdot B\|_{\partial G',\gamma-(1/2)}=\|A'\cdot B\|_{\partial G',\gamma-(1/2)}$, (Ap. 3) follows from (Ap. 2).

Now, when $A=A(t)$ and $B=B(t)$ depend on $t$ continuously, we give the results corresponding to (Ap. 1)-(Ap. 3). Below, $J$ always refers to a time interval.

(Ap. 4.a) $$A(t)\cdot B(t)\in C^0(J, H^{\gamma}(G))\,;$$

(Ap. 4.b) $$\|A(t)\cdot B(t)\|_{G,\gamma}\leqq C\|A(t)\|_{G,\alpha}\|B(t)\|_{G,\beta}$$

for any $A(t)\in C^0(J, H^{\alpha}(G))$ and $B(t)\in C^0(J, H^{\beta}(G))$ provided that $\alpha$, $\beta$, $\gamma$ are integers such that $\alpha$, $\beta\geqq\gamma\geqq0$ and $\alpha+\beta-\gamma>n/2$.

(Ap. 5.a) $$A(t)\cdot B(t)\in C^0(J, H^{\gamma-(1/2)}(\partial G'))\,;$$

(Ap. 5.b) $$\|A(t)\cdot B(t)\|_{\partial G',\gamma-(1/2)}\leqq C\|A(t)\|_{G',\alpha}\|B(t)\|_{G',\beta}$$

for any $A(t)\in C^0(J, H^{\alpha}(G'))$ and $B(t)\in C^0(J, H^{\beta}(G'))$ provided that $\alpha$, $\beta$, $\gamma$ are integers such that $\alpha$, $\beta\geqq\gamma\geqq1$ and $\alpha+\beta-\gamma>n/2$.

(Ap. 6.a) $$A(t)\cdot B(t)\in C^0(J, H^{\gamma-(1/2)}(\partial G'))\,;$$

(Ap. 6.b) $$\|A(t)\cdot B(t)\|_{\partial G',\gamma-(1/2)}\leqq C\|A(t)\|_{\partial G',\alpha-(1/2)}\|B(t)\|_{G',\beta}$$

for any $A(t)\in C^0(J, H^{\alpha-(1/2)}(\partial G'))$ and $B(t)\in C^0(J, H^{\beta}(G))$ provided that $\alpha, \beta, \gamma$ are integers such that $\alpha, \beta\geqq\gamma\geqq 1$ and $\alpha+\beta-\gamma>n/2$. In fact, by (Ap. 1) we see that

$$\|A(t)\cdot B(t)-A(s)\cdot B(s)\|_{G,\gamma}\leqq$$
$$C\{\|A(t)-A(s)\|\|_{G,\alpha}\|B(t)\|_{G,\beta}+\|A(s)\|_{G,\alpha}\|B(t)-B(s)\|_{G,\beta}\}.$$

From this, (Ap. 4) follows immediately. Employing the same arguments, we see that (Ap. 5) and (Ap. 6) follow from (Ap. 2) and (Ap. 3), respectively,

Now, we give the results on differentiability in $t$. Let $M_1$, $M_2$ and $N$ be integers such that $M_1, M_2\geqq N$ and $M_1+M_2+1-N>n/2$. Let $Z^{L,M}=X^{L,M}$ or $Y^{L,M}$. Then,

(Ap. 7.a) $\quad A(t)\cdot B(t)\in Z^{1,N}(J, \Omega)$;

(Ap. 7.a) $\quad \|\partial_t(A(t)\cdot B(t))\|_N\leqq C\Big(\sum_{l=0}^{1}\|\partial_t^l A(t)\|_{M_1+1-l}\Big)\Big(\sum_{l=0}^{1}\|\partial_t^l B(t)\|_{M_2+1-l}\Big)$

for any $A(t)\in Z^{1,M_1}(J, \Omega)$ and $B(t)\in Z^{1,M_2}(J, \Omega)$.

(Ap. 8.a) $\quad A(t)\cdot B(t)\in Z^{1,N-(1/2)}(J, \Gamma)$;

(Ap. 8.b) $\quad \langle\!\langle\partial_t(A(t)\cdot B(t))\rangle\!\rangle_{N-(1/2)}\leqq C\Big(\sum_{l=0}^{1}\|\partial_t^l A(t)\|_{M_1+1-l}\Big)\Big(\sum_{l=0}^{1}\|\partial_t^l B(t)\|_{M_2+1-l}\Big)$

for any $A(t)\in Z^{1,M_1}(J, \Omega)$ and $B(t)\in Z^{1,M_2}(J, \Omega)$.

(Ap. 9.a) $\quad A(t)\cdot B(t)\in Z^{1,N-(1/2)}(J, \Gamma)$;

(Ap. 9.b) $\quad \langle\!\langle\partial_t(A(t)\cdot B(t))\rangle\!\rangle_{N-(1/2)}\leqq C\Big(\sum_{l=0}^{1}\langle\!\langle\partial_t^l A(t)\rangle\!\rangle_{M_1+(1/2)-l}\Big)\Big(\sum_{l=0}^{1}\|\partial_t^l B(t)\|_{M_2+1-l}\Big)$

for any $A(t)\in Z^{1,M_1-(1/2)}(J, \Gamma)$ and $B(t)\in Z^{1,M_2}(J, \Omega)$. In fact, since

$$\|A(t)\cdot B(t)-A(s)\cdot B(s)\|_N\leqq\|(A(t)-A(s))\cdot B(t)\|_N+\|A(s)\cdot(B(t)-B(s))\|_N$$

applying (Ap. 1) with $\alpha=M_1$, $\beta=M_2+1$ and $\gamma=N$ to the first term of the right-hand side and with $\alpha=M_1+1$, $\beta=M_2$ and $\gamma=N$ to the second term of the right-hand side, we have

$$\|A(t)\cdot B(t)-A(s)\cdot B(s)\|_N\leqq C\{\|A(t)-A(s)\|_{M_1}\|B(t)\|_{M_2+1}$$
$$+\|A(s)\|_{M_1+1}\|B(t)-B(s)\|_{M_2}\}.$$

From this it follows that $A(t)B(t)\in\mathrm{Lip}(J, H^N(\Omega))\cap X^0(J, H^N(\Omega))$. Since $\partial_t(A(t)\cdot B(t))=\partial_t A(t)\cdot B(t)+A(t)\cdot\partial_t B(t)$, by employing the same arguments, we have

$$\|\partial_t(A(t)\cdot B(t))\|_N\leqq C\{\|\partial_t A(t)\|_{M_1}\|B(t)\|_{M_2+1}+\|A(t)\|_{M_1+1}\|B(t)\|_{M_2}\}.$$

Applying (Ap. 1) with $\alpha=M_1+1$, $\beta=M_2+1$ and $\gamma=N+1$, we have also

$$\|A(t)\cdot B(t)\|_{N+1}\leqq C\{\|A(t)\|_{M_1+1}\|B(t)\|_{M_2+1}\},$$

which implites that $A(t)B(t)\in L^\infty(J, H^{N+1}(\Omega))$. Therefore, we have proved $A(t)\cdot B(t)\in Y^{1,N}(J, \Omega)$. Furtheremore, we have

$$\begin{aligned}
&\|(A(t+h)\cdot B(t+h)-A(t)\cdot B(t))h^{-1}-\partial_t(A(t)\cdot B(t))\|_N\\
&\leqq\|\partial_t A(t)\cdot(B(t+h)-B(t))\|_N+\|\{(A(t+h)-A(t))h^{-1}-\partial_t A(t)\}B(t+h)\|_N\\
&\quad+\|A(t)\{(B(t+h)-B(t))h^{-1}-\partial_t B(t)\}\|_N\\
&\leqq C\{\|\partial_t A(t)\|_{M_1}\|B(t+h)-B(t)\|_{M_2+1}+\|(A(t+h)-A(t))h^{-1}-\partial_t A(t)\|_{M_1}\|B(t+h)\|_{M_2+1}\\
&\quad+\|A(t)\|_{M_1+1}\|(B(t+h)-B(t))h^{-1}-\partial_t B(t)\|_{M_2}\}\,.
\end{aligned}$$

From this we see easily that $A(t)\cdot B(t)\in X^{1,N}(J, \Omega)$. Hence, we have (Ap. 7). With the help of Corollaries Ap. 4-(1) and (Ap. 5), we have also (Ap. 8) and (Ap. 9) by the same arguments.

In the text, we need the following facts:

(Ap. 10.a) $P(t)[\vec{u}(t)]\in C^{L-2}(J, L^2(\Omega))$ and $\partial_t^{L-2}(P(t)[\vec{u}(t)])\in \mathrm{Lip}(J, L^2(\Omega))$;

(Ap. 10.b) $Q(t)[\vec{u}(t)]\in C^{L-2}(J, H^{1/2}(\Gamma))$ and $\partial_t^{L-2}(Q(t)[\vec{u}(t)])\in \mathrm{Lip}(J, H^{1/2}(\Gamma))$

provided that $\vec{u}(t)\in C^\infty(J, H^{1/2}(\Omega))$ and $2\leqq L\leqq K$, where $J\subset I$. (Ap. 10) follows immediately from the following facts:

(Ap. 11) $A(t)\cdot B(t)\in C^{L-2}(J, H^1(\Omega))$ and $\partial_t^{L-2}(A(t)\cdot B(t))\in \mathrm{Lip}((J, H^1(\Omega))$;

(Ap. 12) $A(t)\cdot B(t)\in C^{L-2}(J, H^{1/2}(\Gamma))$ and $\partial_t^{L-2}(A(t)\cdot B(t))\in \mathrm{Lip}(J, H^{1/2}(\Gamma))$

provided that $A(t)\in Y^{K-1,1}(J, \Omega)$ and $B(t)\in C^\infty(J, H^{L-1}(\Omega))$.

(Ap. 13) $A(t)\cdot B(t)\in C^{L-2}(J, H^{1/2}(\Gamma))$ and $\partial_t^{L-2}(A(t)\cdot B(t))\in \mathrm{Lip}(J, H^{1/2}(\Gamma))$

provided that $A(t)\in Y^{K-1,1/2}(J, \Gamma)$ and $B(t)\in C^\infty(J, H^{L-1}(\Omega))$. By induction on $L\in[2, K]$ and using (Ap. 1) and (Ap. 4) we see easily (Ap. 11). With the help of Corollary Ap. 4-(1), (Ap. 12) follows from (Ap. 11). With the help of Corollary Ap. 5, (Ap. 13) follows also from (Ap. 11).

In the text, we also need the following facts:

(Ap, 14) $P(t)[\vec{u}(t)]\in X^{L-2,0}(J, \Omega)$ and $Q(t)[\vec{u}(t)]\in X^{L-2,1/2}(J, \Gamma)$

provided that $\vec{u}(t)\in X^{L-2,2}(J, \Omega)$ for $2\leqq L\leqq K$, where $J\subset I$. (Ap. 14) follows immediately from the following facts:

(Ap. 15) $A(t)\cdot B(t)\in X^{L-2,1}(J, \Omega)$;

(Ap. 16) $$A(t)\cdot B(t)\in X^{L-2,1/2}(J, \Gamma)$$

provided that $A(t)\in X^{K-2,1}(J, \Omega)$ and $B(t)\in X^{L-2,1}(J, \Omega)$.

(Ap. 17) $$A(t)\cdot B(t)\in X^{L-2,1/2}(J, \Gamma)$$

provided that $A(t)\in X^{K-2,1/2}(J, \Gamma)$ and $B(t)\in X^{L-2,1}(J, \Omega)$. By induction on $L\in [2, K]$ and using (Ap. 4), we have (Ap. 15) easily. With the help of Corollaries Ap. 4-(1) and Ap. 5, (Ap. 16) and (Ap. 17) follows from (Ap. 15) immediately.

Finally, we shall prove that for any $w_0\in H^2(\Omega)$ and $w_1\in H^1(\Omega)$ (scalar-valued functions now being considered), there exists a $w(t, x)\in X^{2,0}(\boldsymbol{R}, \boldsymbol{R}^n)$ such that

(Ap. 18) $$w(0, x)=w_0(x) \quad \text{and} \quad \partial_t w(0, x)=w_1(x) \quad \text{in } \Omega .$$

By well-known Lions' method of extending functions defined on $\Omega$ to whole $\boldsymbol{R}^n$, we know that there exist $W_k(x)\in H^{2-k}(\boldsymbol{R}^n)$ for $k=0$ and 1 such that $w_k(x)=W_k(x)$ for $x\in\Omega$. Then, let us define $w(t, x)\in X^{2,0}(\boldsymbol{R}, \boldsymbol{R}^n)$ by a solution to the Cauchy problem of the wave operator:

$$\partial_t^2 w(t, x)-\sum_{j=1}^{n}\partial_j^2 w(t, x)=0 \text{ in } \boldsymbol{R}\times\boldsymbol{R}^n ;\ w(0, x)=W_0(x)$$
$$\text{and} \quad \partial_t w(0, x)=W_1(x) \quad \text{in } \boldsymbol{R}^n .$$

Obviously, the $w(t, x)$ has the desired properties.

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