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KILLING VECTOR FIELDS AND THE HOLONOMY ALGEBRA IN SEMIRIEMANNIAN MANIFOLDS

By

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Abstract In this paper we generalize some results of Kostant [2] to semiriemannian manifolds of signature s. We also prove that any Killing vector field on a semiriemannian homogeneous compact flat manifold is parallel.

0. Introduction.

Let (M_s^n, g) be a semiriemannian manifold of dimension n and signature s. Let X be a Killing vector field on M. The A_X-operator provides a skew symmetric endomorphism of TM. It is well known that

$$\nabla_Y A_X = R_{XY}.$$

This fact and the Ambrossse-Singer theorem (Wo) show that the A_X-operator lies infinitesimally in the holonomy algebra $\boldsymbol{h}$ of M.

We ask ourselves whether or not A_X lies in $\boldsymbol{h}$.

In the riemannian case the question has an affirmative answer on compact manifolds [2]. We obtain here a similar result in the semiriemannian case.

Finally we study the holonomicity of a Killing vector field on semiriemannian manifolds of constant curvature. If the curvature is non zero, the holonomy algebra can be represented as $\boldsymbol{po}(n, s)$, that is the skew symmetric endomorphisms of TM. In this case each Killing vector field is holonomic.

There are flat manifolds and Killing vector fields on them such that the A_X-operator does not lie in the holonomy algebra, that is $A_X \notin \boldsymbol{h}$. Take, for instance, R_s^n. In the usual coordinates on R_s^n, X is a Killing vector field if

$$X = \sum_{i,j} \varepsilon_i K_i^j x_i \frac{\partial}{\partial x_j}$$

where $K_i^j = -K_j^i$ are constants, $\varepsilon_i = g(\partial/\partial x_i, \partial/\partial x_i) = \pm 1$ and $x_0 = 1$. There are nonholonomic Killing vector fields on R_s^n: nonparallel vector fields are nonholo-

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nomic because flatness implies $\boldsymbol{h}=0$.

However, the assumption of compactness and homogeneity of M allows us to state that any Killing vector field on a compact homogeneous semiriemannian flat manifold is parallel.

1. Main theorem.

Let (M_s^n, g) be a semiriemannian manifold of dimension n and signature s. If $p\in M$ and A, $B\in \mathrm{End}(T_pM)$ we denote by ϕ the trace form

$$\phi(A, B)=-\mathrm{trace}(A\cdot B).$$

Note that,

i) ϕ is nondegenerate on $\boldsymbol{po}(n, s)$,

ii) ϕ is parallel.

From now on for any $\Omega\subset\boldsymbol{po}(n, s)$, $\Omega^{\perp}$ will denote its orthogonal complementary with respect to ϕ.

THEOREM 1. *Let (M_s^n, g) be a semiriemannian manifold compact orientable manifold and X a Killing vector field on M. If ϕ is nondegenerate on the holonomy algebra $\boldsymbol{h}$, then the A_X-operator decomposes as*

$$A_X=h+B_X$$

where $h\in\boldsymbol{h}$, $B_X\in\boldsymbol{h}^{\perp}$ and $\phi(B_X, B_X)=0$.

PROOF. The nondegenerate character of ϕ on $\boldsymbol{h}$ allows us to decompose

$$\boldsymbol{po}(n, s)=\boldsymbol{h}+\boldsymbol{h}^{\perp} \quad \text{and} \quad A_X=h+B_X$$

in a unique way.

For any field Y on M.

$$R_{XY}=\nabla_Y A_X=\nabla_Y h+\nabla_Y B_X\,.$$

R_{XY} and $\nabla_Y h$ lie in $\boldsymbol{h}$ and $\nabla_Y B_X$ lies in $\boldsymbol{h}^{\perp}$. Thus $\nabla_Y B_X=0$ and B_X is parallel.

And accordingly

$$\mathrm{div}\, B_X X=\mathrm{trace}(B_X\cdot B_X)=\phi(B_X, B_X)\,.$$

But $\phi(B_X, B_X)$ is constant because

$$Y\phi(B_X, B_X)=2\phi(\nabla_Y B_X, B_X)=0\,.$$

Finally, the integral of $\mathrm{div}\, B_X X$ on M gives

$$0=\int_M \operatorname{div}(B_X X)=k\operatorname{vol}(M).$$

That is

$$0=k=\phi(B_X, B_X).$$

(Q. E. D.)

REMARK. This theorem still holds without the assumption of orientability because the covering of the orientations is (2: 1) and is also a local isometry.

We gave in [1] some examples of compact semiriemannian manifolds with nonholonomic Killing vector fields.

2. The flat case.

Let (M_s^n, g) be a compact flat manifold. If X is a Killing vector field on M, by Theorem 1

$$\phi(A_X, A_X)=0.$$

On the assumption of homogeneity, we will see in this section that $A_X=0$.

We recall

LEMMA 2 [2]. *Let (M_s^n, g) be a semiriemannian manifold; if X is a Killing vector field on M, assume that*

$$2f=g(X, X).$$

Then,

i) $\operatorname{grad} f=A_X X$.

ii) $H^f(V, W)=g(\nabla_V X, \nabla_W X)+g(R_{XV}X, W)=g(\nabla_V(A_X X), W)$

iii) $\Delta f=-\phi(A_X, A_X)-\operatorname{Ric}(X, X)$.

PROPOSITION 3 (Marsden) [2]. *A homogeneous compact semiriemannian manifold is complete.*

COROLLARY 4. *A homogeneous compact flat semiriemannian manifold is geodesically convex. (i.e. given any two points there is a geodesic joining them).*

PROOF. By Proposition 3 the universal covering of M is a flat complete simply connected manifold; thus it is R_s^n. In order to obtain a geodesic σ joining $p\in M$ and $q\in M$, take $\tilde{p}$, in the fiber of p and $\tilde{q}$ in the fiber of q and project on M the straight line $\tilde{p}\tilde{q}$. (Q. E. D.)

PROPOSITION 5. *Let X be a Killing vector field on a homogeneous compact flat semiriemannian manifold (M_s^n, g). The product $g(X, X)$ is constant on M.*

PROOF. $A_X X$ is a Jacobi field.

Let γ be a geodesic; then

$$\begin{aligned}\nabla_{\gamma^{\cdot}}(\nabla_{\gamma^{\cdot}}(A_X X)) &= \nabla_{\gamma^{\cdot}}((\nabla_{\gamma^{\cdot}}A_X)X + A_X(\nabla_{\gamma^{\cdot}}X)) = \nabla_{\gamma^{\cdot}}(R_{X\gamma^{\cdot}}X - A_X A_X\gamma^{\cdot}) \\ &= -\nabla_{\gamma^{\cdot}}(A_X(A_X\gamma^{\cdot})) = -(\nabla_{\gamma^{\cdot}}A_X)(A_X\gamma^{\cdot}) - A_X(\nabla_{\gamma^{\cdot}}(A_X\gamma^{\cdot})) \\ &= R_{X\gamma^{\cdot}}(A_X\gamma^{\cdot}) - A_X((\nabla_{\gamma^{\cdot}}A_X)(\gamma) - A_X A_X(\nabla_{\gamma^{\cdot}}\gamma^{\cdot})) = 0\,.\end{aligned}$$

Assume that $2f = g(X, X)$. Since M is compact, f reaches at least a maximum and a minimum at p and q respectively. By Corollary 4 there is a geodesic joining p and q. Call it σ. $A_X X$ is a Jacobi field on σ which cancels at p and q (Lemma 2). Because of the flatness of M, $A_X X = 0$.

Then $f(p) = f(q)$ and f must be constant on M. (Q. E. D.)

THEOREM 6. *Let X be a Killing vector field on a semiriemannian homogeneous compact flat manifold M_s^n. Then X is parallel.*

PROOF. 1st step. The universal covering of M_s^n is R_s^n. Then $M \cong R_s^n/\Gamma$ where Γ is a properly discontinuous subgroup of the motions of R_s^n. Let $\tilde{X}$ be the lift of X on R_s^n; $\tilde{X}$ is a Killing vector field on R_s^n and it is Γ-invariant.

2nd step. Take $f = (1/2)g(X, X) = (1/2)g(\tilde{X}, \tilde{X})$.

From Lemma 2 and because of the flatness of M,

$$H^f(V, W) = g(\nabla_V X, \nabla_W X) = g(A_X V, A_X W) = -g(A_X A_X V, W)\,.$$

On the other hand, by Proposition 5

$$H^f(V, W) = 0 \qquad \forall V, W\,.$$

Thus $A_X \cdot A_X = 0$ and $A_{\tilde{X}} \cdot A_{\tilde{X}} = 0$.

3rd step. Let p be a point of R_s^n. We can choose a basis of T_pM $v_1, w_1, \cdots, v_r, w_r, u_1, \cdots, u_t$ in which the $A_{\tilde{X}}$ matrix has the form

$$(*)\qquad \begin{array}{c} \begin{array}{cccccc} v_1 & w_1 & v_r & w_r & u_1 & u_t \end{array} \\ \left(\begin{array}{cccc|ccc} 0 & & & & & & \\ 1 & 0 & & & & & \\ & & \ddots & & & & \\ & & & 0 & & & \\ & & & 1 \;\; 0 & & & \\ \hline & & & & 0 & & \\ & & & & & \ddots & \\ & & & & & & 0 \end{array}\right) \end{array}$$

Using parallel transport and because of the flatness we can assume that we have a coordinate system $x_1, y_1, \cdots, x_r, y_r, z_1, \cdots, z_t$ on R_s^n such that the matrix of the $A_{\tilde{X}}$-operator in the associated frame is (*). In this coordinate system the nonparallel part of $\tilde{X}$ is

$$x_1\frac{\partial}{\partial y_1}+\cdots+x_r\frac{\partial}{\partial y_1}.$$

There is no lost of generality in assuming that

$$\tilde{X}=x_1\frac{\partial}{\partial y_1}+\cdots+x_r\frac{\partial}{\partial y_r}.$$

4$^{\text{th}}$ Step. Let us now consider the new system

$$(x_1, \cdots, x_r, y_1, \cdots, y_r, z_1, \cdots, z_t).$$

Let μ be an element of Γ; because of the homogeneity of M_s^n, Γ is a group of pure translations (see [4] pg. 135). In our new coordinate system

$$\begin{array}{c} \\ \mu= \end{array}\begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} \end{array}\begin{array}{c} \\ +\begin{pmatrix} M \\ N \\ U \end{pmatrix} \end{array}$$

where I is the identity matrix.

The Γ-invariance of $\tilde{X}$ is reflected on the μ-matrix by the fact that $M=0$, so that the dimension of the subspace spanned by the translation components of the elements of Γ is not greater than $n-r$. If $r\neq 0$ the translation components of the elements of Γ do not generate R_s^n. But this is impossible because M is compact.

Consequently, $r=0$ and $\tilde{X}$ and X are parallel. (Q.E.D.)

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