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ON THE CAUCHY PROBLEM FOR THE NONLINEAR KLEIN-GORDON EQUATION WITH A CUBIC CONVOLUTION

By

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Abstract. We study the Cauchy problem for the nonlinear Klein-Gordon equation with a cubic convolution $\{V_{\gamma}*(w(t))^{2}\}w(t)$, where $V_{\gamma}(x)=|x|^{-\gamma}$, in $(x, t)\in \boldsymbol{R}^{n}\times \boldsymbol{R}$. We prove the existence of weak solutions for $0<\gamma<n$. We also prove that for $0<\gamma<\mathrm{Min}\{4, n\}$ the weak solution is unique and there exists a regular solution.

Key Words. nonlinear Klein-Gordon equation, cubic convolution, Cauchy problem, global solution, uniqueness.

1. Introduction and Results.

We consider the Cauchy problem for the nonlinear Klein-Gordon equation;

$$\begin{cases} \partial_{t}^{2}w(t)-\Delta w(t)+w(t)+F(w(t))=0 \\ w(0)=\phi(x), \qquad \partial_{t}w(0)=\psi(x) \end{cases} \tag{1.1}$$

in $(x, t)\in \boldsymbol{R}^{n}\times \boldsymbol{R}$. Here $w(t)$ is a real valued function and

$$F(w(t))=\{V_{\gamma}*f(w(t))\}w(t), \tag{1.2}$$

where $f(w)=w^{2}$, $V_{\gamma}(x)=|x|^{-\gamma}$ $(0<\gamma<n)$ and $*$ denotes the spatial convolution. The study of this equation was begun in Strauss [13] and Menzala and Strauss [9]. In [9] they proved the existence of a global regular solution of (1.1) for $0<\gamma\leqq 3$. The main purpose of the present paper is to prove the same result for $0<\gamma<\mathrm{Min}\{4, n\}$. The upper bound $\mathrm{Min}\{4, n\}$ of γ has been already appeared in the case of nonlinear Schrödinger equation with the same nonlinear term. The case of Schrödinger equation has been studied by Chadam and Glassey [2], Glassey [6], Ginibre and Velo [4] and Hayashi and Tsutsumi [7]. It seems that $\mathrm{Min}\{4, n\}$ is a critical value caused by the Sobolev embedding theorem.

In order to state our results, we give the main notations used in this paper. We denote by $\|\cdot\|_{p}$ the norm in $L_{p}=L_{p}(\boldsymbol{R}^{n})$. Let $H_{p}^{s}=H_{p}^{s}(\boldsymbol{R}^{n})$ with $s\in \boldsymbol{R}$ and

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$1 \leqq p < \infty$ (especially $H^s = H^s(\boldsymbol{R}^n)$ for $p=2$) be the Sobolev spaces which are the completion of $C_0^\infty(\boldsymbol{R}^n)$ with norms

$$\|u\|_{s,p} = \|\mathcal{F}^{-1}((1+|\xi|^2)^{s/2}\hat{u}(\xi))\|_p .$$

Here $\hat{}$ denotes the Fourier transformation and $\mathcal{F}^{-1}$ is its inverse. For any interval $I \subset \boldsymbol{R}$ and any Banach space B, we denote by $C^k(I; B)$ the space of B-valued C^k-functions over I, and by $C_w(I; B)$ the space of weakly continuous functions from I to B, and by $C_L(I; B)$ the space of functions from I to B that are strongly Lipschitz continuous. We denote by $C^k(I; \mathcal{D}')$ the space of $\mathcal{D}'$-valued functions $u(t)$ such that $\langle u(t), v\rangle$ is in $C^k(I)$ for any $v \in \mathcal{D}$.

We shall use the operator $\zeta(H)$ for suitable functions $\zeta(\cdot)$ as follows:

$$\zeta(H)u = \mathcal{F}^{-1}(\zeta(\langle\xi\rangle)\hat{u}(\xi)) \qquad \text{in } \mathcal{S}' .$$

where $\langle\xi\rangle = (1+|\xi|^2)^{1/2}$ and $\mathcal{S}'$ means the tempered distribution.

Now we are ready to state our results.

THEOREM 1. *Let $0<\gamma<n$ $(n \geqq 1)$. Assume that $(\phi, \psi) \in H^1 \cap L_{4n/(2n-\gamma)} \times L_2$. Then there exists a weak solution $w(t)$ of* (1.1) *which satisfies the following:*

$$w(t) \in L_\infty(\boldsymbol{R}; H^1) \cap C_w(\boldsymbol{R}; H^1) \cap C_L(\boldsymbol{R}; L_2) \cap C^2(\boldsymbol{R}; \mathcal{D}'), \tag{1.3}$$

$$F(w(t)) \in L_\infty(\boldsymbol{R}; L_{2n/(n+\gamma)}) \cap C(\boldsymbol{R}; \mathcal{D}') \tag{1.4}$$

$$\begin{aligned}(w(t), v) = {} & (\phi, \cos\{Ht\}v) + (\psi, H^{-1}\sin\{Ht\}v) \\ & -\int_0^t (F(w(\tau)), H^{-1}\sin\{H(t-\tau)\}v)d\tau ,\end{aligned} \tag{1.5}$$

$$\begin{cases} \dfrac{d^2}{dt^2}(w(t), v) + (w(t), (-\Delta+1)v) + (F(w(t), v) = 0 \\ (w(0), v) = (\phi, v), \ \dfrac{d}{dt}(w(0), v) = (\psi, v). \end{cases} \tag{1.6}$$

Here $v \in C_0^\infty(\boldsymbol{R}^n)$ and $(\ ,\ )$ is L_2-inner product. And we have the energy inequality

$$E(w(t), \partial_t w(t)) \leqq E(\phi, \phi) \qquad \textit{for} \quad t \in \boldsymbol{R}. \tag{1.7}$$

where

$$E(\phi, \psi) = \frac{1}{2}\|\psi\|_2^2 + \frac{1}{2}\|\phi\|_{1,2}^2 + \frac{1}{4}V_{(n+\gamma)/2} * f(\phi)\|_2^2 . \tag{1.8}$$

THEOREM 2. *Let $0<\gamma<\mathrm{Min}\{4, n\}$ $(n \geqq 1)$ and $(\phi, \psi) \in H^1 \times L_2$. Let I be an open interval in $\boldsymbol{R}$ and $0 \in I$. Then there exists at most one $w(t)$ which satisfies* (1.5) *and*

$$w(t) \in L_\infty^{\mathrm{loc}}(I; H^1) \qquad \textit{for} \quad 0<\gamma \leqq 3, \tag{1.9}$$

(1.10) $$w(t)\in L_{\infty}^{\mathrm{loc}}(I;H^{1})\cap L_{r}^{\mathrm{loc}}(I;L_{p'}) \quad \textit{for} \quad 3<\gamma<4,$$

wher $1/p'=1/2-(\gamma-1)/2n$ *and* $1/r=(\gamma-3)/2$.

THEOREM 3. *Let* $0<\gamma<\mathrm{Min}\{4, n\}$ $(n\geqq 1)$.

(i) *Let* $(\phi, \psi)\in H^{1}\times L_{2}$. *Then* $w(t)$ *which is obtained by Theorem* 1 *is unique and satisfies the following:*

(1.11) $$w(t)\in C(\boldsymbol{R};H^{1})\cap C^{1}(\boldsymbol{R};L_{2}) \quad \textit{for} \quad 0<\gamma\leqq 3,$$

(1.12) $$w(t)\in C(\boldsymbol{R};H^{1})\cap C^{1}(\boldsymbol{R};L_{2})\cap L_{r}^{\mathrm{loc}}(\boldsymbol{R};L_{p'}) \quad \textit{for} \quad 3<\gamma<4,$$

(1.13) $$E(w(t), \partial_{t}w(t))=E(\phi, \psi) \quad \textit{for} \quad t\in\boldsymbol{R},$$

where r *and* p' *are given in Theorem* 2.

(ii) *Let* $(\phi, \psi)\in H^{k}\times H^{k-1}$ $(k\in\boldsymbol{N}$ *(natural number) and* $k\geqq 2)$. *Then* (1.1) *has a unique solution* $w(t)$ *which satisfies*

(1.14) $$w(t)\in \bigcap_{i=0}^{k} C^{i}(\boldsymbol{R};H^{k-i}).$$

COROLLARY. (i) *If* $k>n/2+2$, $w(t)$ *is in* $C^{2}(\boldsymbol{R}^{n}\times\boldsymbol{R})$.

(ii) *If* $k=\infty$, $w(t)$ *is in* $C^{\infty}(\boldsymbol{R}^{n}\times\boldsymbol{R})$.

REMARK. (i) If $1<\gamma<\mathrm{Min}\{4, n\}$, we have $H^{1}\hookrightarrow L_{4n/(2n-\gamma)}$ by the Sobolev embedding theorem. So the initial condition $\phi\in H^{1}\cap L_{4n/(2n-\gamma)}$ becomes $\phi\in H^{1}$ in Theorem 2 and 3.

(ii) The upper bound $\mathrm{Min}\{4, n\}$ of γ has been already appeared in the case of the nonlinear Schrödinger equation. (See [4] and [7].)

Theorem 1 is proved by the compactness method which were used by Segal in [12]. He used this method for the nonlinear Klein-Gordon equation with the power nonlinearity. (See also Reed [11] 5.) We can choose a convergent subsequence from solutions of the equation which approximate (1.1) by the double convolution mollifier due to Ginibre and Velo [3].

In the case $0<\gamma\leqq 3$ the same results of Theorem 2 and 3 have been already proved by [9]. Thus, we shall prove Theorem 2 and 3 in the case $3<\gamma<4$.

Theorem 2 is proved by the contraction method.

In order to prove Theorem 3, we show that a weak solution obtained by Theorem 1 becomes a regular solution. For this purpose we estimate the solutions of the approximating equation used for the proof of Theorem 1. This method has been already used by Ginibre and Velo [5] and Motai [10] in the case where $F(w)$ is the power nonlinearity.

2. Proof of Theorem 1.

First we approximate the nonlinear term by the double convolution mollifier due to Ginibre and Velo [3]. We choose an even non-negative function $h\in C_0^\infty(\boldsymbol{R}^n)$ such that $\|h\|_1=1$. For any $j\in\boldsymbol{N}$(natural number) we put

$$F_j(u)=h_j*\{V_\gamma*f(h_j*u)h_j*u\}, \tag{2.1}$$

where $h_j(x)=j^n h(jx)$. Coresponding to (2.1), we consider the Cauchy problem;

$$\begin{cases} \partial_t^2 w_j(t)-\Delta w_j(t)+w_j(t)+F_j(w_j(t))=0 \\ w_j(0)=h_j*\phi, \qquad \partial_t w_j(0)=h_j*\psi. \end{cases} \tag{2.2}$$

LEMMA 2.1. *Let* $0<\gamma<n$ $(n\geqq 1)$. *Assume that* $(\phi,\psi)\in H^1\cap L_{4n/(2n-\gamma)}\times L_2$. *Then for all* $j\in\boldsymbol{N}$ (2.2) *has a unique solution* $w_j(t)$ *such that*

$$w_j(t)\in\bigcap_{i=0}^{k}C^i(\boldsymbol{R};H^{k-i}) \qquad \textit{for any} \quad k\in\boldsymbol{N}. \tag{2.3}$$

And $w_j(t)$ *satisfies the integral equation in* H^k;

$$w_j(t)=w_j^0(t)-\int_0^t H^{-1}\sin\{H(t-\tau)\}F_j(w_j(\tau))d\tau, \tag{2.4}$$

where

$$w_j^0(t)=\cos\{Ht\}h_j*\phi+H^{-1}\sin\{Ht\}h_j*\psi. \tag{2.5}$$

In addition the conservation of energy holds;

$$E_j(w_j(t),\partial_t w_j(t))=E_j(h_j*\phi,h_j*\psi) \qquad \textit{for} \quad t\in\boldsymbol{R}, \tag{2.6}$$

where

$$E_j(\phi,\psi)=\frac{1}{2}\|\psi\|_2^2+\frac{1}{2}\|\phi\|_{1,2}^2+\frac{1}{4}\|V_{(n+\gamma)/2}*f(h_j*\phi)\|_2^2. \tag{2.7}$$

PROOF. Applying Reed [11] Theorem 2 in section 1 to (2.2), we can show the existence of a unique global solution. Employing the same arguments as in Ginibre and Velo [3] Proposition 3.3, we can also prove (2.6). □

We obtain the following lemma by the compactness method.

LEMMA2.2. *Let* $w_j(t)$ $(j\in\boldsymbol{N})$ *be a solution of* (2.2) *obtained by Lemma* 2.1. *Then* $\{w_j(t)\}$ *has a convergent subsequence (again denoted by* $\{w_j(t)\}$*) as follows: For any compact interval* $I\subset\boldsymbol{R}$ *and any comsact subset* $K\subset\boldsymbol{R}^n$

$$w_j(t)\longrightarrow w(t) \qquad \textit{in} \quad C(I;L_2(K)) \qquad \textit{as } j\to\infty. \tag{2.8}$$

Here $w(t)$ *satisfies*

(2.9) $$w(t)\in L_\infty(\boldsymbol{R};\, H^1)\cap C_w(\boldsymbol{R};\, H^1)\cap C_L(\boldsymbol{R};\, L_2).$$

PROOF. Noting (2.6), the Ascoli-Arzela theorem yields (2.8) and (2.9). For details please refer to Segal [12] and Reed [11] 5. □

The following lemma is the well-known Sobolev's inequality.

LEMMA 2.3. *Let* $1<q<p<\infty$ *and* $0<\gamma<n$ $(n\geqq 1)$. *Then we have*

(2.10) $$\|V_\gamma * u\|_p \leqq C\|u\|_q$$

provided that

(2.11) $$\frac{1}{p}=\frac{1}{q}+\frac{\gamma}{n}-1.$$

PROOF. See Hörmander [8] Theorem 4.5.3 for a proof. □

LEMMA 2.4. *Let* $0<\gamma<n$ $(n\geqq 1)$. *We have*

(2.12) $$\left|\int V_\gamma * f(w)(x)u(x)v(x)dx\right| \leqq C\|V_{(n+\gamma)/2} * f(w)\|_2 \|uv\|_{2n/(2n-\gamma)}$$
$$\leqq C\|V_{(n+\gamma)/2} * f(w)\|_2 \|u\|_2 \|v\|_{2n/(n-\gamma)}$$

for suitable functions u, v *and* w.

PROOF. Using the Plancherel theorem and the Schwartz inequality we have

(2.13) $$\int V_\gamma * f(w)(x)u(x)v(x)dx=(2n)^{-n}\int |\xi|^{(\gamma-n)/2}\hat{f}(w)(\xi)|\xi|^{(\gamma-n)/2}\widehat{uv}(\xi)d\xi$$
$$\leqq \|V_{(n+\gamma)/2} * f(w)\|_2 \|V_{(n+\gamma)/2} * (uv)\|_2 .$$

It follows from Lemma 2.3 and the Hölder inequality that

(2.14) $$\|V_{(n+\gamma)/2} * (uv)\|_2 \leqq C\|uv\|_{2n/(2n-\gamma)} \leqq C\|u\|_2 \|v\|_{2n/(n-\gamma)} .$$

(2.13) and (2.14) show that (2.12) holds. □

LEMMA 2.5. *Let* $0<\gamma<n$ $(n\geqq 1)$. *Let* $w_j(t)$ *be a solution of* (2.2) *obtained by Lemma* 2.1. *Then the following estimates holds:*

(2.15) $$\|V_{(n+\gamma)/2} * f(h_j * w_j(t))\|_2 \leqq C(\phi, \psi),$$

(2.16) $$\|V_\gamma * f(h_j * w_j(t))\|_{2n/\gamma} \leqq C(\phi, \psi),$$

(2.17) $$\|F_j(w_j(t))\|_{2n/(n+\gamma)} \leqq C(\phi, \psi)$$

for $j\in \boldsymbol{N}$ *and* $t\in \boldsymbol{R}$, *where* $C(\phi, \psi)$ *is a positive constant which is dependent on* (ϕ, ψ) *but independent of* t *and* j.

PROOF. Noting (2.6), we have (2.15) by Lemma 2.3.

From Lemma 2.4 it follows that

$$(2.18)\qquad \left|\int V_\gamma * f(h_j * w_j(t)) v(x) dx\right| \leqq C\| V_{(n+\gamma)/2} * f(h_j * w_j(t))\|_2 \|v\|_{2n/(2n-\gamma)}$$

for $v\in C_0^\infty(\boldsymbol{R}^n)$. Therefore we obtain (2.16) by (2.15), the density and the duality. Noting $\|w_j(t)\|_2 \leqq C(\phi, \psi)$, (2.17) follows from (2.16) and the Hölder inequality. □

LEMMA 2.6. *Let I be any compact interval in $\boldsymbol{R}$. Let $\{w_j(t)\}$ be a convergent subsequence obtained by Lemma 2.2. Then it has the following properties:*

$$(2.19)\qquad V_{(n+\gamma)/2} * f(h_j * w_j(t)) \longrightarrow V_{(n+\gamma)/2} * f(w(t))$$

weakly in L_2 and uniformly on I and

$$(2.20)\qquad F_j(w_j(t)) \longrightarrow F(w(t))$$

weakly in $L_{2n/(n+\gamma)}$ for $t\in I$ as $j\to\infty$.

In order to prove this lemma, we prepare two lemmas.

LEMMA 2.7. *For any compact interval $I\subset\boldsymbol{R}$ and any compact subset $K\subset\boldsymbol{R}^n$ we have*

$$(2.21)\qquad h_j * w_j(t) \longrightarrow w(t) \quad \text{in} \quad C(I; L_2(K)) \quad \text{as } j\to\infty .$$

PROOF. Noting (2.8), we can prove (2.21) easily. So we may omit the proof. □

LEMMA 2.8. *Let $0<\gamma<n$. For any compact interval $I\subset\boldsymbol{R}$ we have*

$$(2.22)\qquad V_\gamma * f(h_j * w_j(t)) \longrightarrow V_\gamma * f(w(t)) \quad \text{in } \mathcal{D}'$$

uniformly on I as $j\to\infty$.

PROOF. Let $v\in C_0^\infty(\boldsymbol{R}^n)$ and supp $v\subset\{x; |x|\leqq R\}$. By the Fubini theorem we have

$$(2.23)\qquad \int V_\gamma * \{f(h_j * w_j(t)) - f(w(t))\} v(x) dx = \int \{f(h_j * w_j(t)) - f(w(t))\} V_\gamma * v(x) dx$$
$$= \int_{|x|\leqq R+m} + \int_{|x|\geqq R+m}$$
$$= I_1 + I_2 .$$

Here m is a suitable number which will be chosen later. If $|x|\geqq R+m$, we have $|x-y|\geqq m$ for $|y|\leqq R$. Noting this, we obtain

$$|I_2|\leqq m^{-\gamma}\int|f(h_j*w_j(t))-f(w(t))|dx\int|v(y)|dy \tag{2.24}$$
$$\leqq m^{-\gamma}(\|h_j*w_j(t)\|_2^2+\|w(t)\|_2^2)\|v\|_1.$$

Next we estimate I_1. We have

$$|I_1|\leqq\int_{|x|\leqq R+m}\left\{|f(h_j*w_j(t))-f(w(t))|\int_{|y|\leqq R}|x-y|^{-\gamma}|v(y)|dy\right\}dx. \tag{2.25}$$

It follows from $n-1-\gamma>-1$ that

$$\int_{|y|\leqq R}|x-y|^{-\gamma}|v(y)|dy\leqq C(2R+m)^{n-\gamma}\|v\|_\infty. \tag{2.26}$$

This implies that

$$|I_1|\leqq C(2R+m)^{n-\gamma}(\|w_j(t)\|_2+\|w(t)\|_2)\|v\|_\infty\|h_j*w_j(t)-w(t)\|_{L_2(|x|\leqq R+m)}. \tag{2.27}$$

Choosing m sufficiently large, we have (2.22) by (2.6), (2.9), (2.24), (2.27) and Lemma 2.7. □

We are ready to prove Lemma 2.6.

PROOF OF LEMMA 2.6. As $0<(n+\gamma)/2<n$, we have (2.19) by (2.15) and Lemma 2.8.

By (2.17) we obtain (2.20) if we can show that

$$F_j(w_j(t))\longrightarrow F(w(t))\qquad \text{in } \mathcal{D}' \text{ for } t\in I \tag{2.28}$$

as $j\to\infty$. For $v\in C_0^\infty(\boldsymbol{R}^n)$ we have

$$\begin{aligned}(F_j(w_j(t))-F(w(t)),\ v)&=(V_\gamma*f(h_j*w_j(t))h_j*w_j(t),\ h_j*v-v)\\&\quad+(F(h_j*w_j(t))-F(w(t)),\ v)\\&=I_1+I_2.\end{aligned} \tag{2.29}$$

Lemma 2.4, (2.15) and (2.6) imply that

$$\begin{aligned}|I_1|&\leqq C\|V_{(n+\gamma)/2}*f(h_j*w_j(t))\|_2\|w_j(t)\|_2\|h_j*v-v\|_{2n/(n-\gamma)}\\&\leqq C(\phi,\ \psi)\|h_j*v-v\|_{2n/(n-\gamma)}.\end{aligned} \tag{2.30}$$

We put

$$\begin{aligned}I_2&=(V_\gamma*f(h_j*w_j(t))\{h_j*w_j(t)-w(t)\},\ v)\\&\quad+(V_\gamma*\{f(h_j*w_j(t))-f(w(t))\}w(t),\ v)\\&=I_{21}+I_{22}.\end{aligned} \tag{2.31}$$

Again by Lemma 2.4 and (2.15) we have

$$|I_{21}|\leqq C(\phi,\psi)\|h_j*w_j(t)-w(t)\|_{L_2(\mathrm{supp}\,v)}\|v\|_{2n/(n-\gamma)}. \tag{2.32}$$

We can rewrite I_{22} as follows:

$$I_{22}=(V_\gamma*\{f(h_j*w_j(t))-f(w(t))\},\ w(t)v). \tag{2.33}$$

On the other hand it follows from (2.16) and Lemma 2.8 that

$$V_\gamma*f(h_j*w_j(t))\longrightarrow V_\gamma*f(w(t)) \tag{2.34}$$

weakly in $L_{2n/\gamma}$ and uniformly on I as $j\to\infty$. By the Hölder inequality and (2.6) we have $w(t)v\in L_{2n/(2n-\gamma)}$. Noting this, (2.34) implies that $I_{22}\to 0$ as $j\to\infty$. So (2.30), (2.32) and Lemma 2.7 show that (2.28) holds. □

Now we are in a position to prove Theorem 1.

PROOF OF THEOREM 1. Let $\{w_j(t)\}$ be a convergent subsequence obtained by Lemma 2.2. We multiply $v\in C_0^\infty(\boldsymbol{R}^n)$ by (2.4) and integrate on $\boldsymbol{R}^n$. Then we have

$$\begin{aligned}(w_j(t),v)=&(h_j*\phi,\cos\{Ht\}v)+(h_j*\psi,H^{-1}\sin\{Ht\}v)\\&-\int_0^t(F_j(w_j(\tau)),H^{-1}\sin\{H(t-\tau)\}v)d\tau.\end{aligned} \tag{2.35}$$

Using the Hausdroff-Young inequality, we can show that $H^{-1}\sin\{H(t-\tau)\}v\in L_{2n/(n-\gamma)}$. Thus it follows from (2.20) that

$$(F_j(w_j(\tau)),H^{-1}\sin\{H(t-\tau)\}v)\longrightarrow(F(w(\tau)),H^{-1}\sin\{H(t-\tau)\}v) \tag{2.36}$$

as $j\to\infty$. By the Hölder inequality, (2.17) and the Hausdroff-Young inequality we have

$$\begin{aligned}(F_j(w_j(t)),H^{-1}\sin\{H(t-\tau)\}v)&\leqq\|F_j(w_j(\tau))\|_{2n/(n+\gamma)}\|H^{-1}\sin\{H(t-\tau)\}v\|_{2n/(n-\gamma)}\\&\leqq C(\phi,\psi)\|\hat{v}\|_{2n/(n+\gamma)}.\end{aligned} \tag{2.37}$$

(2.36) and (2.37) mean that we can use the Lebesgue dominated convergence theorem. Thus letting $j\to\infty$ in (2.35), we obtain (1.5).

Noting $\phi\in L_{4n/(2n-\gamma)}$, (2.6) and (2.19) imply (1.7).

Next we show that

$$(w(t),v)\in C^2(\boldsymbol{R})\qquad\text{for any}\quad v\in C_0^\infty(\boldsymbol{R}^n). \tag{2.38}$$

From (1.5) it follows that $(w(t),v)\in C^1(\boldsymbol{R})$ and

$$\begin{aligned}\frac{d}{dt}(w(t),v)=&-(\phi,H^{-1}\sin\{Ht\}v+(\psi,\cos\{Ht\}v)\\&-\int_0^t(F(w(\tau)),\cos\{H(t-\tau)\}v)d\tau.\end{aligned} \tag{2.39}$$

If we show that

$$(F(w(t)),\ v)\in C(\boldsymbol{R}). \tag{2.40}$$

(2.38) can be proved. Let $t\in\boldsymbol{R}$ and be fixed. Put

$$\begin{aligned} J(\eta)&=(F(w(t+\eta))-F(w(t)),\ v)\\ &=(V_\gamma*\{f(w(t+\eta))-f(w(t))\}w(t),\ v)\\ &\quad+(V_\gamma*f(w(t+\eta))\{w(t+\eta)-w(t)\},\ v)\\ &=J_1(\eta)+J_2(\eta). \end{aligned} \tag{2.41}$$

By (2.12) we obtain

$$|J_2(\eta)|\leqq C\|V_{(n+\gamma)/2}*f(w(t+\eta))\|_2\|w(t+\eta)-w(t)\|_2\|v\|_{2n/(n-\gamma)}. \tag{2.42}$$

From (1.7) and (2.9) it follows that $|J_2(\eta)|\to 0$ as $\eta\to 0$. By (2.3) and (2.16) we can show that

$$V_\gamma*f(h_j*w_j(t))\in C_w(\boldsymbol{R};\ L_{2n/\gamma}). \tag{2.43}$$

(2.34) and (2.43) imply that

$$V_\gamma*f(w(t))\in C_w(\boldsymbol{R};\ L_{2n/\gamma}). \tag{2.44}$$

Noting $w(t)v\in L_{2n/(2n-\gamma)}$, by (2.44) we have $|J_1(\eta)|\to 0$ as $\eta\to 0$. Then (2.40) is proved. Noting (2.9), (2.17) and (2.20), (1.3) and (1.4) have already been proved. (1.5) implies (1.6). Thus the proof of Theorem 1 is completed.

3. Proof of Theosem 2.

We begin with the well known estimates for the elementary solution of the linear Klein-Gordon equation.

PROPOSITION 3.1. *Lht* $1<p\leqq 2$ *and* $1/p+1/p'=1$. *Put* $\delta(p')=1/2-1/p'$.

(i) *Let* p', s' *and* s *satisfy*

$$(n+1)\delta(p')\leqq 1+s-s'. \tag{3.1}$$

Then we have for $g\in C_0^\infty(\boldsymbol{R}^n)$

$$\|H^{-1}\sin\{Ht\}g\|_{s',p'}\leqq C|t|^{1+s-s'-2n\delta(p')}\|g\|_{s,p}. \tag{3.2}$$

(ii) *Put* $1/r=s'+n\delta(p')-1$. *Let* p', r *and* s' *satisfy*

$$0\leqq\frac{1}{r}<\frac{1}{2}\quad\text{and}\quad s'\leqq 1-\frac{(n+1)}{2}\delta(p'). \tag{3.3}$$

Then we have for $g\in C_0^\infty(\boldsymbol{R}^n)$

$$\|H^{-1}\sin\{Ht\}g\|_{L_r(\boldsymbol{R};\ H_{p'}^{s'})}\leqq C\|g\|_2. \tag{3.4}$$

PROOF. (i) See Brenner [1] Appendix 2 for a proof.
(ii) See Ginibre and Velo [5] Lemma 3.1 for a proof. □

The following lemma is useful to estimate the nonlinear term.

LEMMA 3.2. *Let p, a, b and q satisfy*

$$\frac{1}{p}=\frac{1}{a}+\frac{1}{b}+\frac{1}{q}+\frac{\gamma}{n}-1 \quad \text{and} \quad 1-\frac{\gamma}{n}<\frac{1}{a}+\frac{1}{b}<1. \tag{3.5}$$

Then we have

$$\|F(u)-F(v)\|_p \leqq C(\|u-v\|_a\|u+v\|_b\|u\|_q+\|v\|_a\|v\|_b\|u-v\|_q) \tag{3.6}$$

for suitable functions u and v.

PROOF. By the Hölder inequality and Lemma 2.3 we have (3.6). (2.11) yields (3.5). □

PROOF OF THEOREM 2. As mentioned in the introduction, we will prove in the case $3<\gamma<4$ ($n\geqq 4$). Let I be an open interval and J be any finite interval such that $0\in J\subset I$. Let I_0 be an interval such that $0\in I_0\subset J$. Put

$$X(I_0)=L_\infty(I_0\,;\,H^1)\cap L_r(I_0\,;\,L_{p'}).$$

The norm of $X(I_0)$ is given by

$$\|u\|_{X(I_0)}=\mathrm{Max}\{\|u\|_{L_\infty(I_0;\,H^1)},\ \|u\|_{L_r(I_0;\,L_{p'})}\}.$$

From Lemma 2.4, Lemma 2.3 and the embedding $H^1\hookrightarrow L_{4n/(2n-\gamma)}$ it follows that

$$\left|\int F(w(t))v(x)dx\right| \leqq \|w(t)\|_{1,2}^3\|v\|_{1,2} \tag{3.7}$$
$$\leqq \|w\|_{X(J)}^3\|v\|_{1,2}.$$

This means that $F(w(t))\in H^{-1}$ for $t\in J$. Thus by (1.4) we have

$$w(t)=w^0(t)-\int_0^t H^{-1}\sin\{H(t-\tau)\}F(w(\tau))d\tau \tag{3.8}$$

in L_2 for $t\in J$.

Let $w_1(t)$ and $w_2(t)$ be two solutions which satisfy the assumptioms of Theorem 2. From (3.8) we obtain

$$w_1(t)-w_2(t)=-\int_0^t H^{-1}\sin\{H(t-\tau)\}[F(w_1(\tau))-F(w_2(\tau))]d\tau. \tag{3.9}$$

By Proposition 3.1 (i) we have

$$\|w_1(t)-w_2(t)\|_{p'} \leqq C\left|\int_0^t |t-\tau|^{3-\gamma}\|F(w_1(\tau))-F(w_2(\tau)\|_{1,p}d\tau\right|. \tag{3.10}$$

Lemma 3.2 and the Sobolev embedding theorem yield that

$$(3.11)\quad \|F(w_1(\tau))-F(w_2(\tau))\|_{1,p} \leqq C(\|w_1(\tau)\|_{1,2}+\|w_2(\tau)\|_{1,2})(\|w_1(\tau)\|_{p'}+\|w_2(\tau)\|_{p'})\|w_1(\tau)-w_2(\tau)\|_{1,2} + C(\|w_1(\tau)\|_{1,2}+\|w_2(\tau)\|_{1,2})^2\|w_1(\tau)-w_2(\tau)\|_{p'}.$$

By (3.10) we have

$$(3.12)\quad \|w_1(t)-w_2(t)\|_{p'} \leqq C\|w_1-w_2\|_{X(I_0)}(\|w_1\|_{X(J)}+\|w_2\|_{X(J)}) \times\left|\int_0^t |t-\tau|^{3-\gamma}(\|w_1(\tau)\|_{p'}+\|w_2(\tau)\|_{p'})d\tau\right| + C(\|w_1\|_{X(I_0)}+\|w_2\|_{X(I_0)})^2 \times\left|\int_0^t |t-\tau|^{3-\gamma}\|w_1(\tau)-w_2(\tau)\|_{p'}d\tau\right|$$

As $3-\gamma>-1$, from the Young inequality we obtain

$$(3.13)\quad \|w_1(t)-w_2(t)\|_{L_r(I_0;\, L_{p'})} \leqq C|I_0|^{4-\gamma}(\|w_1\|_{X(J)}+\|w_2\|_{X(J)})^2\|w_1-w_2\|_{X(I_0)}.$$

Employing the same arguments as we obtain (3.11), we have

$$(3.14)\quad \|F(w_1(\tau))-F(w_2(\tau))\|_2 \leqq C(\|w_1(\tau)\|_{1,2}+\|w_2(\tau)\|_{1,2})(\|w_1(\tau)\|_{p'}+\|w_2(\tau)\|_{p'})\|w_1(\tau)-w_2(\tau)\|_{p'}.$$

Hence it follows that

$$(3.15)\quad \|w_1(t)-w_2(t)\|_{1,2} \leqq C(\|w_1\|_{X(J)}+\|w_2\|_{X(J)})\left|\int_0^t (\|w_1(\tau)\|_{p'}+\|w_2(\tau)\|_{p'})\|w_1(\tau)-w_2(\tau)\|_{p'}d\tau\right|.$$

Noting $r>2$, from the Hölder inequality we obtain

$$(3.16)\quad \|w_1(t)-w_2(t)\|_{1,2} \leqq C|I_0|^{(r-2)/r}(\|w_1\|_{X(J)}+\|w_2\|_{X(J)})^2\|w_1-w_2\|_{X(I_0)}.$$

(3.13) and (3.16) show that

$$(3.17)\quad \|w_1-w_2\|_{X(I_0)} \leqq C|I_0|^{4-\gamma}(\|w_1\|_{X(J)}+\|w_2\|_{X(J)})^2\|w_1-w_2\|_{X(I_0)}.$$

Taking $|I_0|$ sufficiently small in (3.17), we obtain a inequality which implies that $w_1=w_2$ on I_0. Iterating this process, we can show that $w_1=w_2$ on J. As J arbitrary, Theorem 2 is proved.

4. Proof of Theorem 3.

In this section we restrict our attention to $3<\gamma<4$ ($n\geqq 4$), too. In order to investigate the regularity of a weak solution, we estimate the solutions of the approximating equation.

LEMMA 4.1. *Let* $3<\gamma<4$ $(n\geqq 4)$. *Let* $(\phi, \psi)\in H^1\times L_2$ *and* $w_j(t)$ $(j\in \boldsymbol{N})$ *be a solution of* (2.2) *obtained by Lemma* 2.1. *Let* p' *and* r *be given in Theorem* 2. *Then for any compact interval* $I\subset \boldsymbol{R}$ *there exists a positive constant* $C(\phi, \psi, I)$ *which is dependent on* (ϕ, ψ) *and* I *but independent of* j *such that*

$$\|w_j\|_{L_r(I;\, L_{p'})}\leqq C(\phi, \psi, I) \quad \text{for} \quad j\in \boldsymbol{N}. \tag{4.1}$$

PROOF. It is sufficient to prove (4.1) in the case $I=[0, \alpha]$. In the same way as we obtain (3.12) we have

$$\|w_j(t)\|_{p'}\leqq \|w_j^0(t)\|_{p'}+C(\phi, \psi)\int_0^t |t-\tau|^{3-\gamma}\|w_j(\tau)\|_{p'}\,d\tau \tag{4.2}$$

Here we have used (2.6). By Propositon 3.1 (ii) and the Young inequality we have

$$\begin{aligned}\|w_j\|_{L_r(I;\, L_{p'})}&\leqq C(\|\phi\|_{1,2}+\|\psi\|_2)+C(\phi, \psi)\Big\|\int_0^t |t-\tau|^{3-\gamma}\|w_j(\tau)\|_{p'}\,d\tau\Big\|_{L_r(I)}\\ &\leqq C(\|\phi\|_{1,2}+\|\psi\|_2)+C(\phi, \psi)\alpha^{4-\gamma}\|w_j\|_{L_r(I;\, L_{p'})}.\end{aligned} \tag{4.3}$$

We can verify the condition (3.3) easily. Choosing α to satisfy $C(\phi, \psi)\alpha^{4-\gamma}\leqq 1/2$, we have

$$\|w_j\|_{L_r(I;\, L_{p'})}\leqq C(\phi, \psi, I) \quad \text{for} \quad j\in \boldsymbol{N}. \tag{4.4}$$

Next we show that (4.1) holds for any number $\alpha\in[0, \infty)$. Let M be the supremum of the number $\alpha\in[0, \infty)$ so that (4.1) holds with $I=[0, \alpha]$. We have already showed that $M>0$. If $M=\infty$, the lemma is proved. We assume that $M<\infty$. Let $\alpha<M$ and $I_1=[0, \alpha]$. From the definition of M it follows that

$$\|w_j\|_{L_r(I_1;\, L_{p'})}\leqq C(\phi, \psi, I_1) \quad \text{for} \quad j\in \boldsymbol{N}. \tag{4.5}$$

Let $\alpha<\beta$ and $I_2=[\alpha, \beta]$. Employing the same arguments as we obtain (4.3), we have

$$\begin{aligned}\|w_j\|_{L_r(I_2;\, L_{p'})}&\leqq C(\|\phi\|_{1,2}+\|\psi\|_2)\\ &\quad +C(\phi, \psi)\Big\|\int_\alpha^t |t-\tau|^{3-\gamma}\|w_j(\tau)\|_{p'}\,d\tau\Big\|_{L_r(I_2)}\\ &\quad +C(\phi, \psi)\Big\|\int_0^\alpha |t-\tau|^{3-\gamma}\|w_j(\tau)\|_{p'}\,d\tau\Big\|_{L_r(I_2)}\\ &=J_1+J_2+J_3.\end{aligned} \tag{4.6}$$

From the same arguments of a proof of the Young inequality we obtain

$$J_2\leqq C(\phi, \psi)(\beta-\alpha)^{4-\gamma}\|w_j\|_{L_r(I_2;\, L_{p'})}, \tag{4.7}$$

$$J_3\leqq C(\phi, \psi)\beta^{4-\gamma}\|w_j\|_{L_r(I_1;\, L_{p'})}. \tag{4.8}$$

Choosing β near α to satisfy $C(\phi, \psi)(\beta-\alpha)^{4-\gamma}\leqq 1/2$, by (4.5)~(4.8) we have

$$\|w_j\|_{L_r([0,\beta];\, L_{p'})}\leqq C(\phi, \psi, \beta) \quad \text{for} \quad j\in \boldsymbol{N}. \tag{4.9}$$

Since the distence between α and β depends on $C(\phi, \psi)$ only, we can choose α near M to satisfy $M-\alpha<\beta-\alpha$. Hence (4.9) contradicts the definition of M. □

LEMMA 4.2. *Let* $3<\gamma<4$ $(n\geqq 4)$. *Let* $(\phi, \psi)\in H^2\times H^1$ *and* $w_j(t)$ $(j\in \boldsymbol{N})$ *be a solution of* (2.2) *obtained by Lemma* 2.1. *Let* $1/q'=1/2-1/2n$. *Then for any compact interval* $I\subset \boldsymbol{R}$ *there exists a positive constant* $C(\phi, \psi, I)$ *which is dependent on* (ϕ, ψ) *and* I *but independent of* j *such that*

$$\|w_j\|_{L_\infty(I;\, H^1_{q'})}\leqq C(\phi, \psi, I) \quad \textit{for} \quad j\in \boldsymbol{N}. \tag{4.10}$$

PROOF. Let $I=[0, \alpha]$. From (2.4) and Proposition 3.1 (i) it follows that

$$\|w_j(t)\|_{1,q'}\leqq \|w_j^0(t)\|_{1,q'}+\int_0^t \|F_j(w_j(\tau))\|_{1,q}d\tau. \tag{4.11}$$

We can verify (3.1) easily. Applying Lemma 3.2 to $\|F_j(w_j(\tau))\|_{1,q}$, we have

$$\|F_j(w_j(\tau))\|_{1,q}\leqq C\|w_j(\tau)\|^2_{p'}\|w_j(\tau)\|_{1,q'}, \tag{4.12}$$

where p' is given by Lemma 4.1. As the embedding $H^2\hookrightarrow H^1_{q'}$ holds, from (4.11) and (4.12) we obtain

$$\|w_j(t)\|_{1,q'}\leqq C(\|\phi\|_{2,2}+\|\psi\|_{1,2})+C\|w_j\|_{L_\infty(I;\, H^1_{q'})}\int_0^t \|w_j(\tau)\|^2_{p'}d\tau. \tag{4.13}$$

From the Hölder inequality and Lemma 4.1 it follows that

$$\|w_j\|_{L_\infty(I;\, H^1_{q'})}\leqq C(\|\phi\|_{2,2}+\|\psi\|_{1,2})+C(\phi, \psi, I)\alpha^{(r-2)/r}\|w_j\|_{L_\infty(I;\, H^1_{q'})}. \tag{4.14}$$

Here choosing α sufficiently small, we have

$$\|w_j\|_{L_\infty(I;\, H^1_{q'})}\leqq C(\phi, \psi, I). \tag{4.15}$$

Employing the same arguments of the proof of Lemma 4.1, we can show that (4.10) holds for any $\alpha\in[0, \infty)$. So we may omit its proof. □

LEMMA 4.3. *Under the same assumptions of Lemma* 4.2. *we have*

$$\|w_j\|_{L_\infty(I;\, H^2)}\leqq C(\phi, \psi, I) \quad \textit{for} \quad j\in \boldsymbol{N} \tag{4.16}$$

for any compact interval $I\subset \boldsymbol{R}$. *Here* $C(\phi, \psi, I)$ *is a positive constant which is dependent on* (ϕ, ψ) *and* I *but independent of* j.

PROOF. From (2.4) it follows that

$$\|w_j(t)\|_{2,2}\leqq C(\|\phi\|_{2,2}+\|\psi\|_{1,2})+\int_0^t \|F_j(w_j(\tau))\|_{1,2}d\tau. \tag{4.17}$$

Applying Lemma 3.2 to $\|F_j(w_j(\tau))\|_{1,2}$, we obtain

$$\|F_j(w_j(\tau))\|_{1,2} \leqq C\|w_j(\tau)\|_{1,q'}^2 \|w_j(\tau)\|_{2,2}, \tag{4.18}$$

where q' is given by Lemma 4.2. To note Lemma 4.2, we have

$$\|w_j(t)\|_{2,2} \leqq C(\|\phi\|_{2,2}+\|\psi\|_{1,2})+C(\phi, \psi, I)\int_0^t \|w_j(\tau)\|_{2,2} d\tau. \tag{4.19}$$

The Gronwall inequality implies (4.16). □

Now we give the estimates of the weak solution.

LEMMA 4.4. *Let $w(t)$ be a weak solution of* (1.1) *obtained by Theorem* 1. *Let* $3<\gamma<4$ $(n\geqq 4)$ *and I be any compact interval in $\boldsymbol{R}$.*

(i) *Let $(\phi, \psi)\in H^1\times L_2$. Then we have*

$$\|w\|_{L_r(I;\, L_{p'})} \leqq C(\phi, \psi, I), \tag{4.20}$$

where $C(\phi, \psi, I)$ is a positive constant which is dependent on (ϕ, ψ) and I, provided that

$$\frac{1}{p'}=\frac{1}{2}-\frac{\gamma-1}{2n} \quad \text{and} \quad \frac{1}{r}=\frac{\gamma-3}{2}. \tag{4.21}$$

(ii) *Let $(\phi, \psi)\in H^2\times H^1$. Then we have*

$$\|w\|_{L_\infty(I;\, H^2)} \leqq C(\phi, \psi, I), \tag{4.22}$$

where $C(\phi, \psi, I)$ is a positive constant which is dependent on (ϕ, ψ) and I.

PROOF. By (4.1), (4.16) and Lemma 2.2 we can choose a covergent subsequence (again denoted by $w_j(t)$) so that

$$w_j(t) \longrightarrow w(t) \quad \text{weakly in } L_r(I; L_{p'}), \tag{4.23}$$

$$w_j(t) \longrightarrow w(t) \quad \text{weakly in } H^2 \text{ and uniformly on } I \tag{4.24}$$

as $j\to\infty$. Thus we have (4.20) and (4.22). □

We prepare three lemmas on the regularity of the integral equation.

LEMMA 4.5. *Assume that for $i=0$ or* 1

$$F(w(t))\in L_1^{\mathrm{loc}}(\boldsymbol{R}; H^i). \tag{4.25}$$

Then we have

$$\int_0^t H^{-1}\sin\{H(t-\tau)\}F(w(\tau))d\tau \in C(\boldsymbol{R}; H^{1+i})\cap C^1(\boldsymbol{R}; H^i). \tag{4.26}$$

PROOF. See Motai [9] Lemma 4.2 for a proof. □

LEMMA 4.6. *Assume that for* $k\in\boldsymbol{N}$

$$w(t)\in\bigcap_{i=0}^{k}C^{i}(\boldsymbol{R};\,H^{k-i}). \tag{4.27}$$

Then we have

$$F(w(t))\in\bigcap_{i=0}^{k}C^{i}(\boldsymbol{R};\,H^{k-i})\qquad \text{for}\quad 0<\gamma<\mathrm{Min}\{2k,\,n\}. \tag{4.28}$$

PROOF. If we use Lemma 3.2 and the Sobolev embedding theorem, we can prove (4.28) easily. So we may omit a proof. □

LEMMA 4.7. *Assume that for* $k\in\boldsymbol{N}$

$$F(w(t))\in\bigcap_{i=0}^{k}C^{i}(\boldsymbol{R};\,H^{k-i}). \tag{4.29}$$

Then we have

$$\int_{0}^{t}H^{-1}\sin\{H(t-\tau)\}F(w(\tau))d\tau\in\bigcap_{i=0}^{k+1}C^{i}(\boldsymbol{R};\,H^{k+1-i}). \tag{4.30}$$

PROOF. This result is well-known. So we may omit the proof. □

We are in a positon to prove Theorem 3.

PROOF OF THEOREM 3. (i) Let $w(t)$ be a weak solution obtained by Theorem 1. Since $w(t)\in L_{\infty}(\boldsymbol{R};\,H^{1})$, from the same argument as we obtain (3.8) it follows that

$$w(t)=w^{0}(t)-\int_{0}^{t}H^{-1}\sin\{H(t-\tau)\}F(w(\tau))d\tau\qquad \text{in}\ \ L_{2} \tag{4.31}$$

for $t\in\boldsymbol{R}$. By $(\phi,\,\psi)\in H^{1}\times L_{2}$ we have

$$w^{0}(t)\in C(\boldsymbol{R};\,H^{1})\cap C^{1}(\boldsymbol{R};\,L_{2}). \tag{4.32}$$

Noting (3.14), from (1.7) we obtain

$$\|F(w(t))\|_{2}\leqq C(\phi,\,\psi)\|w(t)\|_{p'}^{\frac{2}{p'}}. \tag{4.33}$$

As $r>2$, Lemma 4.4 (i) and (4.32) imply (4.25). Hence by Lemma 4.5 we have (1.12).

The uniqueness of $w(t)$ follows from (1.12) and Theorem 2.

If we resolve (1.1) at initial time $t_{0}\in\boldsymbol{R}$ with a initial data $(w(t_{0}),\,\partial_{t}w(t_{0}))$, by Theorem 1 we obtain

$$E(w(t),\,\partial_{t}w(t))\leqq E(w(t_{0}),\,\partial_{t}w(t_{0}))\qquad \text{for}\quad t\in\boldsymbol{R}. \tag{4.34}$$

The uniqueness, (1.7) and (4.34) imply (1.13).

(ii) We first note that for $(\phi, \psi)\in H^k\times H^{k-1}$ $(k\geqq 2)$ we have

$$w_0(t)\in\bigcap_{i=0}^{k}C^i(\boldsymbol{R};\, H^{k-i}). \tag{4.35}$$

In the case $k=2$ we have

$$F(w(t))\leqq C\|w(t)\|_{2,2}^3 \tag{4.36}$$

by Lemma 3.2 and the Sobolev embedding theorem. From Lemma 4.4 (ii) and Lemma 4.5 it follows that

$$w(t)\in C(\boldsymbol{R};\, H^2)\cap C^1(\boldsymbol{R};\, H^1). \tag{4.37}$$

This implies that

$$F(w(t))\in C(\boldsymbol{R};\, H^1)\cap C^1(\boldsymbol{R};\, L_2). \tag{4.38}$$

By Lemma 4.7 we have

$$w(t)\in\bigcap_{i=0}^{2}C^i(\boldsymbol{R};\, H^{2-i}). \tag{4.39}$$

In the case $k>2$ we can first obtain (4.39). Lemma 4.6 shows that

$$F(w(t))\in\bigcap_{i=0}^{2}C^i(\boldsymbol{R};\, H^{2-i}). \tag{4.40}$$

And Lemma 4.7 implies that

$$w(t)\in\bigcap_{i=0}^{3}C^i(\boldsymbol{R};\, H^{3-i}). \tag{4.41}$$

Iterating this process, we can prove (1.14).

Corollary follows from the Sobolev lemma.

The proof Theorem 3 is completed.

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